Objectivity in continuum mechanics, an introduction

Motions, Eulerian and Lagrangian variables and functions, deformation gradient, Lie derivatives, velocity-addition formula, Coriolis.

Gilles LEBORGNE, www.isima.fr/leborgne

January 3, 2023

In classical mechanics, there are two objectivities: 1- The covariant objectivity concerns the universal laws of physics required to be observer independent (true in any reference frame); This is a main topic in this manuscript. 2- The isometric objectivity concerns the constitutive laws of materials once expressed in a reference frame.

Covariant objectivity in continuum mechanics follows Maxwell's requirements, cf. [13] page 1: "2. (...) The formula at which we arrive must be such that a person of any nation, by substituting for the different symbols the numerical value of the quantities as measured by his own national units, would arrive at a true result. (...) 10. (...) The introduction of coordinate axes into geometry by Des Cartes was one of the greatest steps in mathematical progress, for it reduced the methods of geometry to calculations performed on numerical quantities. The position of a point is made to depend on the length of three lines which are always drawn in determinate directions (...) But for many purposes in physical reasoning, as distinguished from calculation, it is desirable to avoid explicitly introducing the Cartesian coordinates, and to fix the mind at once on a point of space instead of its three coordinates, and on the magnitude and direction of a force instead of its three components. This mode of contemplating geometrical and physical quantities is more primitive and more natural than the other,..."

And see the (short) historical note given in the introduction of Abraham and Marsden book "Foundations of Mechanics" [1], about qualitative versus quantitative theory: "Mechanics begins with a long tradition of qualitative investigation culminating with KEPLER and GALILEO. Following this is the period of quantitative theory (1687-1889) characterized by concomitant developments in mechanics, mathematics, and the philosophy of science that are epitomized by the works of NEWTON, EULER, LAGRANGE, LAPLACE, HAMILTON, and JACOBI. (...) For celestial mechanics (...) resolution we owe to the genius of POINCARÉ, who resurrected the qualitative point of view (...) One advantage (...) is that by suppressing unnecessary coordinates the full generality of the theory becomes evident."

After having defined motions, Eulerian and Lagrangian variables and functions, we give the definition of the deformation gradient as a function. We then obtain a simple understanding of the Lie derivatives of vector fields which meet the needs of engineers. Then we get the velocity addition formula and verify that the Lie derivatives are objective. Note that Cauchy would certainly have used the Lie derivatives if they had existed during his lifetime: To get a stress, Cauchy had to compare two vectors, whereas one vector is enough when using the derivatives of Lie.

We systematically start with qualitive definitions (observer independent), before quantifying with bases and/or Euclidean dot products (observer dependent). A fairly long appendix tries to give in one manuscript the definitions, properties and interpretations, usually scattered across several books (and not always that easy to find).

Contents

I Motions, Eulerian and Lagrangian descriptions, flows 11

1 Motions 11
- 1.1 Referential 11
- 1.2 Einstein's convention (duality notation) 12
- 1.3 Motion of an object 12
- 1.4 Virtual and real motion 12
- 1.5 Hypotheses (Newton and Einstein) 13
- 1.6 Configurations 13
- 1.7 Definition of the Eulerian and Lagrangian variables 13
- 1.8 Trajectories 13
- 1.9 Pointed vector, tangent space, fiber, vector field, bundle 13

2 Eulerian description (spatial description at actual time t) 14
- 2.1 The set of configurations 14
- 2.2 Eulerian variables and functions 14
- 2.3 Eulerian velocity (spatial velocity) and speed 15
- 2.4 Spatial derivative of the Eulerian velocity 15
  - 2.4.1 Definition 16
  - 2.4.2 The convective derivative $d\mathcal{E}ul.\vec{v}$ 16
  - 2.4.3 Quantification in a basis: $df.\vec{u}$ is written $(\vec{u}.\vec{\mathrm{grad}})f$ 16
  - 2.4.4 Representation relative to a Euclidean dot product: $\vec{\mathrm{grad}}f$ 17
  - 2.4.5 Vector valued functions 17
- 2.5 Streamline (current line) 17
- 2.6 Material time derivative (dérivées particulaires) 18
  - 2.6.1 Usual definition 18
  - 2.6.2 Remark: About notations 19
  - 2.6.3 Definition bis: Time-space definition 19
  - 2.6.4 The material time derivative is a derivation 20
  - 2.6.5 Commutativity issue 20
- 2.7 Eulerian acceleration 20
- 2.8 Time Taylor expansion of $\widetilde{\Phi}$ 21

3 Motion from an initial configuration: Lagrangian description 21
- 3.1 Initial configuration and Lagrangian "motion" 21
  - 3.1.1 Definition 21
  - 3.1.2 Diffeomorphism between configurations 22
  - 3.1.3 Trajectories 22
  - 3.1.4 Streaklines (lignes d'émission) 22
- 3.2 Lagrangian variables and functions 23
  - 3.2.1 Definition 23
  - 3.2.2 A Lagrangian function is a two point tensor 23
- 3.3 Lagrangian function associated with a Eulerian function 24
  - 3.3.1 Definition 24
  - 3.3.2 Remarks 24
- 3.4 Lagrangian velocity 24
  - 3.4.1 Definition 24
  - 3.4.2 Lagrangian velocity versus Eulerian velocity 24
  - 3.4.3 Relation between differentials 25
  - 3.4.4 Computation of $d\vec{v}$ called $L = \dot{F}.F^{-1}$ wih Lagrangian variables 25
- 3.5 Lagrangian acceleration 25
- 3.6 Time Taylor expansion of Φ^{t_0} 26
- 3.7 A vector field that let itself be deformed by a motion 26

4 Deformation gradient $F := d\Phi$ 26
4.1 Definitions . . . 27
4.1.1 Definition of the deformation gradient F . . . 27
4.1.2 Push-forward (values of F) . . . 27
4.1.3 F is a two point tensors . . . 28
4.1.4 Evolution: Toward the Lie derivative (in continuum mechanics) . . . 28
4.1.5 Pull-back . . . 29
4.2 Quantification with bases . . . 29
4.3 The unfortunate notation $d\vec{x} = F.d\vec{X}$ . . . 30
4.3.1 Issue . . . 30
4.3.2 Where does this unfortunate notation come from? . . . 30
4.3.3 Interpretation: Vector approach . . . 30
4.3.4 Interpretation: Differential approach . . . 30
4.3.5 The ambiguous notation $\dot{d\vec{x}} = \dot{F}.d\vec{X}$ . . . 31
4.4 Tensorial notations, warnings, remarks . . . 31
4.5 Change of coordinate system at t for F . . . 32
4.6 Spatial Taylor expansion of Φ and F . . . 32
4.7 Time Taylor expansion of F . . . 32
4.8 Homogeneous and isotropic material . . . 33
4.9 The inverse of the deformation gradient . . . 33

5 Flow 34
5.1 Introduction: Motion versus flow . . . 34
5.2 Definition . . . 34
5.3 Cauchy–Lipschitz theorem . . . 34
5.4 Examples . . . 36
5.5 Composition of flows . . . 36
5.5.1 Law of composition of flows (determinism) . . . 36
5.5.2 Stationnary case . . . 37
5.6 Velocity on the trajectory traveled in the opposite direction . . . 38
5.7 Variation of the flow as a function of the initial time . . . 38
5.7.1 Ambiguous and non ambiguous notations . . . 38
5.7.2 Variation of the flow as a function of the initial time . . . 39

II Push-forward 40

6 Push-forward 40
6.1 Definition . . . 40
6.2 Push-forward and pull-back of points . . . 40
6.3 Push-forward and pull-back of curves . . . 41
6.4 Push-forward and pull-back of scalar functions . . . 41
6.4.1 Definitions . . . 41
6.4.2 Interpretation: Why is it useful? . . . 42
6.5 Push-forward and pull-back of vector fields . . . 42
6.5.1 A definition by approximation . . . 42
6.5.2 The definition of the push-forward of a vector field . . . 42
6.5.3 Pull-back of a vector field . . . 43
6.6 Quantification with bases . . . 44
6.6.1 Usual result . . . 44
6.6.2 Example: Polar coordinate system . . . 44

7 Push-forward and pull-back of differential forms 46
7.1 Definition . . . 46
7.2 Incompatibility: Riesz representation and push-forward . . . 47

8 Push-forward and pull-back of tensors 48
- 8.1 Push-forward and pull-back of order 1 tensors 48
- 8.2 Push-forward and pull-back of order 2 tensors 48
- 8.3 Push-forward and pull-back of endomorphisms 49
- 8.4 Application to derivatives of vector fields 49
- 8.5 $\Psi_*(d\vec{u})$ versus $d(\Psi_*\vec{u})$: No commutativity 50
- 8.6 Application to derivative of differential forms 50
- 8.7 $\Psi_*(d\alpha)$ versus $d(\Psi_*\alpha)$: No commutativity 50

III Lie derivative 51

9 Lie derivative 51
- 9.0 Purpose and first results 51
  - 9.0.1 Purpose? 51
  - 9.0.2 Basic results 51
- 9.1 Definition 51
  - 9.1.1 Issue (ubiquity gift)... 51
  - 9.1.2 ...Toward a solution (without ubiquity gift)... 52
  - 9.1.3 ... The Lie derivative, first definition 52
  - 9.1.4 A more general definition 52
  - 9.1.5 Equivalent definition (differential geometry) 53
- 9.2 Lie derivative of a scalar function 53
- 9.3 Lie derivative of a vector field 54
  - 9.3.1 Formula 54
  - 9.3.2 Interpretation: Flow resistance measurement 54
  - 9.3.3 Autonomous Lie derivative and Lie bracket 55
- 9.4 Examples 55
  - 9.4.1 Lie Derivative of a vector field along itself 55
  - 9.4.2 Lie derivative along a uniform flow 55
  - 9.4.3 Lie derivative of a uniform vector field 55
  - 9.4.4 Uniaxial stretch of an elastic material 55
  - 9.4.5 Simple shear of an elastic material 56
  - 9.4.6 Shear flow 56
  - 9.4.7 Spin 57
  - 9.4.8 Second order Lie derivative 57
- 9.5 Lie derivative of a differential form 57
- 9.6 Incompatibility with Riesz representation vectors 59
- 9.7 Lie derivative of a tensor 59
  - 9.7.1 Lie derivative of a mixed tensor 59
  - 9.7.2 Lie derivative of a up-tensor 60
  - 9.7.3 Lie derivative of a down-tensor 60

IV Velocity-addition formula 61

10 Change of referential and velocity-addition formula 61
- 10.0 Issue and result (summary) 61
- 10.1 Referentials and "matrix motions" 61
  - 10.1.1 Absolute and relative referentials 61
  - 10.1.2 Motion of a material object *Obj* 62
  - 10.1.3 Quantification: Absolute and relative "motion" of *Obj* 62
  - 10.1.4 Motion of $\mathcal{R}_B$ 63
  - 10.1.5 Quantification: Drive and static "motion" of $\mathcal{R}_B$ 63
- 10.2 The translator Θ_t 64
  - 10.2.1 Definition of Θ_t 64
  - 10.2.2 Translation at t for the motion $\widetilde{\Phi}$ 64
- 10.3 $d\Theta_t$ 64
  - 10.3.1 Push-forward 64
  - 10.3.2 Θ_t is affine in classical mechanics 65

10.4 Translated velocities 65
10.5 Definition of Θ 66
10.6 The "Θ-velocity" is the drive velocity 66
10.7 The velocity-addition formula 66
10.8 Coriolis acceleration, and the acceleration-addition formula 67
10.9 With an initial time 67
10.10 Drive and Coriolis forces 68
  10.10.1 Fundamental principal: requires a Galilean referential 68
  10.10.2 Drive + Coriolis forces = the inertial force 68
10.11 Summary for "Sun and Earth" 69
  10.11.1 Coriolis forces on the Earth 69

11 Objectivities **71**
11.1 "Isometric objectivity" and "Frame Invariance Principle" 71
11.2 Definition and characterization of the covariant objectivity 71
  11.2.1 Framework of classical mechanics 71
  11.2.2 Covariant objectivity of a scalar function 71
  11.2.3 Covariant objectivity of a vector field 72
  11.2.4 Covariant objectivity of differential forms 72
  11.2.5 Covariant objectivity of tensors 72
11.3 Non objectivity of the velocities 73
  11.3.1 Eulerian velocity $\vec{v}$: not covariant (and not isometric) objective 73
  11.3.2 $d\vec{v}$ is not objective 73
  11.3.3 $d\vec{v} + d\vec{v}^T$ is "isometric objective" 74
  11.3.4 Lagrangian velocities 74
11.4 The Lie derivatives are covariant objective 74
  11.4.1 Scalar functions 74
  11.4.2 Vector fields 74
  11.4.3 Tensors 75
11.5 Taylor expansions and ubiquity gift 76
  11.5.1 In $\mathbb{R}^n$ with ubiquity 76
  11.5.2 General case 76

V Appendix **78**

A Classical and duality notations **78**
A.1 Contravariant vector and basis 78
  A.1.1 Contravariant vector 78
  A.1.2 Basis 78
  A.1.3 Canonical basis 78
  A.1.4 Cartesian basis 78
A.2 Representation of a vector relative to a basis 79
A.3 Dual basis 79
  A.3.1 Linear forms = "Covariant vectors" 79
  A.3.2 Covariant dual basis (= the functions that give the components of a vector) 80
  A.3.3 Example: aeronautical units 81
  A.3.4 Matrix representation of a linear form 81
  A.3.5 Example: Thermodynamic 82
A.4 Einstein convention 82
  A.4.1 Definition 82
  A.4.2 Do not mistake yourself 83
A.5 Change of basis formulas 83
  A.5.1 Change of basis endomorphism and transition matrix 83
  A.5.2 Inverse of the transition matrix 84
  A.5.3 Change of dual basis 84
  A.5.4 Change of coordinate system for vectors and linear forms 84
A.6 Bidual basis (and contravariance) 85
A.7 Bilinear forms 85

A.7.1 Definition 85
A.7.2 The transposed of a bilinear form 85
A.7.3 Symmetric and definite positive bilinear forms 86
A.7.4 Inner dot product, and metric 86
A.7.5 Quantification: Matrice $[\beta_{ij}]$ and tensorial representation 86
A.8 Linear maps 87
A.8.1 Definition 87
A.8.2 Quantification: Matrices $[L_{ij}] = [L^i{}_j]$ 88
A.8.3 Trace of an endomorphism 89
A.9 Transposed matrix 89
A.10 A transposed endomorphism: depends on a chosen inner dot product 90
A.10.1 Definition (requires an inner dot product: Not objective) 90
A.10.2 Quantification with bases 90
A.10.3 Symmetric endomorphism 91
A.10.4 The general flat $^\flat$ notation for an endomorphism: Relative to a $(\cdot,\cdot)_g$ 92
A.11 A transposed of a linear map: depends on chosen inner dot products 92
A.11.1 Definition (subjective) 93
A.11.2 Quantification with bases 93
A.11.3 Deformation gradient symmetric: Absurd 93
A.11.4 Isometry 94
A.12 The adjoint of a linear map (objective) 94
A.12.1 Definition 94
A.12.2 Quantification 95
A.12.3 Relation with the transposed when inner dot products are introduced 95
A.13 Tensorial representation of a linear map 95
A.13.1 A tensorial representation 95
A.13.2 Warning: Confusion between transposed and adjoint 96
A.14 Change of basis formulas for bilinear forms and linear maps 96
A.14.1 Notations for transitions matrices for bilinear forms and linear maps 96
A.14.2 Change of coordinate system for bilinear forms $\in \mathcal{L}(A,B;\mathbb{R})$ 97
A.14.3 Change of coordinate system for bilinear forms $\in \mathcal{L}(A^*,B^*;\mathbb{R})$ 97
A.14.4 Change of coordinate system for bilinear forms $\in \mathcal{L}(B^*,A;\mathbb{R})$ 97
A.14.5 Change of coordinate system for tri-linear forms $\in \mathcal{L}(A^*,A,A;\mathbb{R})$ 98
A.14.6 Change of coordinate system for linear maps $\in \mathcal{L}(A;B)$ 98

B Euclidean Frameworks **98**
B.1 Euclidean basis 99
B.2 Euclidean dot product 99
B.3 Change of Euclidean basis 100
B.3.1 Two Euclidean dot products are proportional 100
B.3.2 Counterexample : non existence of a Euclidean dot product 100
B.4 Euclidean transposed of the deformation gradient 100
B.5 The Euclidean transposed for endomorphisms 100
B.6 Unit normal vector, unit normal form 101
B.6.1 Framework 101
B.6.2 Unit normal vector 101
B.6.3 Unit normal form $n^\flat$ associated to $\vec{n}$ 102
B.7 Integration by parts (Green–Gauss–Ostrogradsky) 102

C Rate of deformation tensor and spin tensor **103**
C.1 The symmetric and antisymmetric parts of $d\vec{v}$ 103
C.2 Quantification with a basis 103

D Interpretation of the rate of deformation tensor **104**

E Rigid body motions and the spin tensor **104**
E.1 Affine motions and rigid body motions 104
E.1.1 Affine motions 104
E.1.2 Rigid body motion 105
E.1.3 Alternative definition of a rigid body motion: $d\vec{v}+d\vec{v}^T=0$ 106
E.2 Representation of the spin tensor Ω: vectors, and pseudo-vectors 106
E.2.1 Reminder 106
E.2.2 Definition of the vector product (cross product) 107
E.2.3 Calculation of the vector product 107
E.2.4 Antisymmetric endomorphism represented by a vector 108
E.2.5 Curl 109
E.3 Pseudo-cross product, and pseudo-vector 109
E.3.1 Definition 109
E.3.2 Antisymmetric matrix represented by a pseudo-vector 110
E.3.3 Antisymmetric endomorphism and its pseudo-vectors representations 110
E.4 Examples 110
E.4.1 Rectilinear motion 110
E.4.2 Circular motion 111
E.4.3 Motion of a planet (centripetal acceleration) 112

F Riesz representation theorem **115**
F.1 The Riesz representation theorem 115
F.2 The Riesz representation operator 116
F.3 Quantification with a basis 116
F.4 Change of Riesz representation vector, and Euclidean case 117
F.5 A Riesz representation vector is contravariant 118
F.6 What is a vector versus a $(\cdot,\cdot)_g$-vector? 119
F.7 The “$(\cdot,\cdot)_g$-dual vectorial bases” of one basis (and warnings) 119
F.7.1 A basis and its many associated “dual vectorial basis” 119
F.7.2 Components of $\vec{e}_{jg}$ in the basis $(\vec{e}_i)$ 120
F.7.3 Multiple admissible notations for the components of $\vec{e}_{jg}$ 121
F.7.4 (Huge) differences between “the (covariant) dual basis” and “a dual vectorial basis” 121
F.7.5 About the notation g^{ij} = shorthand notation for $(g^\sharp)^{ij}$ 121

G Cauchy–Green deformation tensor $C=F^T.F$ **122**
G.0 Goal 122
G.1 Transposed F^T: Inner dot products required 122
G.1.1 Definition of the function F^T 122
G.1.2 Quantification with bases (matrix representation) 123
G.1.3 Remark: F^* 123
G.2 Cauchy–Green deformation tensor C 124
G.2.1 Definition of C 124
G.2.2 Quantification 125
G.3 Time Taylor expansion of C 125
G.4 Remark: $C^\flat$ 126
G.4.1 Definition of $C^\flat$... 126
G.4.2 ... and remarks about $C^\flat$... and Jaumann 126
G.5 Stretch ratio and deformed angle 127
G.5.1 Stretch ratio 127
G.5.2 Deformed angle 127
G.6 Decompositions of C 128
G.6.1 Spherical and deviatoric tensors 128
G.6.2 Rigid motion 128
G.6.3 Diagonalization of C 128
G.6.4 Mohr circle 128
G.7 Green–Lagrange deformation tensor E 129
G.8 Small deformations (linearization): The infinitesimal strain tensor $\underline{\underline{\varepsilon}}$ 130
G.8.1 Landau notations big-O and little-o 130
G.8.2 Definition of the infinitesimal strain tensor $\underline{\underline{\varepsilon}}$ 130

H Finger tensor $F.F^T$ (left Cauchy–Green tensor) **131**
- H.1 Definition 131
- H.2 $\underline{\underline{b}}^{-1}$ 132
- H.3 Time derivatives of $\underline{\underline{b}}^{-1}$ 132
- H.4 Euler–Almansi tensor $\underline{\underline{a}}$ 133
- H.5 Time Taylor expansion for $\underline{\underline{a}}$ 133
- H.6 Almansi modified Infinitesimal strain tensor $\widetilde{\underline{\underline{\varepsilon}}}$ 133

I Polar decomposition, elasticity and objectivity **133**
- I.1 Polar decompositions of F (“isometric objectivity”) 133
  - I.1.1 $F = R.U$ (right polar decomposition) 134
  - I.1.2 $F = S.R_0.U$ (shifted right polar decomposition for covariant objectivity) 134
  - I.1.3 $F = V.R$ (left polar decomposition) 135
- I.2 Linear elasticity: A Classical “tensorial” approach 136
  - I.2.1 Classical approach (“isometric objectivity”), and an issue 136
  - I.2.2 A functional (tensorial) formulation (“isometric objectivity”) 136
  - I.2.3 Second functional formulation: With the Finger tensor 138
- I.3 Elasticity with a covariant objective approach? 138

J Displacement **139**
- J.1 The displacement vector $\vec{\mathcal{U}}$ 139
- J.2 The differential of the displacement vector 140
- J.3 Deformation “tensor” $\underline{\underline{\varepsilon}}$ (matrix), bis 140
- J.4 Small displacement hypothesis, bis 141
- J.5 Displacement vector with differential geometry 141
  - J.5.1 The shifter 141
  - J.5.2 The displacement vector 142

K Determinants **142**
- K.1 Alternating multilinear form 142
- K.2 Leibniz formula 142
- K.3 Determinant of vectors 143
- K.4 Determinant of a matrix 144
- K.5 Volume 144
- K.6 Determinant of an endomorphism 145
  - K.6.1 Definition and basic properties 145
  - K.6.2 The determinant of an endomorphism is objective 146
- K.7 Determinant of a linear map 147
  - K.7.1 Definition and first properties 147
  - K.7.2 Jacobian of a motion, and dilatation 147
  - K.7.3 Determinant of the transposed 148
- K.8 Dilatation rate 148
  - K.8.1 $\frac{\partial J^{t_0}}{\partial t}(t,p_{t_0}) = J^{t_0}(t,p_{t_0})\,\mathrm{div}\vec{v}(t,p_t)$ 148
  - K.8.2 Leibniz formula 149
- K.9 $\partial J/\partial F = J\,F^{-T}$ 150
  - K.9.1 Meaning of $\frac{\partial \det}{\partial M_{ij}}$? 150
  - K.9.2 Calculation of $\frac{\partial \det}{\partial M_{ij}}$ 150
  - K.9.3 $\partial J/\partial F = J\,F^{-T}$ usually written $[\frac{\partial J}{\partial F_{ij}}] = J\,F^{-T}$ 150
  - K.9.4 Interpretation of $\frac{\partial J}{\partial F_{ij}}$? 150

L Transport of volumes and areas **151**
- L.1 Transformed parallelepiped 151
- L.2 Transformed volumes 151
- L.3 Transformed parallelogram 151
- L.4 Transformed surface 152
  - L.4.1 Deformation of a surface 152
  - L.4.2 Euclidean dot product and unit normal vectors 152
  - L.4.3 Relations between surfaces 153
- L.5 Piola identity 153

L.6 Piola transformation 154

M Work and power 154
M.1 Definitions 154
M.1.1 Work 154
M.1.2 And its associated power density 155
M.2 Piola–Kirchhoff tensors 155
M.2.1 Objective internal power for the stress: function of $d\vec{v}$ 155
M.2.2 The first Piola–Kirchhoff tensor 156
M.2.3 The second Piola–Kirchhoff tensor 156
M.3 Classical hyper-elasticity: $\partial W/\partial F$ 157
M.3.1 Definition 157
M.3.2 Expression with bases (quantification): The $\partial W/\partial L_{ij}$ 157
M.3.3 Motions and ω-lemma 158
M.3.4 Application to classical hyper-elasticity: $I\!\!K = \partial W/\partial F$ 159
M.3.5 Corollary (hyper-elasticity): $S\!\!K = \partial W/\partial C$ 160

N Conservation of mass 160

O Balance of momentum 161
O.1 Framework 161
O.2 Master balance law 161
O.3 Cauchy theorem $\vec{T} = \underline{\underline{\sigma}}.\vec{n}$ (stress tensor $\underline{\underline{\sigma}}$) 161

P Balance of moment of momentum 163

Q Uniform tensors in $\mathcal{L}^r_s(E)$ 163
Q.1 Tensorial product and multilinear forms 163
Q.1.1 Tensorial product of functions 163
Q.1.2 Tensorial product of linear forms: multilinear forms 163
Q.2 Uniform tensors in $\mathcal{L}^0_s(E)$ 164
Q.2.1 Definition of type $\binom{0}{s}$ uniform tensors 164
Q.2.2 Example: Type $\binom{0}{1}$ uniform tensor = linear forms 164
Q.2.3 Example: Type $\binom{0}{2}$ uniform tensor 164
Q.2.4 Example: Determinant 164
Q.3 Uniform tensors in $\mathcal{L}^r_s(E)$ 164
Q.3.1 Definition of type $\binom{r}{s}$ uniform tensors 165
Q.3.2 Example: Type $\binom{1}{0}$ uniform tensor: Identified with a vector 165
Q.3.3 Example: Type $\binom{1}{1}$ uniform tensor 165
Q.3.4 Example: Type $\binom{1}{2}$ uniform tensor 166
Q.4 Exterior tensorial products 166
Q.5 Contractions 166
Q.5.1 Contraction of a linear form with a vector 166
Q.5.2 Contraction of a $\binom{1}{1}$ tensor and a vector 166
Q.5.3 Contractions of uniform tensors 167
Q.5.4 Objective double contractions of uniform tensors 168
Q.5.5 Non objective double contraction: Double matrix contraction 169
Q.6 Kronecker (contraction) tensor, trace 169

R Tensors in $T^r_s(U)$ 170
R.1 Introduction, module, derivation 170
R.2 Field of functions and vector fields 170
R.2.1 Field of functions 170
R.2.2 Vector fields 171
R.3 Differential forms 171
R.4 Tensors 171
R.5 First Examples 172
R.5.1 Type $\binom{0}{1}$ tensor = differential forms 172
R.5.2 Type $\binom{1}{0}$ tensor (identified to a vector field) 172

R.5.3 A metric is a $\binom{0}{2}$ tensor . . . 172
R.6 $\binom{1}{1}$ tensor, identification with fields of endomorphisms . . . 172
R.7 Unstationary tensor . . . 172

S Differential, its eventual gradients, divergences 173
S.1 Differential . . . 173
S.1.1 Framework . . . 173
S.1.2 Directional derivative and differential (observer independent) . . . 173
S.1.3 Notation for the second order Differential . . . 174
S.2 A basis and the j-th partial derivative . . . 174
S.3 Application 1: Scalar valued functions . . . 175
S.3.1 Differential of a scalar valued function (objective) . . . 175
S.3.2 Quantification . . . 175
S.3.3 Gradients (subjective) associated with a differential through inner dot products . . . 176
S.4 Application 2: Coordinate system basis and Christoffel symbols . . . 176
S.4.1 Coordinate system, and coordinate system basis . . . 176
S.4.2 Parametric expression of the differential of a scalar valued function . . . 177
S.4.3 Christoffel symbols . . . 178
S.5 Application 3: Differential of a vector field . . . 179
S.6 Application 4: Differential of a differential form . . . 180
S.7 Application 5: Differential of a 1 1 tensor . . . 180
S.8 Divergence of a vector field: Invariant . . . 181
S.9 Objective divergence for 1 1 tensors . . . 182
S.9.1 Divergence of a 2 0 tensor . . . 183
S.9.2 Divergence of a 0 2 tensor . . . 183
S.10 Euclidean framework and “classic divergence” of a tensor (subjective) . . . 184

T Natural canonical isomorphisms 184
T.1 The adjoint of a linear map . . . 184
T.2 An isomorphism $E \simeq E^*$ is never natural (never objective) . . . 185
T.3 Natural canonical isomorphism $E \simeq E^{**}$ . . . 186
T.4 Natural canonical isomorphisms $\mathcal{L}(E;F) \simeq \mathcal{L}(F^*,E;\mathbb{R}) \simeq \mathcal{L}(E^*;F^*)$ . . . 186
T.5 Natural canonical isomorphisms $\mathcal{L}(E;\mathcal{L}(E;F)) \simeq \mathcal{L}(E,E;F) \simeq \mathcal{L}(F^*,E,E;\mathbb{R})$ . . . 187

U Distribution in brief: A covariant concept 188
U.1 Definitions . . . 188
U.2 Derivation of a distribution . . . 189
U.3 Hilbert space $H^1(\Omega)$ . . . 190
U.3.1 Motivation . . . 190
U.3.2 Definition of $H^1(\Omega)$ . . . 190
U.3.3 Subspace $H^1_0(\Omega)$ and its dual space $H^{-1}(\Omega)$ . . . 191

A quantity f being given then: g defined by « g equals f » is noted $g := f$.

Part I

Motions, Eulerian and Lagrangian descriptions, flows

1 Motions

The framework is classical mechanics, time being decoupled from space. $\mathbb{R}^3$ is the classical geometric affine space (the space we live in), and $(\vec{\mathbb{R}}^3, +, .) = \{\vec{pq} : p, q \in \mathbb{R}^3\} =^{\text{noted}} \vec{\mathbb{R}}^3$ is the associated vector space of bipoint vectors equipped with its usual rules. We also consider $\mathbb{R}$ and $\mathbb{R}^2$ as subspaces of $\mathbb{R}^3$, i.e. we consider $\mathbb{R}^n$ and $\vec{\mathbb{R}}^n$, $n = 1, 2, 3$.

1.1 Referential

Origin: An observer chooses an origin $\mathcal{O} \in \mathbb{R}^n$; Thus a point $p \in \mathbb{R}^n$ can be located by the observer thanks to the bipoint vector $\overrightarrow{\mathcal{O}p} = \vec{x} \in \vec{\mathbb{R}}^n$; Hence $p = \mathcal{O} + \vec{x}$, and $\vec{x} = \overrightarrow{\mathcal{O}p} =^{\text{noted}} p - \mathcal{O}$.

Another observer chooses an origin $\widetilde{\mathcal{O}} \in \mathbb{R}^n$; Thus the point p can also be located by this observer with the bipoint vector $\overrightarrow{\widetilde{\mathcal{O}}p} = \widetilde{\vec{x}} \in \vec{\mathbb{R}}^n$; So $p = \mathcal{O} + \vec{x} = \widetilde{\mathcal{O}} + \widetilde{\vec{x}}$, and $\widetilde{\vec{x}} = \overrightarrow{\mathcal{O}\widetilde{\mathcal{O}}} + \vec{x}$.

Cartesian coordinate system: A Cartesian coordinate system in the affine space $\mathbb{R}^n$ is a set $\mathcal{R}_{\text{Cart}} = (\mathcal{O}, (\vec{e}_i)_{i=1,\dots,n})$, where $\mathcal{O}$ is an origin and $(\vec{e}_i) := (\vec{e}_i)_{i=1,\dots,n}$ is a basis in $\vec{\mathbb{R}}^n$ chosen by the observer. Thus the location of a point $p \in \mathbb{R}^n$ can quantified by the observer $\exists \vec{x} \in \vec{\mathbb{R}}^n$ s.t.

$$p = \mathcal{O} + \vec{x} \quad \text{with} \quad \vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i, \quad \text{i.e.} \quad [\overrightarrow{\mathcal{O}p}]_{|\vec{e}} = [\vec{x}]_{|\vec{e}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \tag{1.1}$$

$[\vec{x}]_{|\vec{e}} = [\overrightarrow{\mathcal{O}p}]_{|\vec{e}}$ being the column matrix containing the components $x_i \in \mathbb{R}$ of $\overrightarrow{\mathcal{O}p} = \vec{x}$ in the basis $(\vec{e}_i)$. Another observer with his origin O_b and his Cartesian basis $(\vec{b}_i)_{i=1,\dots,n}$ make the Cartesian coordinate system $\mathcal{R}_{\text{Cart,b}} = (\mathcal{O}_b, (\vec{b}_i)_{i=1,\dots,n})$, and gets for the same position p in $\mathbb{R}^n$,

$$p = \mathcal{O}_b + \vec{y} \quad \text{with} \quad \vec{y} = \sum_{i=1}^{n} \widetilde{y}_i \widetilde{\vec{b}}_i, \quad \text{i.e.} \quad [\overrightarrow{\mathcal{O}_b p}]_{|\vec{b}} = [\vec{y}]_{|\vec{b}} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \tag{1.2}$$

$[\vec{y}]_{|\vec{b}} = [\overrightarrow{\mathcal{O}_b p}]_{|\vec{b}}$ being the column matrix containing the components $y_i \in \mathbb{R}$ of $\overrightarrow{\mathcal{O}_b p} = \vec{y}$ in the basis $(\vec{b}_i)$. And $\overrightarrow{\mathcal{O}_b p} = \overrightarrow{\mathcal{O}_b \mathcal{O}} + \overrightarrow{\mathcal{O}p}$, i.e. $\vec{y} = \overrightarrow{\widetilde{\mathcal{O}}\mathcal{O}} + \vec{x}$, gives the relation between $\vec{x}$ and $\vec{y}$ (drawing).

Chronology: A chronology (or temporal coordinate system) is a set $\mathcal{R}_{\text{time}} = (t_0, (\Delta t))$ chosen by an observer, where $t_0 \in \mathbb{R}$ is the time origin, and (Δt) is the time unit (a basis in $\vec{\mathbb{R}}$).

Referentiel: A referential $\mathcal{R}$ is the set

$$\mathcal{R} = (\mathcal{R}_{\text{time}}, \mathcal{R}_{\text{Cart}}) = (t_0, (\Delta t), \mathcal{O}, (\vec{e}_i)_{i=1,\dots,n}) = (\text{“chronologie”}, \text{“Cartesian coordinate system”}), \tag{1.3}$$

made of a chronology and a Cartesian coordinate system, chosen by an observer.

In the following, to simplify the writings, the same implicit chronology is used by all observers, and a referential $\mathcal{R} = (\mathcal{R}_{\text{time}}, \mathcal{R}_{\text{Cart}})$ will simply be noted as the reference frame $\mathcal{R} = (\mathcal{O}, (\vec{e}_i))$ (so $:= \mathcal{R}_{\text{Cart}}$).

1.2 Einstein's convention (duality notation)

Starting point: The classical notation x_i for the components of a vector $\vec{x}$ relative to a basis, cf. (1.1). Then the duality notion is introduced: $x_i =^{\text{noted}} x^i$ (enables to see the difference between a vector and a function when using components). So

$$\vec{x} = \underbrace{\sum_{i=1}^{n} x_i \vec{e}_i}_{\text{classic not.}} = \underbrace{\sum_{i=1}^{n} x^i \vec{e}_i}_{\text{duality not.}} , \quad \text{and} \quad [\vec{x}]_{|\vec{e}} \overset{\text{clas.}}{=} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \overset{\text{dual}}{=} \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} . \tag{1.4}$$

The duality notation is part of the Einstein's convention; Moreover Einstein's convention uses the notation $\sum_{i=1}^n x^i \vec{e}_i =^{\text{noted}} x^i \vec{e}_i$, i.e. the sum sign $\sum_{i=1}^n$ can be omitted when an index (i here) is used twice, once up and once down, details at § A.4. However this omission of the sum sign $\sum$ will not be made in this manuscript (to avoid ambiguities): The TEX-LATEX program makes it easy to print $\sum_{i=1}^n$.

Example 1.1 The height of a child is represented on a wall by a vertical bipoint vector $\vec{x}$ starting from the ground up to a pencil line. Question: What is the size of the child ?

Answer: It depends... on the observer (quantitative value = subjective result). E.g., an English observer chooses a vertical basis vector $\vec{a}_1$ which length is one English foot (ft). So he writes $\vec{x} = x_1 \vec{a}_1$, and for him the size of the child (size of $\vec{x}$) is x_1 in foot. E.g. $x_1 = 4$ means the child is 4 ft tall. A French observer chooses a vertical basis vector $\vec{b}_1$ which length is one metre (m). So he writes $\vec{x} = y_1 \vec{b}_1$, and for him the size of the child (size of $\vec{x}$) is y^1 metre. E.g., if $x_1 = 4$ then $y_1 \simeq 1.22$, since 1 ft := 0.3048 m: The child is both 4 and 1.22 tall... in foot or metre. This quantification is written $\vec{x} = 4$ ft = 1.22 m, where ft means $\vec{a}_1$ and m means $\vec{b}_1$ here. NB: The qualitative vector $\vec{x}$ is the same vector for all observers, not the quantitative values 4 or 1.22 (depends on a choice of a unit of measurement).

With duality notation: $\vec{x} = x^1 \vec{a}_1 = y^1 \vec{b}_1$, so if $x^1 = 4$ then $y^1 \simeq 1.22$. ∴

This manuscript insists on covariant objectivity; Thus an English engineer (and his foot) and a French engineer (and his metre) will be able to work together ... and be able to avoid crashes like that of the Mars Climate Orbiter probe, see remark A.14. And they will be able to use the results of Galileo, Descartes, Newton, Euler... who used their own unit of length, and knew nothing about the metre defined in 1793 and adopted in 1799 in France (after 6 years of measurements), and considered by the scientific community at the end of the ninetieth century... and couldn't explicitly use the "Euclidean dot products" either (which seems to have been defined mathematically by Grassmann around 1844).

1.3 Motion of an object

Let *Obj* be a "real object", or "material object", made of particles (e.g., the Moon: Exists independently of an observer). Let $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$.

Definition 1.2 The motion of *Obj* in $\mathbb{R}^n$ is the map

$$\widetilde{\Phi} : \begin{cases} [t_1, t_2] \times Obj & \to \mathbb{R}^n \\ (t, \underbrace{P_{Obj}}_{\text{particle}}) & \to \underbrace{p = \widetilde{\Phi}(t, P_{Obj})}_{\text{its position at } t \text{ in the Universe}} \end{cases} . \tag{1.5}$$

And t is the time variable, p is the space variable, and $(t,p) \in \mathbb{R} \times \mathbb{R}^n$ is the time-space variable. And $\widetilde{\Phi}$ is supposed to be C^2 in time.

With an origin $\mathcal{O}$ (observer dependent), the motion can be described with the bi-point vector

$$\vec{x} = \overrightarrow{\mathcal{O}\widetilde{\Phi}(t, P_{Obj})} = \overrightarrow{\mathcal{O}p} \overset{\text{noted}}{=} \widetilde{\varphi}(t, P_{Obj}). \tag{1.6}$$

But then, two observers with different origins $\mathcal{O}$ and $\mathcal{O}_b$ have different description of the motion. Therefore, in the following we won't use $\widetilde{\varphi}$. Then (quantification) with a Cartesian basis $(\vec{e}_i)$ to make a referential $\mathcal{R}$, we get (1.1).

1.4 Virtual and real motion

Definition 1.3 A virtual (or possible) motion of *Obj* is a function $\widetilde{\Phi}$ "regular enough for the calculations to be meaningful". Among all the virtual motions, the observed motion is called the real motion.

1.5 Hypotheses (Newton and Einstein)

Hypotheses of Newtonian mechanics (Galileo relativity) and general relativity (Einstein):

1- You can describe a phenomenon only at the actual time t and from the location p you are at (you have **no** gift of ubiquity in time or space);

2- You don't know the future;

3- You can use your memory, so use some past time t_0 and some past position p_{t_0};

4- You can use someone else memory (results of measurements) if you can communicate objectively.

1.6 Configurations

Fix $t \in [t_1, t_2]$, and define $\widetilde{\Phi}_t : \begin{cases} \mathit{Obj} & \to \mathbb{R}^n \\ P_{\mathit{Obj}} & \mapsto p = \widetilde{\Phi}_t(P_{\mathit{Obj}}) := \widetilde{\Phi}(t, P_{\mathit{Obj}}). \end{cases}$

Definition 1.4 The "configuration at t" of *Obj* is the range (or image) of $\widetilde{\Phi}_t$, i.e. is the subset of $\mathbb{R}^n$ (affine space) defined by

$$\Omega_t := \{p \in \mathbb{R}^n : \exists P_{\mathit{Obj}} \in \mathit{Obj} \text{ s.t. } p = \widetilde{\Phi}_t(P_{\mathit{Obj}})\} \overset{\text{noted}}{=} \widetilde{\Phi}_t(\mathit{Obj}) \overset{\text{noted}}{=} \mathrm{Im}(\widetilde{\Phi}_t). \tag{1.7}$$

If t is the actual time then Ω_t is the actual (or current or Eulerian) configuration.

If t_0 is a time in the past then Ω_{t_0} is the past (or initial or Lagrangian) configuration.

Hypothesis: At any time t, Ω_t is supposed to be a "smooth domain" in $\mathbb{R}^n$, and the map $\widetilde{\Phi}_t$ is assumed to be one-to-one (= injective): *Obj* does not crash onto itself.

1.7 Definition of the Eulerian and Lagrangian variables

- If t is the actual time, then $p_t = \widetilde{\Phi}_t(P_{\mathit{Obj}}) \in \Omega_t$ is called the Eulerian variable relative to P_{Obj} and t.

- If t_0 is a time in the past, then $p_{t_0} = \widetilde{\Phi}_{t_0}(P_{\mathit{Obj}}) \in \Omega_{t_0}$ is called the Lagrangian variable relative to P_{Obj} and t_0. (A Lagrangian variable is a "past Eulerian variable"). (Two observers with two different origin of time t_0 and ${t_0}'$ get two different Lagrangian variable while they have the same Eulerian variable.)

1.8 Trajectories

Let $\widetilde{\Phi}$ be a motion of *Obj*, cf. (1.5), and $P_{\mathit{Obj}} \in \mathit{Obj}$ (a particle in *Obj* = e.g. the Moon).

Definition 1.5 The (parametric) trajectory of P_{Obj} is the function

$$\widetilde{\Phi}_{P_{\mathit{Obj}}} : \begin{cases} [t_1, t_2] & \to \mathbb{R}^n, \\ \quad t & \mapsto p(t) = \widetilde{\Phi}_{P_{\mathit{Obj}}}(t) := \widetilde{\Phi}(t, P_{\mathit{Obj}}) \quad \text{(position of } P_{\mathit{Obj}} \text{ at } t \text{ in the Universe).} \end{cases} \tag{1.8}$$

Its geometric trajectory is the range (image) of $\widetilde{\Phi}_{P_{\mathit{Obj}}}$, i.e.

$$\text{geometric trajectory of } P_{\mathit{Obj}} := \{q \in \mathbb{R}^n : \exists t \in [t_1, t_2] \text{ s.t. } q = \widetilde{\Phi}_{P_{\mathit{Obj}}}(t)\} = \mathrm{Im}(\widetilde{\Phi}_{P_{\mathit{Obj}}}) = \widetilde{\Phi}_{P_{\mathit{Obj}}}([t_1, t_2]). \tag{1.9}$$

1.9 Pointed vector, tangent space, fiber, vector field, bundle

(See e.g. Abraham–Marsden [1].) To deal with surfaces S in $\mathbb{R}^3$, e.g. with S = a sphere (and more generally with manifolds in $\mathbb{R}^n$), a vector cannot simply be a "bi-point vector connecting two points of S" (would get "through the surface"). A vector is defined to be tangent to S: Consider a "regular" curve $c : s \in]-\varepsilon, \varepsilon[\to c(s) \in S$ where S is a surface in an affine space, and the vector tangent to S at $c(0)$ is $\vec{w}(c(0)) = \lim_{h \to 0} \frac{c(h) - c(0)}{h}$ (it is defined with a parametrization of c in a general manifold); Considering all the possible curves, we get "all possible vectors on S".

Notation:

$$T_pS := \{\text{tangent vectors } \vec{w}_p \text{ at } S \text{ at } p\} = \text{The tangent space at } p \in S. \tag{1.10}$$

E.g., if S is a sphere in $\mathbb{R}^3$ and $p \in S$, then T_pS is its usual tangent plane at p at S.

E.g., particular case: If $S = \Omega$ is an open set in $\mathbb{R}^n$, then $T_pS = T_p\Omega = \vec{\mathbb{R}}^n$ is independent of p.

Definition 1.6

$$\text{The fiber at } p := \{p\} \times T_pS = \{ \underbrace{(p, \vec{w}_p)}_{\text{pointed vector}} \in \{p\} \times T_pS\}, \tag{1.11}$$

i.e., the fiber at p is the set of "pointed vectors at p", a pointed vector being the couple $(p, \vec{w}_p)$ made of the "base point" p and the vector $\vec{w}_p$ defined at p.

Drawing: A vector in $\vec{\mathbb{R}}^n$ can be drawn anywhere in $\mathbb{R}^n$; While a "pointed vectors at p" has to be drawn at the point p in $\mathbb{R}^n$.

If the context is clear, a pointed vector is simply noted $\widetilde{\vec{w}}(p) =^{\text{noted}} \vec{w}(p)$ (lighten the writing).

Particular case: If $S = \Omega$ is an open set in $\mathbb{R}^n$, then the fiber at p is $T_p\Omega = \{p\} \times \vec{\mathbb{R}}^n$.

Definition 1.7

$$\text{The tangent bundle } TS := \bigcup_{p \in S} (\{p\} \times T_pS), \tag{1.12}$$

that is, is the union of the fibers.

Definition 1.8 A vector field $\widetilde{\vec{w}}$ in S is a C^∞ function (or at least C^2 in the following)

$$\widetilde{\vec{w}} : \begin{cases} S & \to TS \\ p & \to \widetilde{\vec{w}}(p) = (p, \vec{w}(p)). \end{cases} \tag{1.13}$$

If the context is clear, a vector field is simply noted $\widetilde{\vec{w}} =^{\text{noted}} \vec{w}$ (lighten the writing).

2 Eulerian description (spatial description at actual time t)

2.1 The set of configurations

Let $\widetilde{\Phi}$ be a motion of Obj, cf. (1.5), and $\Omega_t = \widetilde{\Phi}_t(Obj) \subset \mathbb{R}^n$ be the configuration at t, cf. (1.7). The set of configurations is the subset $\mathcal{C} \subset \mathbb{R} \times \mathbb{R}^n$ (the "time-space") defined by

$$\begin{aligned} \mathcal{C} &:= \bigcup_{t \in [t_1, t_2]} (\{t\} \times \Omega_t) \quad (= \text{set in which you find particles in "time-space"}) \\ &= \{(t,p) \in \mathbb{R} \times \mathbb{R}^n : \exists (t, P_{Obj}) \in [t_1, t_2] \times Obj,\ p = \widetilde{\Phi}(t, P_{Obj})\}, \end{aligned} \tag{2.1}$$

Question: Why don't we simply use $\bigcup_{t \in [t_1, t_2]} \Omega_t$ instead of $\mathcal{C} = \bigcup_{t \in [t_1, t_2]} (\{t\} \times \Omega_t)$?

Answer: $\mathcal{C}$ gives the film of the life of Obj = the succession of the photos Ω_t taken at each t; And Ω_t is obtained from $\mathcal{C}$ thanks to the pause feature at t. Whereas $\bigcup_{t \in [t_1, t_2]} \Omega_t \subset \mathbb{R}^n$ is the superposition of all the photos on the image $\bigcup_{t \in [t_1, t_2]} \Omega_t$... and we don't distinguish the past from the present.

2.2 Eulerian variables and functions

Definition 2.1 In short: A Eulerian function relative to Obj is a function, with $m \in \mathbb{N}^*$,

$$\mathcal{E}ul : \begin{cases} \mathcal{C} & \to \vec{\mathbb{R}^m} \text{ (or more generally a suitable set of tensors)} \\ (t,p) & \to \mathcal{E}ul(t,p), \end{cases} \tag{2.2}$$

the spatial variable p being the Eulerian variable.

In details: A function $\mathcal{E}ul$ being given as in (2.2), the associated Eulerian function $\widehat{\mathcal{E}ul}$ is the function

$$\widehat{\mathcal{E}ul} : \begin{cases} \mathcal{C} & \to \mathcal{C} \times \vec{\mathbb{R}^m} \text{ (or } \mathcal{C}\times \text{ some suitable set of tensors)} \\ (t,p) & \to \widehat{\mathcal{E}ul}(t,p) = ((t,p), \mathcal{E}ul(t,p)) = (\text{time-space position , value}), \end{cases} \tag{2.3}$$

and is called "a field of functions". So $\widehat{\mathcal{E}ul}(t,p)$ is the "pointed $\mathcal{E}ul(t,p)$" at (t,p) (in time-space).

So, the range $\text{Im}(\widehat{\mathcal{E}ul}) = \widehat{\mathcal{E}ul}(\mathcal{C})$ of an Eulerian function $\widehat{\mathcal{E}ul}$ is the graph of $\mathcal{E}ul$. (Recall: The graph of a function $f : x \in A \to f(x) \in B$ is the subset $\{(x, f(x)) \in A \times B\} \subset A \times B$: gives the "drawing of f").

If there is no ambiguity, $\widehat{\mathcal{E}ul} =^{\text{noted}} \mathcal{E}ul$ for short.

At t, the Eulerian vector field at t is $\widehat{\mathcal{E}ul}_t : \begin{cases} \Omega_t \to \Omega_t \times \vec{\mathbb{R}}^n \\ p \to \widehat{\mathcal{E}ul}_t(p) := (p, \mathcal{E}ul_t(p)) = (\text{position , value}). \end{cases}$

Example 2.2 $\mathcal{E}ul(t,p) = \theta(t,p) \in \mathbb{R}$ = temperature of the particle P_{Obj} which is at t at $p = \widetilde{\Phi}(t, P_{Obj})$; ⁂

Example 2.3 $\mathcal{E}ul(t,p) = \vec{u}(t,p) \in \vec{\mathbb{R}}^n$ = force applied on the particle P_{Obj} which is at t at p. ⁂

Example 2.4 $\mathcal{E}ul(t,p) = d\vec{u}(t,p) \in \mathcal{L}(\vec{\mathbb{R}}^n ; \vec{\mathbb{R}}^n)$ = the differential at t at p of a Eulerian function $\vec{u}$. ⁂

Question: Why introduce $\widehat{\mathcal{E}ul}$? Isn't $\mathcal{E}ul$ sufficient?
Answer: The "pointed value" $\widehat{\mathcal{E}ul}(t,p) = ((t,p), \mathcal{E}ul((t,p)))$ is drawn on the graph of $\mathcal{E}ul$.

E.g., at t at p the velocity vector $\vec{v}(t,p) \in \vec{\mathbb{R}}^3$ can be drawn anywhere, while the "pointed vector" $\widehat{\vec{v}}(t,p) = ((t,p); \vec{v}(t,p))$ is $\vec{v}(t,p)$ drawn at t at p (and $\widehat{\vec{v}}$ is called the velocity field).

Moreover (2.3) emphasizes the difference between a Eulerian vector field and a Lagrangian vector function, see (3.14).

Remark 2.5 E.g., the initial framework of Cauchy for his description of forces is Eulerian: The Cauchy stress vector $\vec{t} = \underline{\underline{\sigma}}.\vec{n}$ is considered at the actual time t at a point $p \in \Omega_t$. (It is not Lagrangian.) ⁂

2.3 Eulerian velocity (spatial velocity) and speed

Definition 2.6 In short: Consider a particle P_{Obj} and its (regular) trajectory $\widetilde{\Phi}_{P_{Obj}} : t \to p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$, cf. (1.8). Its Eulerian velocity at t at $p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$ is

$$\vec{v}(t,p(t)) := \widetilde{\Phi}_{P_{Obj}}{}'(t) \overset{\text{noted}}{=} \frac{\partial \widetilde{\Phi}}{\partial t}(t, P_{Obj}), \quad \text{when} \quad p(t) = \widetilde{\Phi}_{P_{Obj}}(t), \tag{2.4}$$

i.e., $\vec{v}(t,p(t))$ is the tangent vector at t at $p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$ to the trajectory $\widetilde{\Phi}_{P_{Obj}}$. This defines the vector field (in short) $\vec{v} : \left\{ \begin{matrix} \mathcal{C} & \to \vec{\mathbb{R}}^n \\ (t,p_t) & \to \vec{v}(t,p_t) \end{matrix} \right\}$.

In details: cf. (2.3), the Eulerian velocity is the function $\widehat{\vec{v}} : \left\{ \begin{matrix} \mathcal{C} & \to \mathcal{C} \times \vec{\mathbb{R}}^m \\ (t,p) & \to \widehat{\vec{v}}(t,p) = ((t,p), \vec{v}(t,p)) \end{matrix} \right\}$ (pointed vector) where $\vec{v}(t,p)$ is given by (2.4).

Remark 2.7 $\frac{d\widetilde{\Phi}_{P_{Obj}}}{dt}(t) = \vec{v}(t, \widetilde{\Phi}_{P_{Obj}}(t))$, with $p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$, is often written

$$\frac{dp}{dt}(t) = \vec{v}(t,p(t)), \quad \text{or} \quad \frac{d\vec{x}}{dt}(t) = \vec{v}(t,\vec{x}(t)), \quad \text{or} \quad \frac{d\vec{x}}{dt} = \vec{v}(t,\vec{x}), \tag{2.5}$$

the two last notations when an origin O is chosen and $\vec{x}(t) = \overrightarrow{Op(t)}$. Such an equation is the prototype of an ODE (ordinary differential equation) solved with the Cauchy–Lipschitz theorem, see § 5. (A Lagrangian velocity does not produce an ODE, see (3.21).) ⁂

Definition 2.8 If an observer chooses a Euclidean dot product $(\cdot,\cdot)_g$ (e.g. foot or metre built), the associated norm being $||.||_g$, then the length $||\vec{v}(t,p)||_g$ is the speed (or scalar velocity) of P_{Obj} (e.g. in ft/s or in m/s). And the context must remove the ambiguities: the "velocity" is either the vector velocity $\vec{v}(t,p) = \widetilde{\Phi}_{P_{Obj}}{}'(t)$ or the speed (the scalar velocity) $||\vec{v}(t,p)||_g$.

Exercice 2.9 Euclidean dot product $(\cdot,\cdot)_g$, $\vec{x}(t) = \overrightarrow{Op(t)}$, $\vec{T}(t) = \frac{\vec{x}\,'(t)}{||\vec{x}\,'(t)||_g}$, and $f(t) = ||\vec{x}\,'(t)||_g$ (speed). Prove : $\frac{df}{dt}(t) = (\vec{x}\,''(t), \vec{T}(t))_g =^{\text{noted}} \vec{x}\,''(t) \bullet \vec{T}(t)$ (= tangential acceleration).

Answer. 2-D and Euclidean basis: $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ gives $f(t) = (x'(t)^2 + y'(t)^2)^{\frac{1}{2}}$, thus $f'(t) = \frac{x'(t)x''(t)+y'(t)y''(t)}{f(t)} = \frac{\vec{r}\,'(t) \bullet \vec{r}\,''(t)}{||\vec{r}\,'(t)||}$. Idem in n-D. ⁂

2.4 Spatial derivative of the Eulerian velocity

$t \in [t_1,t_2]$ is fixed, $\mathcal{E}ul$ is a given Eulerian function, and $\mathcal{E}ul_t : \left\{ \begin{matrix} \Omega_t & \to \vec{\mathbb{R}}^m \\ p & \to \mathcal{E}ul_t(p) := \mathcal{E}ul(t,p) \end{matrix} \right\}$ is C^1.

2.4.1 Definition

Recall: If Ω is an open set in $\mathbb{R}^n$ and if $f:\Omega\to\mathbb{R}$ is differentiable at p, then its differential at p is the linear form $df(p)\in\mathcal{L}(\vec{\mathbb{R}^n};\mathbb{R})$ (linear map with real values) defined by, for all $\vec{u}\in\vec{\mathbb{R}^n}$ (vector at p),

$$df(p).\vec{u}=\lim_{h\to0}\frac{f(p{+}h\vec{u})-f(p)}{h}. \tag{2.6}$$

This expression is the same for all observers (English, French...: There is no inner dot product here).

Definition 2.10 The space derivative of $\mathcal{E}ul$ at (t,p) is the differential $d\mathcal{E}ul_t$ at p, i.e., for all $t\in[t_1,t_2]$, all $p\in\Omega_t$ and all $\vec{w}_p\in\vec{\mathbb{R}^n_t}$ (vector at p),

$$(d\mathcal{E}ul_t(p).\vec{w}_p=)\quad \boxed{d\mathcal{E}ul(t,p).\vec{w}_p=\lim_{h\to0}\frac{\mathcal{E}ul(t,p{+}h\vec{w}_p)-\mathcal{E}ul(t,p)}{h}}\overset{\text{noted}}{=}\frac{\partial\mathcal{E}ul}{\partial p}(t,p).\vec{w}_p. \tag{2.7}$$

In Ω_t (the photo at t), $d\mathcal{E}ul(t,p).\vec{w}_p$ gives the rate of variations of $\mathcal{E}ul_t$ at p in the direction $\vec{w}_p$.

E.g., at t, the space derivative $d\vec{v}$ of the Eulerian velocity field is defined by

$$d\vec{v}(t,p).\vec{w}_p=\lim_{h\to0}\frac{\vec{v}(t,p{+}h\vec{w}_p)-\vec{v}(t,p)}{h}\qquad(=d\vec{v}_t(p).\vec{w}_p). \tag{2.8}$$

Remark 2.11 In differential geometry, (2.6) is also written $\vec{u}(f)(p)=\frac{d}{dh}f(p{+}h\vec{u})_{|h=0}$; Don't use this notation if you are not at ease with differential geometry (where a vector is defined to be a derivation, so $\vec{u}[f]$ is the derivation of f by $\vec{u}$). ∴

2.4.2 The convective derivative $d\mathcal{E}ul.\vec{v}$

Definition 2.12 If $\vec{v}$ is the Eulerian velocity field, then $d\mathcal{E}ul.\vec{v}$ is called the convective derivative of $\mathcal{E}ul$.

2.4.3 Quantification in a basis: $df.\vec{u}$ is written $(\vec{u}.\vec{\text{grad}})f$

Quantification: Let $f:p\in\mathbb{R}^n\to f(p)\in\mathbb{R}$ be C^1. Let $(\vec{e}_i)$ be a basis in $\vec{\mathbb{R}^n}$. Let (usual definition)

$$\frac{\partial f}{\partial x_i}(p):=df(p).\vec{e}_i\quad\text{and}\quad[df(p)]_{|\vec{e}}=(\ \tfrac{\partial f}{\partial x_1}(p)\ \ \dots\ \ \tfrac{\partial f}{\partial x_n}(p)\ )\ \ \text{(line matrix)}. \tag{2.9}$$

(Recall: The matrix which represents a linear form is a line matrix.) And $[df(p)]_{|\vec{e}}$ is the Jacobian matrix of f at p relative to $(\vec{e}_i)$. So, with $\vec{u}=\sum_{i=1}^n u_i\vec{e}_i$ a vector at p, and with the usual matrix multiplication rule, we have

$$df(p).\vec{u}=[df(p)]_{|\vec{e}}.[\vec{u}]_{|\vec{e}}=\sum_{i=1}^n\frac{\partial f}{\partial x_i}(p)u_i=\sum_{i=1}^n u_i\frac{\partial f}{\partial x_i}(p)\overset{\text{noted}}{=}(\vec{u}.\vec{\text{grad}})_{|e}f(p), \tag{2.10}$$

where $(\vec{u}.\vec{\text{grad}})_{|e}:C^1(\Omega;\mathbb{R})\to C^0(\Omega;\mathbb{R})$ is the differential operator defined relative to a basis $(\vec{e}_i)$ by

$$(\vec{u}.\vec{\text{grad}})_{|e}(f)=\sum_{i=1}^n u_i\frac{\partial f}{\partial x_i}. \tag{2.11}$$

If the basis $(\vec{e}_i)$ is unambiguously imposed, then $(\vec{u}.\vec{\text{grad}})_{|e}=^{\text{noted}}\vec{u}.\vec{\text{grad}}$

For vector valued functions $\vec{f}:\Omega\to\vec{\mathbb{R}^m}$, the above steps apply to the components of $\vec{f}$ in a basis $(\vec{b}_i)$ in $\vec{\mathbb{R}^m}$: If $\vec{f}=\sum_{i=1}^m f_i\vec{b}_i$, i.e. $\vec{f}(p)=\sum_{i=1}^m f_i(p)\vec{b}_i$, then

$$(\vec{u}.\vec{\text{grad}})_{|e}(\vec{f})=\sum_{i=1}^m(df_i.\vec{u})\vec{b}_i=\sum_{i=1}^m((\vec{u}.\vec{\text{grad}})_{|e}f_i)\vec{b}_i=\sum_{i=1}^m\sum_{j=1}^n(u_j.\frac{\partial f_i}{\partial x_j})\vec{b}_i. \tag{2.12}$$

2.4.4 Representation relative to a Euclidean dot product: $\overrightarrow{\mathrm{grad}}f$

An observer chooses a distance unit (foot, metre...) and uses the associated Euclidean dot product $(\cdot,\cdot)_g$.

Let Ω be an open set in $\mathbb{R}^n$, $f \in C^1(\Omega;\mathbb{R})$ (scalar valued function), and $p \in \Omega$. Then the $(\cdot,\cdot)_g$-Riesz representation vector of the differential form $df(p)$ is called the gradient of f at p relative to $(\cdot,\cdot)_g$, and named $\overrightarrow{\mathrm{grad}}_g f(p) \in \vec{\mathbb{R}}^n$: It is defined by

$$\forall \vec{u} \in \vec{\mathbb{R}}^n, \quad (\overrightarrow{\mathrm{grad}}_g f(p), \vec{u})_g = df(p).\vec{u}, \quad \text{written} \quad \overrightarrow{\mathrm{grad}}f \bullet \vec{u} = df.\vec{u}, \tag{2.13}$$

the last notation iff a Euclidean dot product $(\cdot,\cdot)_g$ is imposed to all observer (quite subjective: foot, metre ?).

(The first order Taylor expansion $f(p{+}h\vec{u}) = f(p) + h\,df(p).\vec{u} + o(h)$ can therefore, after a choice of an Euclidean dot product, be written $f(p{+}h\vec{u}) = f(p) + h\,\overrightarrow{\mathrm{grad}}_g f(p) \bullet_g \vec{u} + o(h)$.)

Quantification: Let $(\vec{e}_i)$ be a Cartesian basis in $\mathbb{R}^n$. Then (2.13) gives $[df].[\vec{u}] = [\overrightarrow{\mathrm{grad}}f]^T.[g].[\vec{u}]$, for all $\vec{u} \in \vec{\mathbb{R}}^n_t$ (more precisely $[df]_{|\vec{e}}.[\vec{u}]_{|\vec{e}} = [\overrightarrow{\mathrm{grad}}_g f]^T_{|\vec{e}}.[g]_{|\vec{e}}.[\vec{u}]_{|\vec{e}}$), thus (since $[g]_{|\vec{e}}$ is symmetric)

$$[\overrightarrow{\mathrm{grad}}f] = [g].[df]^T \quad \text{(column matrix).} \tag{2.14}$$

I.e., if $\overrightarrow{\mathrm{grad}}f = \sum_{i=1}^n a_i \vec{e}_i$ then $a_i = \sum_{j=1}^n g_{ij} \frac{\partial f}{\partial x_j}$ for all i. In particular, if $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis then $[\overrightarrow{\mathrm{grad}}f] = [df]^T$.

With duality notations, $\overrightarrow{\mathrm{grad}}f = \sum_{i=1}^n a^i \vec{e}_i$ and (2.14) gives $a^i = \sum_{j=1}^n g_{ij} \frac{\partial f}{\partial x^j}$: The Einstein convention is ***not*** satisfied (the index j is twice bottom), which is expected since the definition of $\overrightarrow{\mathrm{grad}}_g f$ depends on a subjective choice (unit of length). In comparison, $df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$ satisfies the Einstein convention (a differential is objective).

Mind the notations: The gradient $\overrightarrow{\mathrm{grad}}_g f \overset{\text{noted}}{=} \overrightarrow{\mathrm{grad}}f$ depends on $(\cdot,\cdot)_g$, cf. (2.13)-(2.14), while $(\vec{u}.\overrightarrow{\mathrm{grad}})f$ does not (only depends on a basis), cf. (2.11) (historical notations...).

2.4.5 Vector valued functions

For vector valued functions $\vec{f} : \Omega \to \vec{\mathbb{R}}^m$, the above steps apply to the components f_i of $\vec{f}$ relative to a basis $(\vec{b}_i)$ in $\vec{\mathbb{R}}^m$... But, depending on the book you read:

1- Ambiguous: $d\vec{f}$, the differential of $\vec{f}$, is unfortunately also sometimes called the "gradient matrix" (although no Euclidean dot product is required).

2- Ambiguous: It could mean the differential... or the Jacobian matrix... or its transposed... because an orthonormal basis relative to an imposed Euclidean dot product is chosen (which one?) and then $[\overrightarrow{\mathrm{grad}}f_i] = [df_i]^T$... And calculations confuses [.] and $[.]^T$...

3- Non ambiguous: In the objective framework of this manuscript, we will use the differential $d\vec{f}$ (objective) to begin with; And only after an explicit choice of bases $(\vec{e}_i)$ for quantitative purposes, the Jacobian matrix, which is $[df]_{|\vec{e}}$, will be used.

Exercice 2.13 A Euclidean framework being chosen, prove: $(\vec{v}.\overrightarrow{\mathrm{grad}})\vec{v} = \frac{1}{2}\overrightarrow{\mathrm{grad}}(||\vec{v}||^2) + \overrightarrow{\mathrm{rot}}\vec{v} \wedge \vec{v}$.

Answer. Euclidean basis $(\vec{E}_i)$, Euclidean dot product $(\cdot,\cdot)_g \overset{\text{noted}}{=} (\cdot,\cdot)$, associated norm $||.||_g \overset{\text{noted}}{=} ||.||$. Thus $\vec{v} = \sum_{i=1}^n v_i \vec{E}_i$ gives $||\vec{v}||^2 = \sum\limits_i v_i^2$, thus $\dfrac{\partial ||\vec{v}||^2}{\partial x_k} = \sum\limits_i 2v_i \dfrac{\partial v_i}{\partial x_k}$, for any $k = 1,2,3$. And, the first component of $\overrightarrow{\mathrm{rot}}\vec{v}$ is $(\overrightarrow{\mathrm{rot}}\vec{v})_1 = \dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3}$, idem for $(\overrightarrow{\mathrm{rot}}\vec{v})_2$ and $(\overrightarrow{\mathrm{rot}}\vec{v})_3$ (circular permutation). Thus (first component) $(\overrightarrow{\mathrm{rot}}\vec{v}{\wedge}\vec{v})_1 = (\dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1})v_3 - (\dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2})v_2$, idem for $(\overrightarrow{\mathrm{rot}}\vec{v}{\wedge}\vec{v})_2$ and $(\overrightarrow{\mathrm{rot}}\vec{v}{\wedge}\vec{v})_2$. Thus $(\frac{1}{2}\overrightarrow{\mathrm{grad}}(||\vec{v}||^2) {+} \overrightarrow{\mathrm{rot}}\vec{v}{\wedge}\vec{v})_1 = v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_1} + v_3 \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} v_3 - \frac{\partial v_3}{\partial x_1} v_3 - \frac{\partial v_2}{\partial x_1} v_2 + \frac{\partial v_1}{\partial x_2} v_2 = v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} = (\vec{v}.\overrightarrow{\mathrm{grad}})v_1$. Idem for the other components. ∎

2.5 Streamline (current line)

Fix $t \in \mathbb{R}$, and consider the photo $\Omega_t = \widetilde{\Phi}_t(\mathit{Obj})$. Let $p_t \in \Omega_t$, $\varepsilon > 0$, and consider the spatial curve in Ω_t at p_t defined by:

$$c_{p_t} : \left\{ \begin{array}{rl}]-\varepsilon,\varepsilon[& \to \Omega_t \\ s & \to q = c_{p_t}(s) \end{array} \right\} \quad \text{s.t.} \quad c_{p_t}(0) = p_t. \tag{2.15}$$

So s is a curvilinear spatial coordinate (dimension of a length), and the graph of c_{p_t} is drawn in the photo Ω_t at t.

Definition 2.14 $\vec{v} : (t,p) \to \vec{v}(t,p)$ being the Eulerian velocity field of *Obj*, a streamline through a point $p_t \in \Omega_t$ is a (parametric) spatial curve c_{p_t} solution of the differential equation

$$\frac{dc_{p_t}}{ds}(s) = \vec{v}_t(c_{p_t}(s)) \quad \text{with} \quad c_{p_t}(0) = p_t. \tag{2.16}$$

And $\mathrm{Im}(c_{p_t})$ is the geometric associated streamline ($\subset \Omega_t$).

NB: (2.16) cannot be confused with (2.5): In (2.5) the variable is the time variable t, while in (2.16) the variable is the space variable s.

Usual notation: If an origin $\mathcal{O}$ is chosen at t by an observer and $\vec{x}(s) := \overrightarrow{\mathcal{O}c_{p_t}(s)}$, then (2.16) is written

$$\frac{d\vec{x}}{ds}(s) = \vec{v}_t(\vec{x}(s)) \quad \text{with} \quad \vec{x}(0) = \overrightarrow{\mathcal{O}p_t}. \tag{2.17}$$

Moreover, with a Cartesian basis $(\vec{e}_i)$) chosen at t by the observer, with $\vec{x}(s) = \sum_{i=1}^n x_i(s)\vec{e}_i$ we get $\frac{d\vec{x}}{ds}(s) = \sum_{i=1}^n \frac{dx_i}{ds}(s)\vec{e}_i$, and (2.17) reads as the differential system of n equations in $\mathbb{R}^n$

$$\forall i = 1,...,n, \quad \frac{dx_i}{ds}(s) = v_i(t, x_1(s), ..., x_n(s)) \quad \text{with} \quad x_i(0) = (\overrightarrow{\mathcal{O}p_t})_i \tag{2.18}$$

(the n functions $x_i : s \to x_i(s)$ are the unknown). Also written

$$\frac{dx_1}{v_1} = ... = \frac{dx_n}{v_n} = ds, \tag{2.19}$$

which means: It is the differential system (2.18) of n equations and n unknowns which must be solved.

(With duality notations, $\frac{dx^i}{ds}(s) = v^i(t, x^1(s), ..., x^n(s))$ and $x^i(0) = (\overrightarrow{\mathcal{O}p_t})^i$ for all i.)

2.6 Material time derivative (dérivées particulaires)

2.6.1 Usual definition

Goal: To compute the variations of a Eulerian function $\mathcal{E}ul$ along the trajectory $\widetilde{\Phi}_{P_{Obj}}$ of a particle P_{Obj} (e.g. the temperature of a particle along its trajectory). So consider the function $g_{P_{Obj}}$ giving the values of $\mathcal{E}ul$ relative to a P_{Obj} along its trajectory:

$$g_{P_{Obj}}(t) := \mathcal{E}ul(t, p(t)) \quad \text{when} \quad p(t) := \widetilde{\Phi}_{P_{Obj}}(t). \tag{2.20}$$

Definition 2.15 The Material time derivative of $\mathcal{E}ul$ at $(t, p(t))$ is $g_{P_{Obj}}{}'(t) =^{\text{noted}} \frac{D\mathcal{E}ul}{Dt}(t, p(t))$.

So:

$$\frac{D\mathcal{E}ul}{Dt}(t, p(t)) := g_{P_{Obj}}{}'(t) \quad (= \lim_{h\to 0} \frac{\mathcal{E}ul(t+h, p(t+h)) - \mathcal{E}ul(t, p(t))}{h}). \tag{2.21}$$

Since $g_{P_{Obj}}(t) := \mathcal{E}ul(t, \widetilde{\Phi}_{P_{Obj}}(t))$ we get $g_{P_{Obj}}{}'(t) = \frac{\partial\mathcal{E}ul}{\partial t}(t, \widetilde{\Phi}_{P_{Obj}}(t)) + d\mathcal{E}ul(t, \widetilde{\Phi}_{P_{Obj}}(t)).\widetilde{\Phi}'_{P_{Obj}}(t)$, thus, having $\widetilde{\Phi}'_{P_{Obj}}(t) = \vec{v}(t, p(t))$ (Eulerian velocity), $\frac{D\mathcal{E}ul}{Dt}(t, p(t)) = \frac{\partial\mathcal{E}ul}{\partial t}(t, p(t)) + d\mathcal{E}ul(t, p(t)).\vec{v}(t, p(t))$:

$$\boxed{\frac{D\mathcal{E}ul}{Dt} := \frac{\partial\mathcal{E}ul}{\partial t} + d\mathcal{E}ul.\vec{v}}. \tag{2.22}$$

Exercice 2.16 Prove, if $\mathcal{E}ul$ is C^2:

$$\frac{D^2\mathcal{E}ul}{Dt^2} = \frac{\partial^2\mathcal{E}ul}{\partial t^2} + 2d\frac{\partial\mathcal{E}ul}{\partial t}.\vec{v} + d\mathcal{E}ul.\frac{\partial\vec{v}}{\partial t} + d^2\mathcal{E}ul(\vec{v}, \vec{v}) + d\mathcal{E}ul.d\vec{v}.\vec{v} \tag{2.23}$$

Answer.

$$\begin{aligned} \frac{D^2\mathcal{E}ul}{Dt^2} = \frac{D\frac{D\mathcal{E}ul}{Dt}}{Dt} = g''_{P_{Obj}}(t) &= \frac{\partial(\frac{\partial\mathcal{E}ul}{\partial t} + d\mathcal{E}ul.\vec{v})}{\partial t} + d(\frac{\partial\mathcal{E}ul}{\partial t} + d\mathcal{E}ul.\vec{v}).\vec{v} \\ &= \frac{\partial^2\mathcal{E}ul}{\partial t^2} + \frac{\partial(d\mathcal{E}ul)}{\partial t}.\vec{v} + d\mathcal{E}ul.\frac{\partial\vec{v}}{\partial t} + d\frac{\partial\mathcal{E}ul}{\partial t}.\vec{v} + d^2\mathcal{E}ul(\vec{v}, \vec{v}) + d\mathcal{E}ul.d\vec{v}.\vec{v}, \end{aligned}$$

and $\mathcal{E}ul$ C^2 gives $\frac{\partial}{\partial t} \circ d = d \circ \frac{\partial}{\partial t}$ (Schwarz theorem), hence (2.23). ∎

Exercice 2.17 Prove, if $\mathcal{E}ul$ is C^2, for any vector field $\vec{w}$,

$$\frac{D(d\mathcal{E}ul.\vec{w})}{Dt} = d\frac{\partial\mathcal{E}ul}{\partial t}.\vec{w} + d\mathcal{E}ul.\frac{\partial\vec{w}}{\partial t} + d^2\mathcal{E}ul(\vec{v},\vec{w}) + d\mathcal{E}ul.d\vec{w}.\vec{v}. \tag{2.24}$$

Answer. $\frac{D(d\mathcal{E}ul.\vec{w})}{Dt} = \frac{\partial(d\mathcal{E}ul.\vec{w})}{\partial t} + d(d\mathcal{E}ul.\vec{w}).\vec{v} = \frac{\partial d\mathcal{E}ul}{\partial t}.\vec{w} + d\mathcal{E}ul.\frac{\partial\vec{w}}{\partial t} + (d(d\mathcal{E}ul).\vec{v}).\vec{w} + d\mathcal{E}ul.d\vec{w}.\vec{v}$. And the Schwarz theorem gives $\frac{\partial(d\mathcal{E}ul)}{\partial t} = d(\frac{\partial\mathcal{E}ul}{\partial t})$ since $\mathcal{E}ul \in C^2$. Hence (2.24). ∎

2.6.2 Remark: About notations

- The notation $\frac{d}{dt}$ (lowercase letters) concerns a function of one variable, e.g. $\frac{dg_{P_{Obj}}}{dt}(t) := g_{P_{Obj}}{}'(t) := \lim_{h\to 0}\frac{g_{P_{Obj}}(t+h))-g_{P_{Obj}}(t)}{h}$;
- The notation $\frac{\partial}{\partial t}$ concerns a function with more than one variable, e.g. $\frac{\partial\mathcal{E}ul}{\partial t}(t,p) = \lim_{h\to 0}\frac{\mathcal{E}ul(t+h,p)-\mathcal{E}ul(t,p)}{h}$;
- The notation $\frac{D}{Dt}$ (capital letters) concerns a Eulerian function differentiated along a motion, cf. (2.21).
- Other notations, often practical but might be ambiguous if composed functions are considered:

$$\frac{d\mathcal{E}ul(t,p(t))}{dt} := \frac{D\mathcal{E}ul}{Dt}(t,p(t)), \quad \text{and} \quad \frac{d\mathcal{E}ul(t,p(t))}{dt}_{|t=t_0} := \frac{D\mathcal{E}ul}{Dt}(t_0,p(t_0)). \tag{2.25}$$

2.6.3 Definition bis: Time-space definition

Consider a C^1 time-space function $f : (t,p) \in \mathbb{R}\times\mathbb{R}^n \to f(t,p)$ where t = time and p = space.

Definition 2.18 The differential of $f : (t,p) \in \mathbb{R}^{n+1} \to f(t,p)$ considered as a function on the Cartesian (time×space) product $\mathbb{R}\times\mathbb{R}^n$ is called the "total differential", or "total derivative", and is written Df (here time and space are of a different nature).

So if $p_+ = (t,p) \in \mathbb{R}\times\mathbb{R}^n$ and $\vec{w}_+ = (w_0,\vec{w}) \in \vec{\mathbb{R}}\times\vec{\mathbb{R}}^n$ (time×space) then, by definition of a differential,

$$Df(p_+).\vec{w}_+ := \lim_{h\to 0}\frac{f(p_+ + h\vec{w}_+) - f(p_+)}{h}, \tag{2.26}$$

i.e.

$$Df(t,p).(w_0,\vec{w}) := \lim_{h\to 0}\frac{f(t+hw_0, p+h\vec{w}) - f(t,p)}{h}. \tag{2.27}$$

Thus

$$Df(t,p) = \frac{\partial f}{\partial t}(t,p)\,dt + df(t,p). \tag{2.28}$$

Along a trajectory $\widetilde{\Phi}_{P_{Obj}} : t \to p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$ with $f = \mathcal{E}ul$ a Eulerian function: Consider the time-space trajectory

$$\widetilde{\Psi}_{P_{Obj}} : \begin{cases} [t_1,t_2] & \to \mathbb{R}\times\mathbb{R}^n \\ \quad t & \to \widetilde{\Psi}_{P_{Obj}}(t) := (t, \widetilde{\Phi}_{P_{Obj}}(t)) \quad (= (t,p(t))). \end{cases} \tag{2.29}$$

(So $\mathrm{Im}(\widetilde{\Psi}_{P_{Obj}}) = \mathrm{graph}(\widetilde{\Phi}_{P_{Obj}})$.) The tangent vector to this curve at t is

$$\widetilde{\Psi}_{P_{Obj}}{}'(t) = (1, \widetilde{\Phi}_{P_{Obj}}{}'(t)) = (1, \vec{v}(t,p(t)) \in \vec{\mathbb{R}}\times\vec{\mathbb{R}}^n \tag{2.30}$$

where $\vec{v}(t,p(t))$ the Eulerian velocity at $p_+ = (t,p(t))$. And (2.20) reads

$$g_{P_{Obj}}(t) = (\mathcal{E}ul \circ \widetilde{\Psi}_{P_{Obj}})(t) = \mathcal{E}ul(\widetilde{\Psi}_{P_{Obj}}(t)), \tag{2.31}$$

thus

$$g'_{P_{Obj}}(t) = D\mathcal{E}ul(\widetilde{\Psi}(t)).\widetilde{\Psi}_{P_{Obj}}{}'(t). \tag{2.32}$$

And we recover (2.22): $g'_{P_{Obj}}(t) =^{(2.28)} \frac{\partial\mathcal{E}ul}{\partial t}(t,p(t)).1 + d\mathcal{E}ul(t,p(t)).\vec{v}(t,p(t)) =^{\text{noted}} \frac{D\mathcal{E}ul}{Dt}(t,p(t))$: The material time derivative is the "total derivative" $D\mathcal{E}ul$ along the time-space trajectory $\widetilde{\Psi}_{P_{Obj}}$.

2.6.4 The material time derivative is a derivation

Proposition 2.19 *All the functions are Eulerian and supposed C^1.*

- *Linearity:*

$$\frac{D(\mathcal{E}ul_1+\lambda\mathcal{E}ul_2)}{Dt}=\frac{D\mathcal{E}ul_1}{Dt}+\lambda\frac{D\mathcal{E}ul_2}{Dt}. \tag{2.33}$$

- *Product rules: If $\mathcal{E}ul_1,\mathcal{E}ul_2$ are scalar valued functions then*

$$\frac{D(\mathcal{E}ul_1\mathcal{E}ul_2)}{Dt}=\frac{D\mathcal{E}ul_1}{Dt}\mathcal{E}ul_2+\mathcal{E}ul_1\frac{D\mathcal{E}ul_2}{Dt}. \tag{2.34}$$

And if $\vec{w}$ is a vector field and T a compatible tensor (so $T.\vec{w}$ is meaningful) then

$$\frac{D(T.\vec{w})}{Dt}=\frac{DT}{Dt}.\vec{w}+T.\frac{D\vec{w}}{Dt}. \tag{2.35}$$

Proof. Let $i=1,2$, and g_i defined by $g_i(t):=\mathcal{E}ul_i(t,p(t))$ where $p(t)=\widetilde{\Phi}_{P_{Obj}}(t)$.

- $(g_1+\lambda g_2)'=g_1'+\lambda g_2'$ gives (2.33).
- On the one hand $\frac{D(T.\vec{w})}{Dt}=\frac{\partial(T.\vec{w})}{\partial t}+d(T.\vec{w}).\vec{v}=\frac{\partial T}{\partial t}.\vec{w}+T.\frac{\partial\vec{w}}{\partial t}+(dT.\vec{v}).\vec{w}+T.(d\vec{w}.\vec{v})$, and on the other hand $\frac{DT}{Dt}.\vec{w}+T.\frac{D\vec{w}}{Dt}=(\frac{\partial T}{\partial t}+dT.\vec{v}).\vec{w}+T.(\frac{\partial\vec{w}}{\partial t}+d\vec{w}.\vec{v})$. Thus (2.34)-(2.35). ∎

2.6.5 Commutativity issue

The Schwarz theorem that, when $\mathcal{E}ul$ is C^2, the derivatives $\frac{\partial\mathcal{E}ul}{\partial t}$ and $d\mathcal{E}ul$ commute. But

Proposition 2.20 *The material time derivative $\frac{D}{Dt}$ does not commute with the temporal derivation $\frac{\partial}{\partial t}$ or with the spatial derivation d: We have $\frac{\partial(\frac{D\mathcal{E}ul}{Dt})}{\partial t}\neq\frac{D(\frac{\partial\mathcal{E}ul}{\partial t})}{Dt}$ and $d(\frac{D\mathcal{E}ul}{Dt})\neq\frac{D(d\mathcal{E}ul)}{Dt}$ in general (because the variables t and p are not independent along a trajectory). In facts:*

$$\left.\begin{aligned}\frac{\partial(\frac{D\mathcal{E}ul}{Dt})}{\partial t}&=\frac{D(\frac{\partial\mathcal{E}ul}{\partial t})}{Dt}+d\mathcal{E}ul.\frac{\partial\vec{v}}{\partial t}\\&=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+d\frac{\partial\mathcal{E}ul}{\partial t}.\vec{v}+d\mathcal{E}ul.\frac{\partial\vec{v}}{\partial t}\end{aligned}\right\},\quad\text{and}\quad\left\{\begin{aligned}d(\frac{D\mathcal{E}ul}{Dt})&=\frac{D(d\mathcal{E}ul)}{Dt}+d\mathcal{E}ul.d\vec{v}\\&=\frac{\partial(d\mathcal{E}ul)}{\partial t}+d^2\mathcal{E}ul.\vec{v}+d\mathcal{E}ul.d\vec{v}\end{aligned}\right. \tag{2.36}$$

Proof. $\dfrac{\partial\frac{D\mathcal{E}ul}{Dt}}{\partial t}=\dfrac{\partial(\frac{\partial\mathcal{E}ul}{\partial t}+d\mathcal{E}ul.\vec{v})}{\partial t}=\dfrac{\partial(\frac{\partial\mathcal{E}ul}{\partial t})}{\partial t}+\dfrac{\partial d\mathcal{E}ul}{\partial t}.\vec{v}+d\mathcal{E}ul.\dfrac{\partial\vec{v}}{\partial t}\overset{\text{Schwarz}}{=}\dfrac{\partial(\frac{\partial\mathcal{E}ul}{\partial t})}{\partial t}+d\dfrac{\partial\mathcal{E}ul}{\partial t}.\vec{v}+d\mathcal{E}ul.\dfrac{\partial\vec{v}}{\partial t}$, thus $(2.36)_1$.

$d\dfrac{D\mathcal{E}ul}{Dt}=d(\dfrac{\partial\mathcal{E}ul}{\partial t}+d\mathcal{E}ul.\vec{v})=\dfrac{\partial(d\mathcal{E}ul)}{\partial t}+d(d\mathcal{E}ul).\vec{v}+d\mathcal{E}ul.d\vec{v}$, thus $(2.36)_2$.

So $\dfrac{\partial(\frac{D\mathcal{E}ul}{Dt})}{\partial t}\neq\dfrac{D(\frac{\partial\mathcal{E}ul}{\partial t})}{Dt}$ and $d(\dfrac{D\mathcal{E}ul}{Dt})\neq\dfrac{D(d\mathcal{E}ul)}{Dt}$ in general. ∎

Exercice 2.21 Prove (2.36) with components.

Answer. $(\vec{e}_i)$ is a Cartesian basis. $\frac{\partial\frac{D\mathcal{E}ul}{Dt}}{\partial t}=\frac{\partial(\frac{\partial\mathcal{E}ul}{\partial t}+\sum_i\frac{\partial\mathcal{E}ul}{\partial x^i}.v^i)}{\partial t}=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+\sum_i\frac{\partial^2\mathcal{E}ul}{\partial t\partial x^i}.v^i+\sum_i\frac{\partial\mathcal{E}ul}{\partial x^i}.\frac{\partial v^i}{\partial t}=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+\sum_i\frac{\partial^2\mathcal{E}ul}{\partial t\partial x^i}.v^i+d\mathcal{E}ul.\frac{\partial\vec{v}}{\partial t}$. And $\frac{D(\frac{\partial\mathcal{E}ul}{\partial t})}{Dt}=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+\sum_i\frac{\partial\frac{\partial\mathcal{E}ul}{\partial t}}{\partial x^i}.v^i=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+\sum_i\frac{\partial^2\mathcal{E}ul}{\partial t\partial x^i}.v^i$.

And $d(\frac{D\mathcal{E}ul}{Dt}).\vec{w}=\sum_j\frac{\partial\frac{D\mathcal{E}ul}{Dt}}{\partial x^j}w^j=\sum_j\frac{\partial(\frac{\partial\mathcal{E}ul}{\partial t}+\sum_i\frac{\partial\mathcal{E}ul}{\partial x^i}v^i)}{\partial x^j}w^j=\sum_j\frac{\partial^2\mathcal{E}ul}{\partial t\partial x^j}w^j+\sum_{ij}\frac{\partial^2\mathcal{E}ul}{\partial x^i\partial x^j}v^iw^j+\sum_{ij}\frac{\partial\mathcal{E}ul}{\partial x^i}\frac{\partial v^i}{\partial x^j}w^j=\sum_j\frac{\partial^2\mathcal{E}ul}{\partial t\partial x^j}w^j+d^2\mathcal{E}ul(\vec{v},\vec{w})+d\mathcal{E}ul.d\vec{v}.\vec{w}$. And $\frac{D(d\mathcal{E}ul)}{Dt}.\vec{w}=(\frac{\partial(d\mathcal{E}ul)}{\partial t}+d(d\mathcal{E}ul).\vec{v}).\vec{w}=\frac{\partial(d\mathcal{E}ul)}{\partial t}.\vec{w}+d^2\mathcal{E}ul(\vec{v},\vec{w})=\sum_i\frac{\partial^2\mathcal{E}ul}{\partial x^i\partial t}w^i+d^2\mathcal{E}ul(\vec{v},\vec{w})$. Thus $d(\frac{D\mathcal{E}ul}{Dt}).\vec{w}=\frac{D(d\mathcal{E}ul)}{Dt}.\vec{w}+d\mathcal{E}ul.d\vec{v}.\vec{w}$ for all $\vec{w}$. ∎

2.7 Eulerian acceleration

Definition 2.22 In short: If $\widetilde{\Phi}_{P_{Obj}}$ is C^2, then the Eulerian acceleration of the particle P_{Obj} which is at t at $p_t=\widetilde{\Phi}(t,P_{Obj})$ is

$$\vec{\gamma}(t,p_t):=\widetilde{\Phi}_{P_{Obj}}{}''(t)\overset{\text{noted}}{=}\frac{\partial^2\widetilde{\Phi}}{\partial t^2}(t,P_{Obj}). \tag{2.37}$$

In details: as in (2.3), the Eulerian acceleration (vector) field $\widehat{\vec{\gamma}}$ is defined with (2.37) by

$$\widehat{\vec{\gamma}}(t,p_t)=((t,p_t),\vec{\gamma}(t,p_t))\in\mathcal{C}\times\vec{\mathbb{R}}^n_t\quad\text{(pointed vector)}. \tag{2.38}$$

Proposition 2.23

$$\boxed{\vec{\gamma} = \frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + d\vec{v}.\vec{v}}. \tag{2.39}$$

And if $\vec{v}$ is C^2 then

$$d\vec{\gamma} = \frac{\partial(d\vec{v})}{\partial t} + d^2\vec{v}.\vec{v} + d\vec{v}.d\vec{v} = \frac{D(d\vec{v})}{Dt} + d\vec{v}.d\vec{v}. \tag{2.40}$$

Proof. With $g(t) = \vec{v}(t,p(t)) = \widetilde{\Phi}_{P_{Obj}}{}'(t)$ and (2.22) we get $\vec{\gamma}(t,p(t)) = g'(t) = \frac{D\vec{v}}{Dt}(t,p(t))$. And $\vec{v}$ being C^2, the Schwarz theorem gives $d\frac{\partial\vec{v}}{\partial t} = \frac{\partial(d\vec{v})}{\partial t}$. ∴

Definition 2.24 If an observer chooses a Euclidean dot product $(\cdot,\cdot)_g$ (based on a foot, a metre...), the associated norm being $||.||_g$, then the length $||\vec{\gamma}(t,p_t)||_g$ is the (scalar) acceleration of P_{Obj}.

2.8 Time Taylor expansion of $\widetilde{\Phi}$

Let $P_{Obj} \in Obj$ and $t \in]t_1,t_2[$. Suppose $\widetilde{\Phi}_{P_{Obj}} \in C^2(]t_1,t_2[;\mathbb{R}^n)$. Its second-order (time) Taylor expansion of $\widetilde{\Phi}_{P_{Obj}}$ is, in the vicinity of a $t \in]t_1,t_2[$,

$$\widetilde{\Phi}_{P_{Obj}}(\tau) = \widetilde{\Phi}_{P_{Obj}}(t) + (\tau-t)\widetilde{\Phi}'_{P_{Obj}}(t) + \frac{(\tau-t)^2}{2}\widetilde{\Phi}''_{P_{Obj}}(t) + o((\tau-t)^2), \tag{2.41}$$

i.e.

$$p(\tau) = p(t) + (\tau-t)\vec{v}(t,p(t)) + \frac{(\tau-t)^2}{2}\vec{\gamma}(t,p(t)) + o((\tau-t)^2). \tag{2.42}$$

3 Motion from an initial configuration: Lagrangian description

Instead of working on Obj, an observer may prefer to work with an initial configuration $\Omega_{t_0} = \widetilde{\Phi}(t_0, Obj)$ of Obj (essential for elasticity): This is the “Lagrangian approach”. This Lagrangian approach is not objective: Two observers may choose two different initial (times and) configurations.

3.1 Initial configuration and Lagrangian “motion”

3.1.1 Definition

Obj is a material object, $\widetilde{\Phi} : [t_1,t_2[\times Obj \to \mathbb{R}^n$ is its motion, $\Omega_\tau = \widetilde{\Phi}_\tau(Obj)$ is its configuration at τ, $t_0 \in]t_1,t_2[$ is an “initial time”, and Ω_{t_0} is the initial configuration for the observer who chose t_0.

Definition 3.1 The motion of Obj relative to the initial configuration $\Omega_{t_0} = \widetilde{\Phi}(t_0, Obj)$ is the function

$$\Phi^{t_0} : \begin{cases} [t_1,t_2] \times \Omega_{t_0} & \to \mathbb{R}^n \\ \qquad (t,p_{t_0}) & \mapsto p_t = \Phi^{t_0}(t,p_{t_0}) := \widetilde{\Phi}(t,P_{Obj}) \quad \text{when} \quad p_{t_0} = \widetilde{\Phi}(t_0,P_{Obj}). \end{cases} \tag{3.1}$$

So, $p_t = \Phi^{t_0}(t,p_{t_0}) := \widetilde{\Phi}(t,P_{Obj})$ is the position at t of the particle P_{Obj} which was at p_{t_0} at t_0. In particular $p_{t_0} = \Phi^{t_0}(t_0,p_{t_0}) := \widetilde{\Phi}(t_0,P_{Obj})$.

Marsden and Hughes notations: Once an initial time t_0 has been chosen by an observer, then $\Phi^{t_0} =^{\text{noted}} \Phi$, then $p_{t_0} =^{\text{noted}} P$ (capital letter for positions at t_0) and $p_t =^{\text{noted}} p$ (lowercase letter for positions at t), so

$$p = \Phi(t,P) \ \in \Omega_t. \tag{3.2}$$

(When objectivity is under concern, we need to switch back to the notations Φ^{t_0}, p_{t_0} and p_t.)

NB: • Talking about the motion of a position p_{t_0} is absurd: A position in $\mathbb{R}^n$ does not move. Thus Φ^{t_0} has no existence without the definition, at first, of the motion $\widetilde{\Phi}$ of particles.

• The domain of definition of Φ^{t_0} depends on t_0 through Ω_{t_0}: The superscript ^{t_0} recalls it. And a late observer with initial time $t_0{}' > t_0$ defines $\Phi^{t_0{}'}$ which domain of definition is $[t_1,t_2] \times \Omega_{t_0{}'}$; And $\Phi^{t_0{}'} \neq \Phi^{t_0}$ in general because $\Omega_{t_0{}'} \neq \Omega_{t_0}$ in general.

- The following notation is also used:

$$\Phi^{t_0}(t,p_{t_0}) = \Phi(t;t_0,p_{t_0}). \tag{3.3}$$

(The couple (t_0,p_{t_0}) is "the initial condition", or t_0 and p_{t_0} are the initial conditions, see the § on flows).

- If a origin $\mathcal{O} \in \mathbb{R}^n$ is chosen by the observer, we may also use, with (1.6),

$$\vec{x}_{t_0} = \overrightarrow{\mathcal{O}p_{t_0}} = \vec{\varphi}^{t_0}(t_0,\vec{x}_{t_0}) = \vec{X} = \overrightarrow{\mathcal{O}P} \quad \text{and} \quad \vec{x}_t = \overrightarrow{\mathcal{O}p_t} = \vec{\varphi}^{t_0}(t,\vec{x}_{t_0}) = \vec{x} = \overrightarrow{\mathcal{O}p}. \tag{3.4}$$

3.1.2 Diffeomorphism between configurations

With (3.1), define

$$\Phi^{t_0}_t : \begin{cases} \Omega_{t_0} & \to \Omega_t \\ p_{t_0} & \to p_t = \Phi^{t_0}_t(p_{t_0}) := \Phi^{t_0}(t,p_{t_0}). \end{cases} \tag{3.5}$$

Hypothesis: For all $t_0, t \in]t_1,t_2[$, the map $\Phi^{t_0}_t : \Omega_{t_0} \to \Omega_t$ is a C^k diffeomorphism (a C^k invertible function whose inverse is C^k), where $k \in \mathbb{N}^*$ depends on the required regularity.

Thus (3.5) gives $\widetilde{\Phi}_t(P_{Obj}) = \Phi^{t_0}_t(\widetilde{\Phi}_{t_0}(P_{Obj}))$, true for all $P_{Obj} \in Obj$, thus $\Phi^{t_0}_t \circ \widetilde{\Phi}_{t_0} = \widetilde{\Phi}_t$, i.e.

$$\boxed{\Phi^{t_0}_t := \widetilde{\Phi}_t \circ (\widetilde{\Phi}_{t_0})^{-1}}. \tag{3.6}$$

Thus, $\Phi^{t_0}_{t_0} = I$ and $\Phi^{t}_{t_0} \circ \Phi^{t_0}_t = (\widetilde{\Phi}_t \circ (\widetilde{\Phi}_{t_0})^{-1}) \circ (\widetilde{\Phi}_{t_0} \circ (\widetilde{\Phi}_t)^{-1}) = I$ give

$$\Phi^{t}_{t_0} = (\Phi^{t_0}_t)^{-1}. \tag{3.7}$$

3.1.3 Trajectories

Let $(t_0,p_{t_0}) \in [t_1,t_2] \times \Omega_{t_0}$ (initial conditions) and with (3.1) define

$$\Phi^{t_0}_{p_{t_0}} : \begin{cases} [t_1,t_2] & \to \mathbb{R}^n \\ \quad t & \mapsto p(t) = \Phi^{t_0}_{p_{t_0}}(t) := \widetilde{\Phi}_{P_{Obj}}(t) = \Phi^{t_0}(t,p_{t_0}) \quad \text{when} \quad p_{t_0} = \widetilde{\Phi}_{P_{Obj}}(t_0). \end{cases} \tag{3.8}$$

Definition 3.2 $\Phi^{t_0}_{p_{t_0}}$ is called the (parametric) "trajectory of p_{t_0}", which means: $\Phi^{t_0}_{p_{t_0}}$ is the trajectory of the particle P_{Obj} that is located at $p_{t_0} = \widetilde{\Phi}(t,P_{Obj})$ at t_0. And the geometric "trajectory of p_{t_0}" is

$$\mathrm{Im}(\Phi^{t_0}_{p_{t_0}}) = \Phi^{t_0}_{p_{t_0}}([t_1,t_2]) = \bigcup_{t\in[t_1,t_2]} \{\Phi^{t_0}_{p_{t_0}}(t)\} \quad (= \mathrm{Im}(\widetilde{\Phi}_{P_{Obj}})). \tag{3.9}$$

NB: The terminology "trajectory of p_{t_0}" is awkward, since a position p_{t_0} does not move: It is indeed the trajectory $\widetilde{\Phi}_{P_{Obj}}$ of a particle P_{Obj} which is at p_{t_0} at t_0 that must be understood.

3.1.4 Streaklines (lignes d'émission)

Take a film between t_0 and T (start and end).

Definition 3.3 Let Q be a fixed point in $\mathbb{R}^n$ (you see the point Q on each photo that make up the film). The streakline through Q is the set

$$\begin{aligned} E_{t_0,T}(Q) &= \{p \in \Omega : \exists \tau \in [t_0,T] : p = \Phi^{\tau}_T(Q) = (\Phi^T_\tau)^{-1}(Q)\} \\ &= \{p \in \Omega : \exists u \in [0,T{-}t_0] : p = \Phi^{T-u}_T(Q) = (\Phi^T_{T-u})^{-1}(Q)\} \end{aligned} \tag{3.10}$$

= the set at T of the positions (a line in $\mathbb{R}^n$) of all the particles which were at Q at a $\tau \in [t_0,T]$.

Example 3.4 Smoke comes out of a chimney. Fix a camera nearby, choose a point Q at the top of the chimney where the particles are colored, and make a film. At T stop filming. Then (at time T) superimpose the photos in the film: The colored curve we see is the streakline. ∴

In other words $= \bigcup_{\tau\in[t_0,T]}\{\Phi^\tau_Q(T)\} = \bigcup_{u\in[0,T-t_0]}\{\Phi^{T-u}_Q(T)\}$.

3.2 Lagrangian variables and functions

3.2.1 Definition

Consider a motion $\widetilde{\Phi}$, cf. (1.5). An observer chose (subjective) a $t_0 \in [t_1, t_2]$ ("in the past"); So $\Omega_{t_0} = \widetilde{\Phi}(t_0, \mathit{Obj})$ is his initial configuration. Let $m \in \mathbb{N}^*$.

Definition 3.5 In short: A Lagrangian function, relative to Obj, $\widetilde{\Phi}$ and t_0, is a function

$$\mathcal{L}ag^{t_0} : \begin{cases} [t_1, t_2] \times \Omega_{t_0} & \rightarrow \overrightarrow{\mathbb{R}^m} \quad \text{(or, more generally, some adequat set)} \\ \qquad (t, p_{t_0}) & \rightarrow \mathcal{L}ag^{t_0}(t, p_{t_0}), \end{cases} \tag{3.11}$$

and p_{t_0} is called the Lagrangian variable relative to the (subjective) choice t_0.
(To compare with (2.2): A Eulerian function does not depend on any t_0.)

Example 3.6 Scalar values: $\mathcal{L}ag^{t_0}(t, p_{t_0}) = \Theta^{t_0}(t, p_{t_0}) =$ temperature at t at $p_t = \Phi_t^{t_0}(p_{t_0}) = \widetilde{\Phi}(t, P_{\mathit{Obj}})$ of the particle P_{Obj} that was at p_{t_0} at t_0. (So, continuing example 2.2, $\Theta^{t_0}(t, p_{t_0}) = \theta(t, p_t)$.) ∴

Example 3.7 Vectorial values: $\mathcal{L}ag^{t_0}(t, p_{t_0}) = \vec{U}^{t_0}(t, p_{t_0}) =$ force at t at $p_t = \Phi_t^{t_0}(p_{t_0}) = \widetilde{\Phi}(t, P_{\mathit{Obj}})$ acting on the particle P_{Obj} that was at p_{t_0} at t_0. (So, continuing example 2.3, $\vec{U}^{t_0}(t, p_{t_0}) = \vec{u}(t, p_t)$.) ∴

If t is fixed or if $p_{t_0} \in \Omega_{t_0}$ is fixed, then we define

$$\mathcal{L}ag_t^{t_0} : \begin{cases} \Omega_{t_0} & \rightarrow \overrightarrow{\mathbb{R}^m} \quad \text{(or, more generally, some adequat set)} \\ p_{t_0} & \rightarrow \mathcal{L}ag_t^{t_0}(p_{t_0}) := \mathcal{L}ag^{t_0}(t, p_{t_0}), \end{cases} \tag{3.12}$$

$$\mathcal{L}ag_{p_{t_0}}^{t_0} : \begin{cases} [t_1, t_2] & \rightarrow \overrightarrow{\mathbb{R}^m} \quad \text{(or, more generally, some adequat set)} \\ \quad t & \rightarrow \mathcal{L}ag_{p_{t_0}}^{t_0}(t) := \mathcal{L}ag^{t_0}(t, p_{t_0}). \end{cases} \tag{3.13}$$

Remark 3.8 The position p_{t_0} is also sometimes called a "material point", which is counter intuitive: P_{Obj} (objective) is the material point, and p_{t_0} is just its spatial position at t_0 (subjective); And a Eulerian variable p_t is not called a "material point" at t...
By the way, the variable p_t is also called the "updated Lagrangian variable"... ∴

3.2.2 A Lagrangian function is a two point tensor

Definition 3.9 In details: $\mathcal{L}ag^{t_0}$ being defined in (3.11), a Lagrangian function is a function

$$\widetilde{\mathcal{L}ag}^{t_0} : \begin{cases} [t_1, t_2] \times \Omega_{t_0} & \rightarrow \mathcal{C} \times \overrightarrow{\mathbb{R}^m} \\ \qquad (t, p_{t_0}) & \rightarrow \widetilde{\mathcal{L}ag}^{t_0}(t, p_{t_0}) = ((t, p_t), \mathcal{L}ag^{t_0}(t, p_{t_0})) \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \end{cases} \tag{3.14}$$

I.e. $\widetilde{\mathcal{L}ag}^{t_0}(t, p_{t_0}) = ((t, \Phi_t^{t_0}(p_{t_0})), \mathcal{L}ag^{t_0}(t, p_{t_0}))$. (And $\overrightarrow{\mathbb{R}^m}$ can be replaced by some set.)

Definition 3.10 (Marsden and Hughes [12].) A Lagrangian function is a "two point vector field" (or more generally a "two point tensor") in reference to the points $p_{t_0} \in \Omega_{t_0}$ (departure set) and $p_t \in \Omega_t$ (arrival set) where the value $\mathcal{L}ag^{t_0}(t, p_{t_0})$ is considered.

Interpretation: (3.14) tells that $\mathcal{L}ag^{t_0}(t, p_{t_0})$ is ***not*** represented at (t, p_{t_0}), but at (t, p_t): That is, having

$$\text{graph}(\mathcal{L}ag^{t_0}) = \{((t, p_{t_0}), \mathcal{L}ag^{t_0}(t, p_{t_0})) \quad \text{and} \quad \text{Im}(\widetilde{\mathcal{L}ag}^{t_0}) = \{((t, p_t), \mathcal{L}ag^{t_0}(t, p_{t_0}))\}, \tag{3.15}$$

we have

$$\text{Im}(\widetilde{\mathcal{L}ag}^{t_0}) \neq \text{graph}(\mathcal{L}ag^{t_0}) : \tag{3.16}$$

So a Lagrangian function does ***not*** define a tensor in the usual sense. To compare with the Eulerian function $\mathcal{E}ul$ which defines a tensor (in particular $\text{Im}(\widehat{\mathcal{E}ul}) = \text{graph}(\mathcal{E}ul)$), cf. (2.3).

3.3 Lagrangian function associated with a Eulerian function

3.3.1 Definition

Let $\widetilde{\Phi}$ be a motion, cf. (1.5). Let $\mathcal{E}ul$ be a Eulerian function, cf. (2.3). Let $t_0 \in [t_1, t_2]$.

Definition 3.11 The Lagrangian function $\mathcal{L}ag^{t_0}$ associated with the Eulerian function $\mathcal{E}ul$ is defined by, for all $(t, P_{Obj}) \in [t_1, t_2] \times Obj$,

$$\mathcal{L}ag^{t_0}(t, \widetilde{\Phi}(t_0, P_{Obj})) := \mathcal{E}ul(t, \widetilde{\Phi}(t, P_{Obj})), \tag{3.17}$$

i.e., for all $(t, p_{t_0}) \in [t_1, t_2] \times \Omega_{t_0}$,

$$\mathcal{L}ag^{t_0}(t, p_{t_0}) := \mathcal{E}ul(t, p_t), \quad \text{when} \quad p_t = \widetilde{\Phi}(t, P_{Obj}) = \Phi_t^{t_0}(p_t) \tag{3.18}$$

i.e., $\mathcal{L}ag^{t_0}(t, p_{t_0}) := \mathcal{E}ul(t, p_t)$ when $p_{t_0} = (\Phi_t^{t_0})^{-1}(p_t)$ for all $(t, p_t) \in \mathcal{C}$. In other words:

$$\boxed{\mathcal{L}ag_t^{t_0} := \mathcal{E}ul_t \circ \Phi_t^{t_0}}. \tag{3.19}$$

3.3.2 Remarks

- If you have a Lagrangian function, then you can associate the function

$$\mathcal{E}ul_t^{t_0} := \mathcal{L}ag_t^{t_0} \circ (\Phi_t^{t_0})^{-1} \tag{3.20}$$

which thus a priori depends on t_0. But, a Eulerian function is independent of any initial time t_0.

- For one measurement, there is only one Eulerian function $\mathcal{E}ul$, while there are as many associated Lagrangian function $\mathcal{L}ag^{t_0}$ as they are t_0 (as many as observers): The Lagrangian function $\mathcal{L}ag^{t_0{}'}$ of a late observer who chooses $t_0{}' > t_0$ is different from $\mathcal{L}ag^{t_0}$ since the domains of definition Ω_{t_0} and $\Omega_{t_0{}'}$ are different (in general).

3.4 Lagrangian velocity

3.4.1 Definition

Definition 3.12 In short: The Lagrangian velocity at t at $p_t = \widetilde{\Phi}(t, P_{Obj})$ of the particle P_{Obj} is the function

$$\vec{V}^{t_0} : \begin{cases} \mathbb{R} \times \Omega_{t_0} & \to \vec{\mathbb{R}}^n \\ (t, p_{t_0}) & \to \vec{V}^{t_0}(t, p_{t_0}) := \widetilde{\Phi}_{P_{Obj}}{}'(t) \quad \text{when} \quad p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj}). \end{cases} \tag{3.21}$$

In details: With (3.21), the Lagrangian velocity is the two point vector field given by

$$\widehat{\vec{V}^{t_0}}(t, p_{t_0}) : \begin{cases} \mathbb{R} \times \Omega_{t_0} & \to \mathcal{C} \times \vec{\mathbb{R}}^n \\ (t, p_{t_0}) & \to \widehat{\vec{V}^{t_0}}(t, p_{t_0}) := ((t, p_t), \vec{V}^{t_0}(t, p_{t_0})), \quad \text{when} \quad p_t = \Phi^{t_0}(t, p_{t_0}). \end{cases} \tag{3.22}$$

Thus $\vec{V}^{t_0}(t, p_{t_0}) = \widetilde{\Phi}_{P_{Obj}}{}'(t) = \vec{v}(t, p_t)$ is the velocity at t at $p_t = \widetilde{\Phi}(t, P_{Obj})$ of the particle P_{Obj} which was at $p_{t_0} = \widetilde{\Phi}(p_{t_0}, P_{Obj})$ at t_0; And $\vec{V}^{t_0}(t, p_{t_0})$ is ***not*** tangent to graph$(\vec{V}^{t_0})$, cf. (3.16): It is tangent to graph$(\vec{v})$ at (t, p_t).

If t is fixed, or if $p_{t_0} \in \Omega_{t_0}$ is fixed, then we define

$$\vec{V}_t^{t_0}(p_{t_0}) := \vec{V}^{t_0}(t, p_{t_0}), \quad \text{or} \quad \vec{V}_{p_{t_0}}^{t_0}(t) := \vec{V}^{t_0}(t, p_{t_0}). \tag{3.23}$$

Remark: A usual definition is given without explicit reference to a particle; It is, instead of (3.21),

$$\vec{V}^{t_0}(t, p_{t_0}) := \frac{\partial \Phi^{t_0}}{\partial t}(t, p_{t_0}), \quad \forall (t, p_{t_0}) \in \mathbb{R} \times \Omega_{t_0}. \tag{3.24}$$

3.4.2 Lagrangian velocity versus Eulerian velocity

(3.21) and (2.4) give (alternative definition), with $p_\tau = \widetilde{\Phi}(\tau, P_{Obj})$,

$$\vec{V}^{t_0}(t, p_{t_0}) = \vec{v}(t, p_t) \quad (= \frac{\partial \Phi^{t_0}}{\partial t}(t, p_{t_0}) = \widetilde{\Phi}_{P_{Obj}}{}'(t) = \text{velocity at } t \text{ at } p_t \text{ of } P_{Obj}). \tag{3.25}$$

In other words,

$$\boxed{\vec{V}_t^{t_0} = \vec{v}_t \circ \Phi_t^{t_0}}. \tag{3.26}$$

3.4.3 Relation between differentials

For C^2 motions (3.26) gives

$$d\vec{V}_t^{t_0}(p_{t_0}) = d\vec{v}_t(p_t).d\Phi_t^{t_0}(p_{t_0}) \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \tag{3.27}$$

I.e., with

$$F_t^{t_0} = d\Phi_t^{t_0} \overset{\text{noted}}{=} \text{the deformation gradient relative to } t_0 \text{ and } t, \tag{3.28}$$

$$\boxed{d\vec{V}_t^{t_0}(p_{t_0}) = d\vec{v}_t(p_t).F_t^{t_0}(p_{t_0})} \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \tag{3.29}$$

Abusively written (dangerous notation: At what points, relative to what times?)

$$d\vec{V} = d\vec{v}.F. \tag{3.30}$$

3.4.4 Computation of $d\vec{v}$ called $L = \overset{\bullet}{F}.F^{-1}$ wih Lagrangian variables

The Lagrangian approach can be introduced before the Eulerian approach: $\vec{V}^{t_0}$ being given, define

$$\vec{v}^{t_0}(t, p_t) := \vec{V}^{t_0}(t, p_{t_0}), \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}), \tag{3.31}$$

cf. (3.20). (I.e. $\vec{v}^{t_0}(t, p_t) := \vec{V}^{t_0}(t, {\Phi_t^{t_0}}^{-1}(p_t))$). So $\vec{v}^{t_0}(t, \Phi_t^{t_0}(p_{t_0})) = \frac{\partial \Phi^{t_0}}{\partial t}(t, p_{t_0})$, thus

$$d\vec{v}^{t_0}(t, p_t).d\Phi^{t_0}(t, p_{t_0}) = d(\frac{\partial \Phi^{t_0}}{\partial t})(t, p_{t_0}) = \frac{\partial (d\Phi^{t_0})}{\partial t}(t, p_{t_0}) = \frac{\partial F^{t_0}}{\partial t}(t, p_{t_0}), \tag{3.32}$$

when Φ^{t_0} is C^2 and $F^{t_0} := d\Phi^{t_0}$. Thus

$$d\vec{v}^{t_0}(t, p_t) = \frac{\partial F^{t_0}}{\partial t}(t, p_{t_0}).F^{t_0}(t, p_{t_0})^{-1}, \quad \text{written in short} \quad d\vec{v} = \overset{\bullet}{F}.F^{-1} \quad \text{(points? times?)}. \tag{3.33}$$

And $d\vec{v}_t^{t_0}$ can be written $L_t^{t_0}$ in classical mechanics books, so you can find

$$L_t^{t_0}(p_t) := \overset{\bullet}{F}{}_{p_{t_0}}^{t_0}(t).F_t^{t_0}(p_{t_0})^{-1}, \quad \text{written in short} \quad L = \overset{\bullet}{F}.F^{-1} \quad \text{(at what points, what times?)}. \tag{3.34}$$

Here it is not obvious that $L_t^{t_0}(p_t)$ does ***not*** depend on t_0, which is indeed the case, cf. (3.29):

$$L_t^{t_0}(p_t) = d\vec{v}_t(p_t). \tag{3.35}$$

Reminder: if possible, use Eulerian quantities as long as possible[1].

3.5 Lagrangian acceleration

Let $P_{Obj} \in Obj$, $t_0, t \in \mathbb{R}$, $p_{t_0} = \widetilde{\Phi}_{P_{Obj}}(t_0)$ and $p_t = \widetilde{\Phi}_{P_{Obj}}(t)$ (positions of P_{Obj} at t_0 and t).

Definition 3.13 In short, the Lagrangian acceleration at t at p_t of the particle P_{Obj} is

$$\vec{\Gamma}^{t_0}(t, p_{t_0}) := \widetilde{\Phi}_{P_{Obj}}''(t) \quad \text{when} \quad p_{t_0} = \widetilde{\Phi}_{P_{Obj}}(t_0). \tag{3.36}$$

In other words

$$\vec{\Gamma}^{t_0}(t, p_{t_0}) := \vec{\gamma}(t, p_t) \quad \text{when} \quad p_t = \Phi^{t_0}(t, p_{t_0}), \tag{3.37}$$

where $\vec{\gamma}(t, p_t) = \widetilde{\Phi}_{P_{Obj}}''(t)$ is the Eulerian acceleration at t at $p_t = \widetilde{\Phi}(t, P_{Obj})$, cf. (2.37).

In details, the Lagrangian acceleration is the “two point vector field” defined on $\mathbb{R} \times \Omega_{t_0}$ by

$$\widetilde{\vec{\Gamma}^{t_0}}(t, p_{t_0}) = ((t, p_t), \widetilde{\Phi}_{P_{Obj}}''(t)), \quad \text{when} \quad p_t = \Phi^{t_0}(t, p_{t_0}). \tag{3.38}$$

(To compare with (2.38).) In particular $\vec{\Gamma}^{t_0}(t, p_{t_0})$ is not drawn on the graph of $\vec{\Gamma}^{t_0}$ at (t, p_{t_0}), but on the graph of $\vec{\gamma}$ at (t, p_t).

---

[1] To get Eulerian results from Lagrangian computations can make the understanding of a Lie derivative quite difficult: To introduce the “so-called” Lie derivatives in classical mechanics you can find the following steps: 1- At t consider the Cauchy stress vector $\vec{t}$ (Eulerian), 2- then with a unit normal vector $\vec{n}$, define the associated Cauchy stress tensor $\underline{\underline{\sigma}}$ (satisfying $\vec{t} = \underline{\underline{\sigma}}.\vec{n}$), 3- then use the virtual power and the change of variables in integrals to be back into Ω_{t_0} to be able to work with Lagrangian variables, 4- then introduce the first Piola–Kirchhoff (two point) tensor PK, 5- then introduce the second Piola–Kirchhoff tensor SK (endomorphism in Ω_{t_0}), 6- then differentiate SK in Ω_{t_0} (in the Lagrangian variables although the initials variables are the Eulerian variables in Ω_t), 7- then back in Ω_t to get back to Eulerian functions (change of variables in integrals), 8- then you get some Jaumann or Truesdell or other so called Lie derivatives type terms, the appropriate choice among all these derivatives being quite obscure because the covariant objectivity has been forgotten en route... While, with simple Eulerian considerations, it requires a few lines to understand the (real) Lie derivative (Eulerian concept) and its simplicity, see § 9, and deduce second order covariant objective results.

If t is fixed, or if $p_{t_0} \in \Omega_{t_0}$ is fixed, then define

$$\vec\Gamma_t^{t_0}(p_{t_0}) := \vec\Gamma^{t_0}(t, p_{t_0}), \quad \text{and} \quad \vec\Gamma_{p_{t_0}}^{t_0}(t) := \vec\Gamma^{t_0}(t, p_{t_0}). \tag{3.39}$$

Thus

$$\vec\Gamma_t^{t_0} = \vec\gamma_t \circ \Phi_t^{t_0}, \quad \text{and} \quad d\vec\Gamma_t^{t_0}(p_{t_0}) = d\vec\gamma_t(p_t).F_t^{t_0}(p_{t_0}), \tag{3.40}$$

when $p_t = \Phi_t^{t_0}(p_{t_0})$ and $F_t^{t_0} := d\Phi_t^{t_0}$ (the deformation gradient).

Risky notation: $d\vec\Gamma = d\vec\gamma.F$ (points? times?).

3.6 Time Taylor expansion of Φ^{t_0}

Let $p_{t_0} \in \Omega_{t_0}$. Then, at second order,

$$\Phi_{p_{t_0}}^{t_0}(\tau) = \Phi_{p_{t_0}}^{t_0}(t) + (\tau - t)\Phi_{p_{t_0}}^{t_0}{}'(t) + \frac{(\tau-t)^2}{2}\Phi_{p_{t_0}}^{t_0}{}''(t) + o((\tau-t)^2), \tag{3.41}$$

that is, with $p(\tau) = \widetilde\Phi_{P_{\mathit{Obj}}}(\tau) = \Phi_\tau^{t_0}(p_{t_0})$,

$$p(\tau) = p(t) + (\tau-t)\vec V^{t_0}(t, p_{t_0}) + \frac{(\tau-t)^2}{2}\vec\Gamma^{t_0}(t, p_{t_0}) + o((\tau-t)^2). \tag{3.42}$$

NB: There are ***three*** times involved: t_0 (observer dependent), t and τ (for the Taylor expansion). To compare with (2.41)-(2.42): $p(\tau) = p(t) + (\tau-t)\vec v(t, p(t)) + \frac{(\tau-t)^2}{2}\vec\gamma(t, p(t)) + o((\tau-t)^2)$, independent of t_0.

3.7 A vector field that let itself be deformed by a motion

Consider a C^0 Eulerian vector field $\vec w : \left\{\begin{array}{cl} \mathcal{C} & \to \vec{\mathbb{R}}^n \\ (t, p_t) & \to \vec w(t, p_t) \end{array}\right\}$. Let $t_0 \in [t_1, t_2[$ and let $\vec w_{t_0} : \left\{\begin{array}{ll} \Omega_{t_0} & \to \vec{\mathbb{R}}^n \\ p_{t_0} & \to \vec w_{t_0}(p_{t_0}) := \vec w(t_0, p_{t_0}) \end{array}\right\}$ (vector field in Ω_{t_0}). Then define the (virtual) vector field

$$\vec w_{t_0 *} : \left\{\begin{array}{cl} \mathcal{C} & \to \vec{\mathbb{R}}^n \\ (t, p_t) & \to \vec w_{t_0 *}(t, p_t) := d\Phi^{t_0}(t, p_{t_0}).\vec w_{t_0}(p_{t_0}), \quad \text{when} \quad p(t) = \Phi^{t_0}(t, p_{t_0}). \end{array}\right. \tag{3.43}$$

(The push-forward = result of the deformation of $\vec w_{t_0}$ by the motion, see figure 4.1.)

Proposition 3.14 *For C^2 motions, we have (time variation rate along a virtual trajectory)*

$$\frac{D\vec w_{t_0 *}}{Dt} = d\vec v.\vec w_{t_0 *}, \tag{3.44}$$

i.e. $\mathcal{L}_{\vec v}\vec w_{t_0 *} = \vec 0$, where $\mathcal{L}_{\vec v}\vec u := \frac{D\vec u}{Dt} - d\vec v.\vec u$ $(= \frac{\partial \vec u}{\partial t} + d\vec u.\vec v - d\vec v.\vec u)$ is the Lie derivative of a (unsteady) vector field $\vec u : \mathcal{C} \to \vec{\mathbb{R}}^n$ along $\vec v$.

Interpretation: *We will see that $\mathcal{L}_{\vec v}\vec w(t_0, p_{t_0}) = \lim_{t\to t_0} \frac{\vec w(t,p(t)) - \vec w_{t_0 *}(t,p(t))}{h}$ measures the "resistance of $\vec w$ to a motion", see § 9.3.2; Thus the result $\mathcal{L}_{\vec v}\vec w_{t_0 *}(t_0, p_{t_0}) = \vec 0$ is "obvious" $(= \lim_{t\to t_0} \frac{\vec w_{t_0 *}(t,p(t)) - \vec w_{t_0 *}(t,p(t))}{h})$: If $\vec w = \vec w_{t_0 *}$ then the vector ("force") field $\vec w$ does not oppose any resistance to the flow.*

Proof. p_{t_0} being fixed, with $d\Phi^{t_0}(t, p_{t_0}) =^{\text{noted}} F(t)$ we have $\vec w_{t_0 *}(t, p(t)) =^{(3.43)} F(t).\vec w_{t_0}(p_{t_0})$, thus $\frac{D\vec w_{t_0 *}}{Dt}(t, p(t)) = F'(t).\vec w_{t_0}(p_{t_0}) = F'(t).F(t)^{-1}.\vec w_{t_0 *}(t, p(t)) =^{(3.33)} d\vec v(t, p(t)).\vec w_{t_0 *}(t, p(t))$, i.e. (3.44). ■

4 Deformation gradient $F := d\Phi$

Consider a motion $\widetilde\Phi : \left\{\begin{array}{ll} \mathbb{R} \times \mathit{Obj} & \to \mathbb{R}^n \\ (t, P_{\mathit{Obj}}) & \to p_t = \widetilde\Phi(t, P_{\mathit{Obj}}) \end{array}\right\}$, $\Omega_t := \widetilde\Phi(t, \mathit{Obj})$ the configuration of *Obj* at any t, fix t_0, t in $\mathbb{R}$, and let $\Phi_t^{t_0} : \left\{\begin{array}{cl} \Omega_{t_0} & \to \Omega_t \\ p_{t_0} = \widetilde\Phi(t_0, P_{\mathit{Obj}}) & \to p_t = \Phi_t^{t_0}(p_{t_0}) := \widetilde\Phi(t, P_{\mathit{Obj}}) \end{array}\right\}$, supposed to be a C^1 diffeomorphism. Notations for calculations (quantification), to comply with practices:

1- Classical (unambiguous) notations as in Arnold, Germain: E.g., $(\vec{a}_i)$ and $(\vec{b}_i)$ are bases resp. in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$, $\vec{w}_{t_0}(p_{t_0}) = \sum_i w_{t_0,i}(p_{t_0})\vec{a}_i \in \vec{\mathbb{R}}^n_{t_0}$, $\vec{w}_{t,i}(p_t) = \sum_i w_{t,i}(p_t)\vec{b}_i \in \vec{\mathbb{R}}^n_t$; And

2- Marsden–Hughes duality notations: Capital letters at t_0, lower case letters at t, duality notation, e.g. $(\vec{E}_I)$ and $(\vec{e}_i)$ are bases resp. in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$, $\vec{W}(P) = \sum_I W^I(P)\vec{E}_I \in \vec{\mathbb{R}}^n_{t_0}$, $\vec{w}(p) = \sum_i w^i(p)\vec{e}_i \in \vec{\mathbb{R}}^n_t$.

4.1 Definitions

4.1.1 Definition of the deformation gradient F

Definition 4.1 The differential $d\Phi^{t_0}_t =^{\text{noted}} F^{t_0}_t : \left\{\begin{array}{ll} \Omega_{t_0} & \to \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t) \\ p_{t_0} & \to F^{t_0}_t(p_{t_0}) := d\Phi^{t_0}_t(p_{t_0}) \end{array}\right\}$ is called "the covariant deformation gradient between t_0 and t", or simply "the deformation gradient". And "the covariant deformation gradient at p_{t_0} between t_0 and t", or in short "the deformation gradient at p_{t_0}" is the linear map $F^{t_0}_t(p_{t_0}) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$, so defined by, for all $\vec{w}_{t_0}(p_{t_0}) \in \vec{\mathbb{R}}^n_{t_0}$ (vector at p_{t_0}),

$$\boxed{F^{t_0}_t(p_{t_0}).\vec{w}_{t_0}(p_{t_0}) := \lim_{h\to 0} \frac{\Phi^{t_0}_t(p_{t_0}+h\vec{w}_{t_0}(p_{t_0})) - \Phi^{t_0}_t(p_{t_0})}{h}} \overset{\text{noted}}{=} (\Phi^{t_0}_t)_*(\vec{w}_{t_0})(p_t) \overset{\text{noted}}{=} \vec{w}_{t_0*}(t,p_t), \tag{4.1}$$

with $p_t = \Phi^{t_0}_t(p_{t_0})$. See figure 4.1.

Marsden–Hughes notations: $\Phi := \Phi^{t_0}_t$, $F := d\Phi$, $P := p_{t_0}$, $\vec{W}(P) := \vec{w}_{t_0}(p_{t_0})$, $p = \Phi(P)$, thus

$$\boxed{F(P).\vec{W}(P) := \lim_{h\to 0} \frac{\Phi(P+h\vec{W}(P)) - \Phi(P)}{h}} \overset{\text{noted}}{=} \Phi_*\vec{W}(p) \overset{\text{noted}}{=} \vec{w}_*(p). \tag{4.2}$$

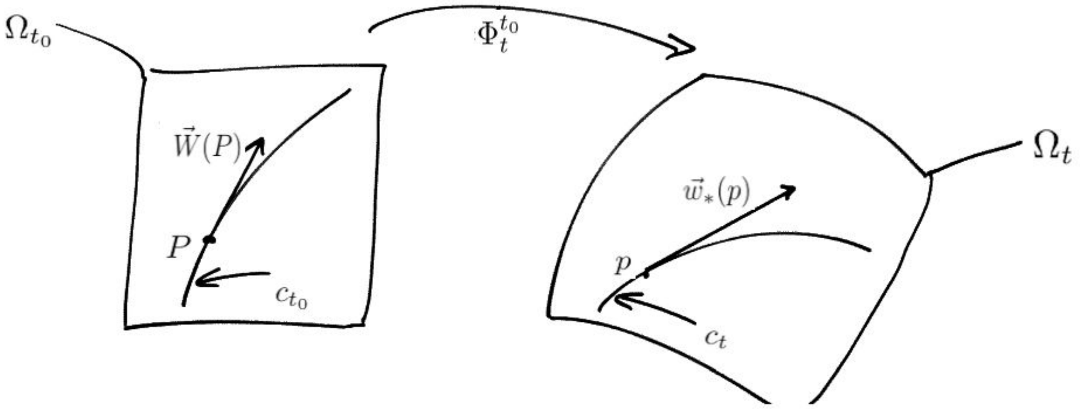

Figure 4.1: $\vec{w}$ is a Eulerian vector field. At t_0 define vector field $\vec{w}_{t_0}$ in Ω_{t_0} by $\vec{w}_{t_0}(p_{t_0}) := \vec{w}(t_0,p_{t_0})$. The (spatial) curve $c_{t_0} : s \to p_{t_0} = c_{t_0}(s)$ in Ω_{t_0} is an integral curve of $\vec{w}_{t_0}$, i.e. satisfies $c_{t_0}{}'(s) = \vec{w}_{t_0}(c_{t_0}(s))$. It is transformed by $\Phi^{t_0}_t$ into the (spatial) curve $c_t = \Phi^{t_0}_t \circ c_{t_0} : s \to p_t = c_t(s)=\Phi^{t_0}_t(c_{t_0}(s))$ in Ω_t; Hence $c_t{}'(s) = d\Phi^{t_0}_t(p_{t_0}).c_{t_0}{}'(s) = d\Phi^{t_0}_t(p_{t_0}).\vec{w}_{t_0}(p_{t_0}) =^{\text{noted}} \vec{w}_{t_0*}(t,p_t)$ is the tangent vector at c_t at p_t (the push-forward of $\vec{w}_{t_0}$ by $\Phi^{t_0}_t$). And $\vec{w}(t,p(t))$ (actual value) is also drawn.

NB: The "deformation gradient" $F^{t_0}_t = d\Phi^{t_0}_t$ is ***not*** a "gradient" (its definition does ***not*** need a Euclidean dot product); This lead to confusions when covariance-contravariance and objectivity are at stake. It would be simpler to stick to the name "$F^{t_0}_t$ = the differential of $\Phi^{t_0}_t$", but it is not the standard usage, except in thermodynamics: E.g., the differential dU of the internal energy U is ***not*** called "the gradient of U" (there is no meaningful inner dot product): It is just called "the differential of U"...

4.1.2 Push-forward (values of F)

Definition 4.2 Let $\vec{w}_{t_0} : \left\{\begin{array}{ll} \Omega_{t_0} & \to \vec{\mathbb{R}}^n_{t_0} \\ p_{t_0} & \to \vec{w}_{t_0}(p_{t_0}) \end{array}\right\}$ be a vector field in Ω_{t_0}. Its push-forward by $\Phi^{t_0}_t$ is the vector field $(\Phi^{t_0}_t)_*(\vec{w}_{t_0})$ in Ω_t defined by

$$(\Phi^{t_0}_t)_*\vec{w}_{t_0}(p_t) = F^{t_0}_t(p_{t_0}).\vec{w}_{t_0}(p_{t_0}) \overset{\text{noted}}{=} \vec{w}_{t_0*}(t,p_t) \quad \text{when} \quad p_t = \Phi^{t_0}_t(p_{t_0}). \tag{4.3}$$

See figure 4.1. Marsden notation: $\Phi_*\vec{W}(p) = F(P).\vec{W}(P) =^{\text{noted}} \vec{w}_*(p)$ when $p = \Phi^{t_0}_t(P)$.

In other words

$$(\Phi_t^{t_0})_*\vec{w}_{t_0} := (F_t^{t_0}.\vec{w}_{t_0}) \circ (\Phi_t^{t_0})^{-1}. \tag{4.4}$$

Marsden notation: $\Phi_*\vec{W} = (F.\vec{W}) \circ \Phi^{-1} = \vec{w}_*$.

4.1.3 F is a two point tensors

With (4.1), "the tangent map" is

$$\widehat{F_t^{t_0}} : \begin{cases} \Omega_{t_0} \to \Omega_t \times \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t) \\ p_{t_0} \to \widehat{F_t^{t_0}}(p_{t_0}) = (p_t, F_t^{t_0}(p_{t_0})) \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \end{cases} \tag{4.5}$$

Definition 4.3 (Marsden–Hughes [12].) The function $\widehat{F_t^{t_0}}$ is called a two point tensor, referring to the points $p_{t_0} \in \Omega_{t_0}$ (departure set) and $p_t = \Phi_t^{t_0}(p_{t_0}) \in \Omega_t$ (arrival set where $\vec{w}_{t_0*}(t,p_t) = F_t^{t_0}(p_{t_0}).\vec{w}_{t_0}(p_{t_0})$ is drawn). And in short $\widehat{F_t^{t_0}} =^{\text{noted}} F_t^{t_0}$ is said to be a two point tensor.

Remark 4.4 The name "two point tensor" is a shortcut than can create confusions and errors when dealing with the transposed: $F_t^{t_0}$ is not immediately a "tensor": A tensor is a multilinear form, so gives scalar results ($\in \mathbb{R}$), while $F(P) := F_t^{t_0}(P) =^{\text{noted}} F_P \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ gives vector results (in $\vec{\mathbb{R}}^n_t$). However F_P can be naturally and canonically associated with the bilinear form $\widetilde{F}_P \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t, \vec{\mathbb{R}}^n_{t_0};\mathbb{R})$ defined by, for all $\vec{u}_P \in \vec{\mathbb{R}}^n_{t_0}$ and $\ell_p \in \vec{\mathbb{R}}^{n*}_t$, with $p = \Phi_t^{t_0}(P)$,

$$\widetilde{F}_P(\ell_p, \vec{u}_P) := \ell_p.F_P.\vec{u}_P \ (\in \mathbb{R}), \tag{4.6}$$

see § A.13, and it is $\widetilde{F}_P$ which defines the so-called "two point tensor".

But don't forget that the transposed of a linear form (F_P here) is ***not*** deduced from the transposed of the associated bilinear form ($\widetilde{F}_P$ here). So be careful with the word "transposed" and its two distinct definitions. Indeed, the transposed of a bilinear form $b(\cdot,\cdot)$ is intrinsic to $b(\cdot,\cdot)$ (is objective), given by $b^T(\vec{u},\vec{w}) = b(\vec{w},\vec{u})$, while the transposed of a linear function L is not intrinsic to L (is subjective), given by $(L^T.\vec{u},\vec{w})_g = (L.\vec{w},\vec{u})_h$ where $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ are inner dot products chosen by human beings. (details in § A.7.2 and § A.11.1). ∴

Remark 4.5 More generally for manifolds, the differential of $\Phi := \Phi_t^{t_0}$ at $P \in \Omega_{t_0}$ is $F(P) := d\Phi(P) : \left\{\begin{array}{ll} T_P\Omega_{t_0} & \to T_p\Omega_t \\ \vec{W}(P) & \to \vec{w}_*(p) := d\Phi(P).\vec{W}(P) \end{array}\right\}$ with $p = \Phi_t^{t_0}(P)$. And the tangent map is

$$T\Phi : \begin{cases} \qquad T\Omega_{t_0} \to T\Omega_t \\ (P,\vec{W}(P)) \to T\Phi(P,\vec{W}(P)) := (p, d\Phi(P).\vec{W}(P)) = (p,\vec{w}_*(p)), \quad \text{where} \quad p = \Phi_t^{t_0}(P), \end{cases} \tag{4.7}$$

called the associated two point tensor. ∴

4.1.4 Evolution: Toward the Lie derivative (in continuum mechanics)

Consider a Eulerian vector field $\vec{w} : \left\{\begin{array}{ll} \mathcal{C} = \bigcup_t(\{t\}\times\Omega_t) & \to \vec{\mathbb{R}}^n \\ (t,p) & \to \vec{w}(t,p) \end{array}\right\}$, e.g. a "force field". Then, at t_0 consider $\vec{w}_{t_0} : \left\{\begin{array}{ll} \Omega_{t_0} & \to \vec{\mathbb{R}}^n_{t_0} \\ p_{t_0} & \to \vec{w}_{t_0}(p_{t_0}) := \vec{w}(t_0,p_{t_0}) \end{array}\right\}$. The push-forward of $\vec{w}_{t_0}$ by $\Phi_t^{t_0}$ is, cf. (4.2),

$$\vec{w}_{t_0*}(t,p(t)) = F_t^{t_0}(p_{t_0}).\vec{w}_{t_0}(p_{t_0}), \quad \text{where} \quad p(t) = \Phi^{t_0}(t,p_{t_0}). \tag{4.8}$$

See figure 4.1. Then, without any ubiquity gift, at t at $p(t)$ we can compare $\vec{w}(t,p(t))$ (real value of $\vec{w}$ at t at $p(t)$) with $\vec{w}_{t_0*}(t,p(t))$ (transported memory along the trajectory). Thus the rate

$$\frac{\vec{w}(t,p(t)) - \vec{w}_{t_0*}(t,p(t))}{t-t_0} = \frac{\text{actual}(t,p(t)) - \text{memory}(t,p(t))}{t-t_0} \quad \text{is meaningful at } (t,p(t)) \tag{4.9}$$

(no ubiquity gift required). This rate gives, as $h \to 0$, the Lie derivative $\mathcal{L}_{\vec{v}}\vec{w}$ (the rate of stress), and we will see at § 9.3 that $\mathcal{L}_{\vec{v}}\vec{w} = \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}$ (the $d\vec{v}$ term tells that a "non-uniform flow" acts on the stress).

4.1.5 Pull-back

Formally the pull-back is the push-forward with $(\Phi_t^{t_0})^{-1}$:

Definition 4.6 The pull-back $(\Phi_t^{t_0})^*\vec{w}_t$ of a vector field $\vec{w}_t$ defined on Ω_t is the vector field defined on Ω_{t_0} by, with $p_{t_0} = (\Phi_t^{t_0})^{-1}(p_t)$,

$$\vec{w}_t^*(t_0,p_{t_0}) = (\Phi_t^{t_0})^*\vec{w}_t(p_{t_0}) := (F_t^{t_0})^{-1}(p_t).\vec{w}_t(p_t), \quad \text{written} \quad \vec{W}^*(P) = F^{-1}(p).\vec{w}(p). \tag{4.10}$$

4.2 Quantification with bases

(Simple Cartesian framework.) $(\vec{a}_i)$ is a Cartesian basis in $\vec{\mathbb{R}}^n_{t_0}$, $(\vec{b}_i)$ is a Cartesian basis in $\vec{\mathbb{R}}^n_t$, o_t is an origin in $\mathbb{R}^n$ at t, $\Phi_t^{t_0} =^{\text{noted}} \Phi$ supposed C^1, $\varphi_i : \Omega_{t_0} \to \mathbb{R}$ is its components in the referential $(o_t,(\vec{b}_i))$:

$$\Phi(p_{t_0}) = o_t + \sum_{i=1}^n \varphi_i(p_{t_0})\vec{b}_i, \quad \text{i.e.} \quad \overrightarrow{o_t\Phi(p_{t_0})} = \sum_{i=1}^n \varphi_i(p_{t_0})\vec{b}_i. \tag{4.11}$$

Thus, with the classic notation $d\varphi_i(p_{t_0}).\vec{a}_j =^{\text{noted}} \frac{\partial\varphi_i}{\partial X_j}(p_{t_0})$ since $(\vec{a}_i)$ is a Cartesian basis, and $(\vec{b}_i)$ being a Cartesian basis,

$$d\Phi(p_{t_0}).\vec{a}_j = \sum_{i=1}^n (d\varphi_i(p_{t_0}).\vec{a}_j)\vec{b}_i = \sum_{i=1}^n \frac{\partial\varphi_i}{\partial X_j}(p_{t_0})\vec{b}_i, \quad \text{thus} \quad [d\Phi(p_{t_0})]_{[\vec{a},\vec{b}]} = [\frac{\partial\varphi_i}{\partial X_j}(p_{t_0})] = [F(p_{t_0})]_{[\vec{a},\vec{b}]},$$

$[d\Phi(p_{t_0})]_{[\vec{a},\vec{b}]} = [F(p_{t_0})]_{[\vec{a},\vec{b}]}$ being the Jacobian matrix of Φ at p_{t_0} relative to the chosen bases. In short:

$$d\Phi.\vec{a}_j = \sum_{i=1}^n \frac{\partial\varphi_i}{\partial X_j}\vec{b}_i, \quad \text{thus} \quad [d\Phi]_{[\vec{a},\vec{b}]} = [\frac{\partial\varphi_i}{\partial X_j}] = [F]_{[\vec{a},\vec{b}]} = [F_{ij}], \tag{4.12}$$

Thus, if $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$ is a vector at p_{t_0} and $\vec{W} = \sum_{j=1}^n W_j\vec{a}_j$ then, by linearity of differentials,

$$d\Phi.\vec{W} = F.\vec{W} = \sum_{i=1}^n F_{ij}W_j\vec{b}_i, \quad \text{i.e.} \quad [F.\vec{W}]_{|\vec{b}} = [F]_{|\vec{a},\vec{b}}.[\vec{W}]_{|\vec{a}} \tag{4.13}$$

(more precisely: $F_t^{t_0}(p_{t_0}).\vec{W}(p_{t_0}) = \sum_{i=1}^n F_{ij}(p_{t_0})W_j(p_{t_0})\vec{b}_i$).

Similarly, for the second order derivative $d^2\Phi = dF$ (when Φ is C^2): With $\vec{U} = \sum_{j=1}^n U_j\vec{a}_j$ and $\vec{W} = \sum_{k=1}^n W_k\vec{a}_k$, and with $(\vec{a}_i)$ and $(\vec{b}_i)$ Cartesian bases, we get

$$dF(\vec{U},\vec{W}) = d^2\Phi(\vec{U},\vec{W}) = \sum_{i=1}^n d^2\varphi_i(\vec{U},\vec{W})\vec{b}_i = \sum_{i,j,k=1}^n \frac{\partial^2\varphi_i}{\partial X_j\partial X_k}U_jW_k\vec{b}_i = \sum_{i=1}^n \left([\vec{U}]^T_{|\vec{a}}.[d^2\varphi_i]_{|\vec{a}}.[\vec{W}]_{|\vec{a}}\right)\vec{b}_i, \tag{4.14}$$

$[d^2\varphi_i(p_{t_0})]_{|\vec{a}} = [\frac{\partial^2\varphi_i}{\partial X_j\partial X_k}(p_{t_0})]_{\substack{j=1,\dots,n\\k=1,\dots,n}}$ being the Hessian matrix of φ_i at p_{t_0} relative to the basis $(\vec{a}_i)$.

With Marsden duality notations:

- $p = \Phi(P) = o_t + \sum_{i=1}^n \varphi^i(P)\vec{e}_i, \quad F^i{}_J(P) = \frac{\partial\varphi^i}{\partial X^J}(P) \quad (= d\varphi^i(P).\vec{E}_J),$
- $F(P).\vec{W} = \sum_{i,J=1}^n F^i{}_J(P)\,W^J\vec{e}_i, \quad [F] = [F^i{}_J] = [d\Phi],$. (4.15)
- $dF(\vec{U},\vec{W}) = d^2\Phi(\vec{U},\vec{W}) = \sum_{i,J,K=1}^n \frac{\partial^2\varphi^i}{\partial X^J\partial X^K}U^JW^K\vec{e}_i = \sum_{i=1}^n \left([\vec{U}]^T.[d^2\varphi^i].[\vec{W}]\right)\vec{e}_i.$

Remark 4.7 J,j are dummy variables when used in a summation: E.g., $df.\vec{W} = \sum_{j=1}^n \frac{\partial f}{\partial X^j}W^j = \sum_{J=1}^n \frac{\partial f}{\partial X^J}W^J = \sum_{\alpha=1}^n \frac{\partial f}{\partial X^\alpha}W^\alpha = \frac{\partial f}{\partial X^1}W^1 + \frac{\partial f}{\partial X^2}W^2 + \dots$ (there is no uppercase for 1, 2...). And Marsden–Hughes notations (capital letters for the past) are not at all compulsory, classical notations being just as good and even preferable if you hesitate (because they are not misleading). See § A. ⁂

4.3 The unfortunate notation $d\vec{x} = F.d\vec{X}$

4.3.1 Issue

(4.3), i.e. $\vec{w}_*(p) := F(P).\vec{W}(P)$, is sometimes written

$$d\vec{x} = F.d\vec{X} \ : \text{“a very unfortunate and misleading notation”} \tag{4.16}$$

which amounts to “confuse a length and a speed”... And you also the phrase “(4.16) is still true if $||d\vec{X}|| = 1$”... while $d\vec{X}$ is supposed to be small...

4.3.2 Where does this unfortunate notation come from?

The notation (4.16) comes from the first order Taylor expansion $\Phi(Q) = \Phi(P) + d\Phi_t^{t_0}(P).(Q{-}P) + o(||Q{-}P||)$, where $P, Q \in \Omega_{t_0}$, i.e., with $p = \Phi_t^{t_0}(P)$ and $q = \Phi_t^{t_0}(Q)$ and $h = ||Q{-}P||$,

$$q - p = F(P).(Q{-}P) + o(h), \quad \text{written} \quad \delta\vec{x} = F.\delta\vec{X} + o(\delta\vec{X}), \tag{4.17}$$

or $\vec{pq} = F(P).\overrightarrow{PQ} + o(h)$. So as $Q \to P$ we get $0 = 0$... Quite useless, isn't it?

While

$$\frac{q-p}{h} = F(P).\frac{Q-P}{h} + o(1) \quad \text{is useful:} \tag{4.18}$$

As $Q \to P$ we get $\vec{w}_* = F(P).\vec{W}$ which relates tangent vectors, see figure 4.1 Details:

4.3.3 Interpretation: Vector approach

Consider a spatial curve $c_{t_0} : \left\{ \begin{array}{rl} [s_1, s_2] & \to \Omega_{t_0} \\ s & \to P := c_{t_0}(s) \end{array} \right\}$ in Ω_{t_0}, cf. figure 4.1. It is deformed by $\Phi_t^{t_0}$ to become the spatial curve defined by $c_t := \Phi_t^{t_0} \circ c_{t_0} : \left\{ \begin{array}{rl} [s_1, s_2] & \to \Omega_t \\ s & \to p := c_t(s) = \Phi_t^{t_0}(c_{t_0}(s)). \end{array} \right.$ in Ω_t. Hence, relation between tangent vectors:

$$\frac{dc_t}{ds}(s) = d\Phi_t^{t_0}(c_{t_0}(s)).\frac{dc_{t_0}}{ds}(s),, \quad \text{i.e.} \quad \vec{w}_*(p) = F(P).\vec{W}(P) \quad \text{written} \quad \boxed{\frac{d\vec{x}}{ds} = F.\frac{d\vec{X}}{ds}}, \tag{4.19}$$

But you ***can't*** simplify by ds to get $d\vec{x} = F.d\vec{X}$: It is absurd to confuse “a slope $\frac{d\vec{X}}{ds}(s)$” and “a length $\delta\vec{X}$”.

NB: $||\frac{dc_t}{ds}(s)|| = ||\frac{d\vec{X}}{ds}(s)|| = 1$ is meaningful in (4.19): It means that the parametrization of the curve c_{t_0} in Ω_{t_0} uses a spatial parameter s such that $||{c_{t_0}}'(s)|| = 1$ for all s, i.e. s.t. $||\vec{W}_P|| = 1$ in figure 4.1. You ***cannot*** simplify by ds: $||d\vec{X}|| = 1$ is absurd together with $d\vec{X}$ “small”.

4.3.4 Interpretation: Differential approach

(4.16) is a relation between differentials... if you adopt the correct notations; Let us do it: With (4.11),

$$\vec{x} = \vec{o_t p} = \overrightarrow{o_t\Phi_t^{t_0}(P)} = \sum_{i=1}^{n} \varphi_i(P)\vec{b}_i \overset{\text{noted}}{=} \sum_{i=1}^{n} x_i(P)\vec{b}_i, \quad \text{where} \quad \varphi_i \overset{\text{noted}}{=} x_i \quad \text{(function of } P\text{)}. \tag{4.20}$$

Thus, with $(\pi_{ai}) = (dX_i)$ the (covariant) dual basis of $(\vec{a}_i)$ we get the system of n equations (functions):

$$d\Phi = F, \quad \text{i.e.} \quad \left\{ \begin{array}{l} d\varphi_1(P) = \sum_{j=1}^{n} \frac{\partial\varphi_1}{\partial X_j}(P)\, dX_j \\ \vdots \\ d\varphi_n(P) = \sum_{j=1}^{n} \frac{\partial\varphi_n}{\partial X_j}(P)\, dX_j \end{array} \right\}, \quad \text{which is noted} \quad d\vec{x} = F.d\vec{X}, \tag{4.21}$$

this last notation being often misunderstood[2]: It is nothing more than $d\Phi = F$ (coordinate free notation).

---

[2] Spivak [17] chapter 4: Classical differential geometers (and classical analysts) did not hesitate to talk about “infinitely small” changes dx^i of the coordinates x^i, just as Leibnitz had. No one wanted to admit that this was nonsense, because true results were obtained when these infinitely small quantities were divided into each other (provided one did it in the right way). Eventually it was realized that the closest one can come to describing an infinitely small change is to describe a direction in which this change is supposed to occur, i.e., a tangent vector. Since *df* is supposed to be the infinitesimal change of *f* under an infinitesimal change of the point, *df* must be a function of this change, which means that *df* should be a function on tangent vectors. The dX_i themselves then metamorphosed into functions, and it became clear that they must be distinguished from the tangent vectors $\partial/\partial X_i$. Once this realization came, it was only a matter of making new definitions, which preserved the old notation, and waiting for everybody to catch up.

4.3.5 The ambiguous notation $\overset{\bullet}{d\vec{x}} = \overset{\bullet}{F}.d\vec{X}$

The bad notation $d\vec{x} = F.d\vec{X}$ gives the unfortunate and misunderstood notations $\overset{\bullet}{d\vec{x}} = \overset{\bullet}{F}.d\vec{X}$, and then

$$\overset{\bullet}{d\vec{x}} = L.d\vec{x} \quad \text{where} \quad L = \overset{\bullet}{F}.F^{-1}. \tag{4.22}$$

Question: What is the meaning (and legitimate notation) of (4.22)?

Answer: $\overset{\bullet}{d\vec{x}} = L.d\vec{x}$ means

$$\boxed{\frac{D\vec{w}_{t_0*}}{Dt} = d\vec{v}.\vec{w}_{t_0*}} \quad = \text{evolution rate of tangent vectors along a trajectory} \tag{4.23}$$

see figure 4.1. Indeed, $\vec{w}_{t_0*}(t,p(t)) =^{(4.8)} F^{t_0}(t,p_{t_0}).\vec{w}_{t_0}(p_{t_0})$ gives

$$\frac{D\vec{w}_{t_0*}}{Dt}(t,p(t)) = \frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}).\vec{w}_{t_0}(p_{t_0}) = \frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}).(F^{t_0}_t(p_{t_0})^{-1}.\vec{w}_{t_0*}(t,p(t))), \tag{4.24}$$

i.e. $\frac{D\vec{w}_{t_0*}}{Dt}(t,p(t)) = d\vec{v}(t,p(t)).\vec{w}_{t_0*}(t,p(t))$, i.e. (4.23). In particular $\frac{D\vec{w}_{t_0*}}{Dt}(t_0,p_{t_0}) = d\vec{v}(t_0,p_{t_0}).\vec{w}_{t_0}(p_{t_0})$ is the evolution rate of tangent vectors at t_0 at p_{t_0}.

4.4 Tensorial notations, warnings, remarks

As already noted, cf. (4.6), the linear map $F := d\Phi^{t_0}_t(p_{t_0}) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ is naturally canonically associated with the bipoint tensor $\widetilde{F} \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t,\vec{\mathbb{R}}^n_{t_0};\mathbb{R})$ defined by, for all $(\ell,\vec{W}) \in \vec{\mathbb{R}}^{n*}_t \times \vec{\mathbb{R}}^n_{t_0}$,

$$\widetilde{F}(\ell,\vec{W}) := \ell.F.\vec{W}, \tag{4.25}$$

Quantification of $\widetilde{F}$: basis $(\vec{a}_i)$ with dual basis (π_{ai}) in $\vec{\mathbb{R}}^n_{t_0}$, basis $(\vec{b}_i)$ in $\vec{\mathbb{R}}^n_t$:

$$\text{if} \quad F.\vec{a}_j = \sum_{i=1}^n \frac{\partial\varphi^i}{\partial X_j}\vec{b}_j \quad \text{then} \quad \widetilde{F} = \sum_{i,j=1}^n \frac{\partial\varphi^i}{\partial X_j}\,\vec{b}_i \otimes \pi_{aj} = \sum_{i=1}^n \vec{b}_i \otimes d\varphi_i. \tag{4.26}$$

And similarly

$$d\widetilde{F} = \sum_{i,j,k=1}^n \frac{\partial^2\varphi^i}{\partial X_j \partial X_k}\,\vec{b}_i \otimes (\pi_{aj} \otimes \pi_{ak}) = \sum_{i=1}^n \vec{b}_i \otimes d^2\varphi_i. \tag{4.27}$$

Warning: The tensorial notation can be misleading, in particular if you use the transposed, see remark 4.4. So, you should always use the standard notation for the linear form $F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ to begin with, i.e. use $F.\vec{a}_j = \sum_{j=1}^n F_{ij}\vec{b}_i$ or $F.\vec{E}_J = \sum_{i,j=1}^n F^i{}_J\vec{e}_i$ (Marsden notations). And only use the tensorial notations for calculations purposes at the end (after application of the proper definitions).

Remark 4.8 In some manuscripts you find the notation $F = d\Phi =^{\text{noted}} \Phi \otimes \nabla_X$. It does not help to understand what F is (it is the differential $d\Phi$), and should not be used as far as objectivity is concerned:

• A differentiation is ***not*** a tensorial operation, see example R.1, so why use the tensor product notation $\Phi \otimes \nabla_X$, when the standard notation $d\Phi \simeq \widetilde{F} = \sum_{i=1}^n \vec{e}_i \otimes d\varphi^i$ is legitimate, explicit, objective and easy to manipulate?

• And it could be misinterpreted, since, in mechanics, ∇f is often understood to be the vector $\sum_i \frac{\partial f}{\partial x_i}\vec{e}_i$ (contravariant) which needs a Euclidean dot product to be defined (which one?), while the differential df is covariant (a differential is unmissable in thermodynamics because you can't use gradients).

• It gives the confusing notation $\Phi \otimes \nabla_X \otimes \nabla_X$, instead of the legitimate $d^2\Phi = \sum_{i=1}^n \vec{b}_i \otimes d^2\varphi_i$ which is explicit, objective and easy to manipulate: $d^2\Phi(\vec{U},\vec{W}) = \sum_{i=1}^n d^2\varphi_i(\vec{U},\vec{W})\,\vec{b}_i$. ∴

Exercice 4.9 Use Marsden duality notations for (4.26)-(4.27).

Answer. Cartesian bases, with (dX^i) the (covariant) dual basis of $(\vec{E}_i)$: with $F^i{}_J = \frac{\partial\varphi^i}{\partial X^J}$, we get $d\Phi =^{\text{noted}} \widetilde{F} = \sum_{i=1}^n \vec{e}_i \otimes d\varphi^i = \sum_{i,J=1}^n F^i{}_J\,\vec{e}_i \otimes dX^J$, and $d^2\Phi = \sum_{i=1}^n \vec{e}_i \otimes d^2\varphi^i = \sum_{i,J,K=1}^n \frac{\partial^2\varphi^i}{\partial X^J \partial X^K}\,\vec{e}_i \otimes (dX^J \otimes dX^K)$. ∴

4.5 Change of coordinate system at t for F

Let $p_{t_0}\in\Omega_{t_0}$, $p_t=\Phi_t^{t_0}(p_{t_0})\in\Omega_t$, $\vec W(p_{t_0})\in\vec{\mathbb{R}}^n_{t_0}$, $\vec w(p_t)=F_t^{t_0}(p_{t_0}).\vec W(p_{t_0})\in\vec{\mathbb{R}}^n_t$ (its push-forward), written $\vec w=F.\vec W$ for short. The observer at t_0 used a basis $(\vec a_i)$ in $\vec{\mathbb{R}}^n_{t_0}$. At t, in $\vec{\mathbb{R}}^n_t$, a first observer chooses a Cartesian basis $(\vec b_{old,i})$, and a second observer chooses a Cartesian basis $(\vec b_{new,i})$. Let $P=[P_{ij}]$ be the transition matrix from $(\vec b_{old,i})$ to $(\vec b_{new,i})$, i.e. $\vec b_{new,j}=\sum_{i=1}^n P_{ij}\vec b_{old,i}$ for all j. The change of basis formula for vectors from $(\vec b_{old,i})$ to $(\vec b_{new,i})$ (in $\vec{\mathbb{R}}^n_t$) gives

$$[\vec w]_{|\vec b_{new}}=P^{-1}.[\vec w]_{|\vec b_{old}},\quad\text{thus}\quad [F.\vec W]_{|\vec b_{new}}=P^{-1}.[F.\vec W]_{|\vec b_{old}}. \tag{4.28}$$

Thus

$$[F]_{|\vec a,\vec b_{new}}.[\vec W]_{|\vec a}=P^{-1}.[F]_{|\vec a,\vec b_{old}}.[\vec W]_{|\vec a}, \tag{4.29}$$

true for all $\vec W$, thus

$$\boxed{[F]_{|\vec a,\vec b_{new}}=P^{-1}.[F]_{|\vec a,\vec b_{old}}}. \tag{4.30}$$

NB: (4.30) is ***not*** the change of basis formula $[L]_{|new}=P^{-1}.[L]_{|old}.P$ for endomorphisms, which would be nonsense since $F:=F_t^{t_0}(p_{t_0}):\vec{\mathbb{R}}^n_{t_0}\to\vec{\mathbb{R}}^n_t$ is not an endomorphism; (4.30) is just the usual change of basis formula for vectors $\vec w$ in $\vec{\mathbb{R}}^n_t$, cf. (4.28).

4.6 Spatial Taylor expansion of Φ and F

$\Phi_t^{t_0}=^{\text{noted}}\Phi$ is supposed to be C^2 for all t_0,t. Let $P\in\Omega_{t_0}$, $d\Phi=F$, and $\vec W\in\vec{\mathbb{R}}^n_{t_0}$ vector at P. Then, in Ω_t,

$$\Phi(P+h\vec W)=\Phi(P)+h\,F(P).\vec W+\frac{h^2}{2}\,dF(P)(\vec W,\vec W)+o(h). \tag{4.31}$$

4.7 Time Taylor expansion of F

The motion $\widetilde\Phi$ is supposed to be C^3. $t_0\in\mathbb{R}$, Φ^{t_0} be the associated motion, $p(t)=\widetilde\Phi(t,P_{Obj})$ and $p_{t_0}=\widetilde\Phi(t_0,P_{Obj})$, with $\vec v(t,p_t)=\frac{\partial\widetilde\Phi}{\partial t}(t,P_{Obj})$ the Eulerian velocity and $\vec V^{t_0}(t,p_{t_0}):=\frac{\partial\Phi^{t_0}}{\partial t}(t,p_{t_0})=\vec v(t,p_t)$ the Lagrangian velocity, and $F^{t_0}_{p_{t_0}}(t)=F^{t_0}(t,p_{t_0})=d\Phi^{t_0}(t,p_{t_0})=^{\text{noted}}F(t)$. We have

$$\begin{aligned}\frac{\partial F^{t_0}}{\partial t}(t,p_{t_0})&=\frac{\partial(d\Phi^{t_0})}{\partial t}(t,p_{t_0})=d(\frac{\partial\Phi^{t_0}}{\partial t})(t,p_{t_0})\\&=d\vec V^{t_0}(t,p_{t_0})=d\vec v(t,p(t)).F(t),\quad\text{in short}\quad\boxed{\dot F=d\vec V=d\vec v.F}.\end{aligned} \tag{4.32}$$

Then $\frac{\partial^2\Phi^{t_0}}{\partial t^2}(t,p_{t_0})=\vec A^{t_0}(t,p_{t_0})=\vec\gamma(t,p(t))$ (Lagrangian and Eulerian accelerations), hence

$$\frac{\partial^2F^{t_0}}{\partial t^2}(t,p_{t_0})=d\vec A^{t_0}(t,p_{t_0})=d\vec\gamma(t,p_t).F(t),\quad\text{in short}\quad\ddot F=d\vec A=d\vec\gamma.F. \tag{4.33}$$

Thus, the second order time Taylor expansion of $F^{t_0}_{p_{t_0}}=^{\text{noted}}F$ is, in the vicinity of t,

$$\begin{aligned}F(t+h)&=F(t)+h\,d\vec V(t)+\frac{h^2}{2}\,d\vec A(t)+o(h^2)\\&=\Big(I+h\,d\vec v+\frac{h^2}{2}\,d\vec\gamma\Big)(t,p(t)).F(t)+o(h^2)\quad\text{when}\quad p(t)=\Phi^{t_0}(t,p_{t_0}).\end{aligned} \tag{4.34}$$

NB: They are ***three*** times are involved: t and $t+h$ as usual, and t_0 through $F:=F^{t_0}_{p_{t_0}}$, $\vec V:=\vec V^{t_0}_{p_{t_0}}$ and $\vec A:=\vec A^{t_0}_{p_{t_0}}$ (observer dependent), as for (3.41).

In particular $F(p_{t_0}):=F^{t_0}_{t_0}(p_{t_0})=I$ gives, in the vicinity of t_0,

$$F(t_0+h)=\big(I+h\,d\vec v+\frac{h^2}{2}\,d\vec\gamma\big)(t_0,p_{t_0})+o(h^2). \tag{4.35}$$

Remark 4.10 $\gamma=\frac{\partial\vec v}{\partial t}+d\vec v.\vec v$ is not linear in $\vec v$. Idem,

$$d\vec\gamma=d(\frac{D\vec v}{Dt})=d(\frac{\partial\vec v}{\partial t}+d\vec v.\vec v)=d\frac{\partial\vec v}{\partial t}+d^2\vec v.\vec v+d\vec v.d\vec v\quad(=\frac{D(d\vec v)}{Dt}+d\vec v.d\vec v) \tag{4.36}$$

is non linear in $\vec v$, and gives ${F^{t_0}_{p_{t_0}}}''(t)=(d\frac{\partial\vec v}{\partial t}+d^2\vec v.\vec v+d\vec v.d\vec v)(t,p_t).F^{t_0}_{p_{t_0}}(t)$, non linear in $\vec v$. ▪

Exercice 4.11 Directly check that $F' = d\vec{v}.F$ gives $F'' = d\vec{\gamma}.F$.

Answer. $F'(t) = d\vec{v}(t,p(t)).F(t)$ gives $F''(t) = \frac{D(d\vec{v})}{Dt}(t,p(t)).F(t) + d\vec{v}(t,p(t)).F'(t)$ with $\frac{D(d\vec{v})}{Dt} = d\vec{\gamma} - d\vec{v}.\vec{v}$, cf. (4.36), thus $F''(t) = (d\vec{\gamma} - d\vec{v}.d\vec{v})(t,p(t)).F(t) + d\vec{v}(t,p(t)).d\vec{v}(t,p(t)).F(t) = d\vec{\gamma}(t,p(t)).F(t)$. ▪

4.8 Homogeneous and isotropic material

Let $P \in \Omega_{t_0}$, let $F_t^{t_0}(P) := d\Phi_t^{t_0}(P)$; Suppose that the "Cauchy stress vector" $\vec{f}_t(p_t)$ à t at $p_t = \Phi_t^{t_0}(P)$ only depends on P, i.e. there exists a function $\vec{\text{fun}}$ such that

$$\vec{f}_t(p_t) = \vec{\text{fun}}(P, F_t^{t_0}(P)). \tag{4.37}$$

Definition 4.12 A material is homogeneous iff $\vec{\text{fun}}$ doesn't depend on the first variable P, i.e., iff, for all $P \in \Omega_{t_0}$,

$$\vec{\text{fun}}(P, F_t^{t_0}(P)) = \vec{\text{fun}}(F_t^{t_0}(P)). \tag{4.38}$$

(Same mechanical property at any point.)

Definition 4.13 (Isotropy.) Consider a Euclidean dot product, the same at all time. A material is isotropic at $P \in \Omega_{t_0}$ iff $\vec{\text{fun}}$ is independent of the direction you consider, i.e., iff, for any rotation $R_{t_0}(P)$ in $\vec{\mathbb{R}}^n_{t_0}$,

$$\vec{\text{fun}}(P, F_t^{t_0}(P) = \vec{\text{fun}}(P, F_t^{t_0}(P).R_{t_0}(P)). \tag{4.39}$$

(Mechanical property unchanged when rotating the material first.)

Definition 4.14 A material is isotropic homogeneous iff it is isotropic and homogeneous.

4.9 The inverse of the deformation gradient

$((\Phi_t^{t_0})^{-1} \circ \Phi_t^{t_0})(P) = P$ gives, with $p = \Phi_t^{t_0}(P)$,

$$d(\Phi_t^{t_0})^{-1}(p).d\Phi_t^{t_0}(P) = I_{t_0}, \quad \text{thus} \quad d(\Phi_t^{t_0})^{-1}(p) = d\Phi_t^{t_0}(P)^{-1} = F_t^{t_0}(P)^{-1}, \tag{4.40}$$

where $F_t^{t_0} = d\Phi_t^{t_0}$ is the deformation gradient. We have thus define the two point tensor

$$H_t^{t_0} := (F_t^{t_0})^{-1} : \begin{cases} \Omega_t & \to \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0}) \\ p & \to \boxed{H_t^{t_0}(p) = (F_t^{t_0})^{-1}(p) := (F_t^{t_0}(P))^{-1}} \quad \text{when} \quad p = \Phi_t^{t_0}(P). \end{cases} \tag{4.41}$$

So

$$H_t^{t_0}(p).\vec{w}(p) = (F_t^{t_0})^{-1}(p).\vec{w}(p) := F_t^{t_0}(P)^{-1}.\vec{w}(p) \in \vec{\mathbb{R}}^n_{t_0}, \quad \text{in short} \quad H.\vec{w} = F^{-1}.\vec{w}, \tag{4.42}$$

for all $\vec{w}(p) \in \vec{\mathbb{R}}^n_t$ vector at p. This defines, with $p_t = \Phi^{t_0}(t,P)$,

$$H^{t_0} : \begin{cases} \mathcal{C} = \bigcup_t (\{t\} \times \Omega_t) & \to \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0}) \\ (t, p_t) & \to H^{t_0}(t,p_t) := H_t^{t_0}(p_t) = (F^{t_0}(t,P))^{-1}. \end{cases} \tag{4.43}$$

NB: H^{t_0} looks like a Eulerian map, but isn't: H^{t_0} depends on a initial time t_0 and is a two point tensor (starts in $\vec{\mathbb{R}}^n_t$, arrives in $\vec{\mathbb{R}}^n_{t_0}$). We will however use the material time derivative $\frac{D}{Dt}$ notation in this case, that is, we define, along a trajectory $t \to p(t) = \Phi^{t_0}(t,P)$,

$$\frac{DH^{t_0}}{Dt}(t,p(t)) := \frac{\partial H^{t_0}}{\partial t}(t,p(t)) + dH^{t_0}(t,p(t)).\vec{v}(t,p(t)), \quad \text{i.e.} \quad \frac{DH^{t_0}}{Dt} = \frac{\partial H^{t_0}}{\partial t} + dH^{t_0}.\vec{v}, \tag{4.44}$$

which is the time derivative $g'(t)$ of the function $g : t \to g(t) = H^{t_0}(t, \Phi^{t_0}(t,P))$ (i.e. $g(t) = H^{t_0}(t,p(t))$).

Hence, with $p(t) = \Phi^{t_0}(t,P)$ and $H^{t_0}(t,p(t)).F^{t_0}(t,P) = I_{t_0}$, written $H.F = I$, we get

$$\frac{DH}{Dt}.F + H.\frac{\partial F}{\partial t} = 0, \quad \text{thus} \quad \frac{DH}{Dt} = -H.d\vec{v}, \tag{4.45}$$

since $\frac{\partial F}{\partial t}(t,P).F^{-1}(t,p(t)) = d\vec{v}(t,p(t))$ cf. (4.32).

Exercice 4.15 With $\vec{w}_{t_0*}(t,p(t)) = F^{t_0}(t,P).\vec{W}(P)$, i.e. $H^{t_0}(t,p(t)).\vec{w}_{t_0*}(t,p(t)) = \vec{W}(P)$, when $p(t) = \Phi^{t_0}(t,P)$, prove (4.45).

Answer. $\frac{D\vec{w}_{t_0*}}{Dt}(t,p(t)) = d\vec{v}(t,p(t)).\vec{w}_{t_0*}(t,p(t))$, cf. (4.23); And $(H^{t_0}.\vec{w}_{t_0*})(t,p(t)) = \vec{W}(P)$ gives $\frac{DH^{t_0}}{Dt}.\vec{w}_{t_0*} + H^{t_0}.\frac{D\vec{w}_{t_0*}}{Dt} = 0$; Thus $\frac{DH^{t_0}}{Dt}.\vec{w}_{t_0*} + H^{t_0}.d\vec{v}.\vec{w}_{t_0*} = 0$, thus $\frac{DH}{Dt} = -H.d\vec{v}$. ∴

Exercice 4.16 Prove: $H_t^{t_0} = H_{t_1}^{t_0} \circ H_t^{t_1}$ and $\frac{DH^{t_0}}{Dt}(t,p(t)) = H_{t_1}^{t_0}(p_{t_1}).\frac{DH^{t_1}}{Dt}(t,p(t))$ for all t_0, t_1 with $p_{t_1} = \Phi_{t_1}^{t_0}(p_{t_0})$.

Answer. We have $\Phi_t^{t_0}(p_{t_0}) = \Phi_t^{t_1}(\Phi_{t_1}^{t_0}(p_{t_0}))$, cf. (5.18), hence $F_t^{t_0}(p_{t_0}) = F_t^{t_1}(p_{t_1}).F_{t_1}^{t_0}(p_{t_0})$, thus $F_t^{t_0}(p_{t_0})^{-1} = F_{t_1}^{t_0}(p_{t_0})^{-1}.F_t^{t_1}(p_{t_1})^{-1}$, i.e. $H_t^{t_0}(p_t) = H_{t_1}^{t_0}(p_{t_1}).H_t^{t_1}(p(t))$, thus, $H^{t_0}(t,p(t)) = H_{t_1}^{t_0}(p_{t_1}).H^{t_1}(t,p(t))$, thus $\frac{DH^{t_0}}{Dt}(t,p(t)) = H_{t_1}^{t_0}(p_{t_1}).\frac{DH^{t_1}}{Dt}(t,p(t))$. ∴

5 Flow

5.1 Introduction: Motion versus flow

- Motion: A motion $\widetilde{\Phi} : (t, P_{Obj}) \to p_t = \widetilde{\Phi}(t, P_{Obj})$ locates at t a particle P_{Obj} in the affine space $\mathbb{R}^n$, cf. (1.5); From which the Eulerian velocity field $\vec{v}$ is deduced: $\vec{v}(t,p_t) := \frac{d\widetilde{\Phi}_{P_{Obj}}}{dt}(t, P_{Obj})$, cf. (2.4).
- Flow: A flow starts with a Eulerian velocity field $\vec{v}$, from which we deduce a motion by solving the ODE (ordinary differential equation) $\frac{d\Phi}{dt}(t) = \vec{v}(t,\Phi(t))$.

5.2 Definition

Let $\vec{v} : \begin{cases} \mathbb{R}\times\mathbb{R}^n & \to \vec{\mathbb{R}}^n \\ (t,p) & \to \vec{v}(t,p) \end{cases}$ be a unstationary vector field (e.g., a Eulerian velocity field which definition domain is $\mathcal{C} = \bigcup_{t\in[t_1,t_2]}(\{t\}\times\Omega_t))$. We look for maps $\Phi : \begin{cases} \mathbb{R} & \to \mathbb{R}^n \\ t & \to p = \Phi(t) \end{cases}$ which are locally (i.e. in the vicinity of some t_0) solutions of the ODE (ordinary differential equation)

$$\frac{d\Phi}{dt}(t) = \vec{v}(t,\Phi(t)), \quad \text{also written} \quad \frac{dp}{dt}(t) = \vec{v}(t,p(t)), \quad \text{or} \quad \frac{d\vec{x}}{dt}(t) = \vec{v}(t,\vec{x}(t)) \tag{5.1}$$

where $\vec{x}(t) = \overrightarrow{\mathcal{O}p(t)}$ after a choice of an origin. Also written $\frac{dp}{dt} = \vec{v}(t,p)$ or $\frac{d\vec{x}}{dt} = \vec{v}(t,\vec{x})$.

Definition 5.1 A solution Φ of (5.1) is a flow of $\vec{v}$; Also called an integral curve of $\vec{v}$ since (5.1) also reads $\Phi(t) = \int_{\tau=t_1}^{t} \vec{v}(\tau,\Phi(\tau))\,d\tau + \Phi(t_1)$.

Remark 5.2 Improper notation for (5.1):

$$\frac{dp}{dt}(t) \overset{\text{noted}}{=} \frac{dp(t)}{dt} \qquad (= \vec{v}(t,p(t))). \tag{5.2}$$

Question: If the notation $\frac{dp(t)}{dt}$ is used, then what is the meaning of $\frac{dp(f(t))}{dt}$?

Answer: It means, either $\frac{dp}{dt}(f(t))$, or $\frac{d(p\circ f)}{dt}(t) = \frac{dp}{dt}(f(t))\frac{df}{dt}(t)$: Ambiguous. So it is better to use $\frac{dp}{dt}(t)$, and to avoid $\frac{dp(t)}{dt}$, unless the context is clear (no composite functions). ∴

5.3 Cauchy–Lipschitz theorem

Let (t_0, p_{t_0}) be in the definition domain of $\vec{v}$. We look for Φ solution of "the ODE with initial condition (t_0,p_{t_0})", in some vicinity of t_0, i.e. such that

$$\frac{d\Phi}{dt}(t) = \vec{v}(t,\Phi(t)) \quad \text{and} \quad \Phi(t_0) = p_{t_0}. \tag{5.3}$$

(The couple (t_0,p_{t_0}) is the initial condition, and the values t_0 and p_{t_0} are the initial conditions.)

Definition 5.3 Let $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$. Let Ω be an open set in $\mathbb{R}^n$ and $\overline{\Omega}$ its closure supposed to be a regular domain. Let $||.||$ be a norm in $\vec{\mathbb{R}}^n$. A continuous map $\vec{v} : [t_1,t_2]\times\overline{\Omega} \to \vec{\mathbb{R}}^n$ is Lipschitzian iff it is "space Lipschitzian, uniformly in time", that is, iff

$$\exists k>0,\ \forall t\in[t_1,t_2],\ \forall p,q\in\overline{\Omega},\ ||\vec{v}(t,q)-\vec{v}(t,p)|| \le k||q-p||. \tag{5.4}$$

So, $\frac{||\vec{v}_t(q)-\vec{v}_t(p)||}{||q-p||} \le k$, for all t and all $p \ne q$ (the variations of $\vec{v}$ are bounded in space, uniformly in time).

Theorem 5.4 (and definifion) *(Cauchy-Lipschitz). If $\vec{v} : [t_1,t_2]\times\overline{\Omega}\to\vec{\mathbb{R}}^n$ is Lipschitzian and $(t_0,p_{t_0})\in]t_1,t_2[\times\Omega$, then there exists $\varepsilon=\varepsilon_{t_0,p_{t_0}}>0$ s.t. (5.3) has a unique solution $\Phi :]t_0-\varepsilon,t_0+\varepsilon[\to\mathbb{R}^n$, noted $\Phi^{t_0}_{p_{t_0}}$:*

$$\frac{d\Phi^{t_0}_{p_{t_0}}}{dt}(t)=\vec{v}(t,\Phi^{t_0}_{p_{t_0}}(t))\quad\text{and}\quad\Phi^{t_0}_{p_{t_0}}(t_0)=p_{t_0}. \tag{5.5}$$

Moreover, if $\vec{v}$ is C^k then Φ is C^{k+1}.

Proof. See e.g. Arnold [2], or any ODE course. In particular $||\vec{v}||_\infty := \sup\limits_{t\in]t_0-\varepsilon,t_0+\varepsilon[,\ p\in\Omega}||\vec{v}(t,p)||_{\mathbb{R}^n}$ (maximum speed) exists since $\vec{v}\in C^0$ on the compact $[t_1,t_2]\times\overline{\Omega})$, see definition 5.3, hence we can choose $\varepsilon=\min(t_0-t_1,t_2-t_0,\frac{d(p_{t_0},\partial\Omega)}{||\vec{v}||_\infty})$ (the time needed to reach the border $\partial\Omega$ from p_{t_0}). ∎

We have thus defined the function, also called "a flow",

$$\Phi:\left\{\begin{array}{ll}]t_1,t_2[\times]t_1,t_2[\times\Omega_{t_0} & \to\Omega\\ \qquad(t,t_0,p_{t_0}) & \to p=\Phi(t,t_0,p_{t_0}):=\Phi^{t_0}_{p_{t_0}}(t)\overset{\text{noted}}{=}\Phi(t;t_0,p_{t_0}).\end{array}\right. \tag{5.6}$$

And (5.5) reads

$$\frac{\partial\Phi}{\partial t}(t;t_0,p_{t_0})=\vec{v}(t,\Phi(t;t_0,p_{t_0})),\quad\text{with}\quad\Phi(t_0;t_0,p_{t_0})=p_{t_0}. \tag{5.7}$$

We have thus defined the function, also called "a flow",

$$\Phi^{t_0}:\left\{\begin{array}{ll} [t_0-\varepsilon,t_0+\varepsilon]\times\Omega_{t_0} & \to\mathbb{R}^n\\ \qquad(t,p_{t_0}) & \to p=\Phi^{t_0}(t,p_{t_0}):=\Phi^{t_0}_{p_{t_0}}(t):\end{array}\right. \tag{5.8}$$

And (5.5) reads

$$\frac{\partial\Phi^{t_0}}{\partial t}(t,p_{t_0})=\vec{v}(t,\Phi^{t_0}(t,p_{t_0})),\quad\text{and}\quad\Phi^{t_0}(t_0,p_{t_0})=p_{t_0}. \tag{5.9}$$

Other definition and notation (can be ambiguous): $\Phi_{t;t_0}=\Phi^{t_0}_t:\Omega_{t_0}\to\mathbb{R}^n$, and (5.7) is written

$$\frac{d\Phi_{t;t_0}(p_{t_0})}{dt}=\vec{v}(t,\Phi_{t;t_0}(p_{t_0})),\quad\text{and}\quad\Phi_{t_0;t_0}(p_{t_0})=p_{t_0}. \tag{5.10}$$

Theorem 5.5 *Let $\vec{v}$ be Lipschitzian, let $t_0\in]t_1,t_2[$, and let Ω_{t_0} be an open set s.t. $\Omega_{t_0}\subset\subset\Omega$ (i.e. there exists a compact set $K\in\mathbb{R}^n$ s.t. $\Omega_{t_0}\subset K\subset\Omega$). Then there exists $\varepsilon>0$ s.t. a flow Φ^{t_0} exists on $]t_0-\varepsilon,t_0+\varepsilon[\times\Omega_{t_0}$.*

Proof. Let $d=d(K,\mathbb{R}^n-\Omega)$ (la distance of K to the border of Ω.

Let $||\vec{v}||_\infty:=\sup\limits_{t\in[t_1,t_2],p\in\overline{\Omega}}||\vec{v}(t,p)||_{\mathbb{R}^n}$ (exists since $\vec{v}\in C^0$ on the compact $[t_1,t_2]\times\overline{\Omega}$).

Let $\varepsilon=\min(t_0-t_1,t_2-t_0,\frac{d}{||\vec{v}||_\infty})$ (less that the minimum time to reach the border from K at maximum speed $||v||_\infty$).

Let $p_{t_0}\in K$ and $t\in]t_0-\varepsilon,t_0+\varepsilon[$. Then $\Phi^{t_0}_{p_{t_0}}$ exists, cf.theorem 5.4, and $||\Phi^{t_0}_{p_{t_0}}(t)-\Phi^{t_0}_{p_{t_0}}(t_0)||_{\mathbb{R}^n}\leq[t-t_0|\sup_{\tau\in]t_0-\varepsilon,t_0+\varepsilon[}(||(\Phi^{t_0}_{p_{t_0}})'(\tau)||_{\mathbb{R}^n})$ (mean value theorem since, $\vec{v}$ being C^0, Φ is C^1). Thus $||\Phi^{t_0}_{p_{t_0}}(t)-\Phi^{t_0}_{p_{t_0}}(t_0)||_{\mathbb{R}^n}\leq[t-t_0|\,||v||_\infty$, thus $\Phi^{t_0}_{p_{t_0}}(t)\in\Omega$. Thus $\Phi^{t_0}_{p_{t_0}}$ exists on $]t_0-\varepsilon,t_0+\varepsilon[$, for all $p_{t_0}\in K$. ∎

Remark 5.6 The definition of a flow starts with a Eulerian velocity (independent of any initial time), and then, due to the introduction of initial conditions, leads to the Lagrangian functions Φ^{t_0}, cf. (5.8). Once again, Lagrangian functions are the result of Eulerian functions. ∎

5.4 Examples

Example 1 $\mathbb{R}^2$ with an origin $\mathcal{O}$, a Euclidean basis $(\vec{e}_1,\vec{e}_2)$ and $\Omega=[0,2]\times[0,1]$ (observation window). Let $p\in\mathbb{R}^2$, $\overrightarrow{\mathcal{O}p} =^{\text{noted}} \vec{x}=x\vec{e}_1+y\vec{e}_2 =^{\text{noted}} (x,y)$. Let $t_1=-1$, $t_2=1$, $t_0\in]t_1,t_2[$, $a,b\in\mathbb{R}$, $a\neq 0$, and

$$\vec{v}(t,p)=\begin{cases} v^1(t,x,y)=ay,\\ v^2(t,x,y)=b\sin(t{-}t_0).\end{cases} \tag{5.11}$$

($b=0$ corresponds to the stationary case = shear flow.) $\vec{x}(t_0)=\begin{pmatrix}x_0\\y_0\end{pmatrix}$, $\vec{x}(t)=\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=\overrightarrow{\mathcal{O}\Phi^{t_0}_{p_{t_0}}(t)}$ and (5.9) give

$$\begin{cases} \dfrac{dx}{dt}(t)=v^1(t,x(t),y(t))=ay(t),\\ \dfrac{dy}{dt}(t)=v^2(t,x(t),y(t))=b\sin(t{-}t_0),\end{cases} \quad\text{with}\quad \begin{cases} x(t_0)=x_0,\\ y(t_0)=y_0.\end{cases} \tag{5.12}$$

Thus

$$\vec{x}(t)=\overrightarrow{\mathcal{O}p(t)}=\overrightarrow{\mathcal{O}\Phi^{t_0}_{p_{t_0}}(t)}=\begin{pmatrix} x(t)=x_0+a(y_0+b)(t{-}t_0)-ab\sin(t{-}t_0)\\ y(t)=y_0+b-b\cos(t{-}t_0)\end{pmatrix}. \tag{5.13}$$

Example 2 Similar framework. Let $\omega>0$ and consider (spin vector field)

$$\vec{v}(t,x,y)=\begin{pmatrix}-\omega y\\ \omega x\end{pmatrix}=\omega\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\overset{\text{noted}}{=}\vec{v}(x,y). \tag{5.14}$$

With $\overrightarrow{\mathcal{O}p_{t_0}}=\vec{x}_{t_0}=\begin{pmatrix}x_{t_0}\\y_{t_0}\end{pmatrix}$, $r_{t_0}=\sqrt{x_{t_0}^2+y_{t_0}^2}$, and θ_0 s.t. $\vec{x}_{t_0}=\begin{pmatrix}x_{t_0}=r_{t_0}\cos(\omega t_0)\\ y_{t_0}=r_{t_0}\sin(\omega t_0)\end{pmatrix}$, the solution $\Phi^{t_0}_{p_{t_0}}$ of (5.9) is

$$\vec{x}(t)=\overrightarrow{\mathcal{O}p(t)}=\overrightarrow{\mathcal{O}\Phi^{t_0}_{p_{t_0}}(t)}=\begin{pmatrix}x(t)=r_{t_0}\cos(\omega t)\\ y(t)=r_{t_0}\sin(\omega t)\end{pmatrix}. \tag{5.15}$$

Indeed, $\begin{pmatrix}\frac{\partial x}{\partial t}(t,\vec{x}_0)\\ \frac{\partial y}{\partial t}(t,\vec{x}_0)\end{pmatrix}=\begin{pmatrix}v^1(t,x(t,\vec{x}_0),y(t,\vec{x}_0))\\ v^2(t,x(t,\vec{x}_0),y(t,\vec{x}_0))\end{pmatrix}=\begin{pmatrix}-\omega y(t,\vec{x}_0)\\ \omega x(t,\vec{x}_0)\end{pmatrix}$, thus $\frac{\partial x}{\partial t}(t,\vec{x}_0)=-\omega y(t,\vec{x}_0)$ and $\frac{\partial y}{\partial t}(t,\vec{x}_0)=\omega x(t,\vec{x}_0)$, thus $\frac{\partial^2 y}{\partial t^2}(t,\vec{x}_0)=-\omega^2 y(t,\vec{x}_0)$, hence y; Idem for x. Here $d\vec{v}(t,x,y)=\omega\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\omega\begin{pmatrix}\cos\frac{\pi}{2}&-\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2}&\cos\frac{\pi}{2}\end{pmatrix}$ is the $\pi/2$-rotation composed with the homothety with ratio ω.

5.5 Composition of flows

Let $\vec{v}$ be a vector field on $\mathbb{R}\times\Omega$ and $\Phi^{t_0}_{p_{t_0}}$ solution of (5.5). We use the notations

$$p_t=\Phi^{t_0}_t(p_{t_0})=\Phi_{t;t_0}(p_{t_0}):=\Phi^{t_0}_{p_{t_0}}(t)=\Phi^{t_0}(t,p_{t_0})=\Phi(t;t_0,p_{t_0})=\Phi_{t_0,p_{t_0}}(t). \tag{5.16}$$

5.5.1 Law of composition of flows (determinism)

Proposition 5.7 *For all $t_0,t_1,t_2\in\mathbb{R}$, we have (determinism)*

$$\Phi^{t_1}_{t_2}\circ\Phi^{t_0}_{t_1}=\Phi^{t_0}_{t_2},\quad\textit{i.e.}\quad \Phi_{t_2;t_1}\circ\Phi_{t_1;t_0}=\Phi_{t_2;t_0}. \tag{5.17}$$

("The composition of the photos gives the film"). So,

$$p_{t_2}=\Phi^{t_1}_{t_2}(p_{t_1})=\Phi^{t_0}_{t_2}(p_{t_0})\quad\textit{when}\quad p_{t_1}=\Phi^{t_0}_{t_1}(p_{t_0}), \tag{5.18}$$

i.e.,

$$p_{t_2}=\Phi_{t_2;t_1}(p_{t_1})=\Phi_{t_2;t_0}(p_{t_0})\quad\textit{when}\quad p_{t_1}=\Phi_{t_1;t_0}(p_{t_0}). \tag{5.19}$$

Thus

$$d\Phi^{t_1}_{t_2}(p_{t_1}).d\Phi^{t_0}_{t_1}(p_{t_0})=d\Phi^{t_0}_{t_2}(p_{t_0}),\quad\text{i.e.}\quad d\Phi_{t_2;t_1}(p_{t_1}).d\Phi_{t_1;t_0}(p_{t_0})=d\Phi_{t_2;t_0}(p_{t_0}). \tag{5.20}$$

Summary with commutative diagrams:

$$\begin{tikzcd} & p_{t_1} \arrow[dr, "\Phi^{t_1}_{t_2}"] & \\ p_{t_0} \arrow[ur, "\Phi^{t_0}_{t_1}"] \arrow[rr, "\Phi^{t_0}_{t_2}"'] & & p_{t_2} \end{tikzcd}$$

i.e.

$$\begin{tikzcd} & p_{t_1} \arrow[dr, "\Phi_{t_2;t_1}"] & \\ p_{t_0} \arrow[ur, "\Phi_{t_1;t_0}"] \arrow[rr, "\Phi_{t_2;t_0}"'] & & p_{t_2} \end{tikzcd}$$

Proof. Let $p_{t_1} = \Phi^{t_0}_{p_{t_0}}(t_1)$. (5.9) gives

$$\left\{\begin{aligned} &\frac{d\Phi^{t_0}_{p_{t_0}}}{dt}(t) = \vec{v}(t, \Phi^{t_0}_{p_{t_0}}(t)),\\ &\frac{d\Phi^{t_1}_{p_{t_1}}}{dt}(t) = \vec{v}(t, \Phi^{t_1}_{p_{t_1}}(t)),\end{aligned}\right\} \quad \text{with} \quad p_{t_1} = \Phi^{t_0}_{p_{t_0}}(t_1) = \Phi^{t_1}_{p_{t_1}}(t_1).$$

Thus $\Phi^{t_0}_{p_{t_0}}$ and $\Phi^{t_1}_{p_{t_1}}$ satisfy the same ODE with the same value at t_1; Thus they are equal (uniqueness thanks to Cauchy–Lipschitz theorem), thus $\Phi^{t_1}_{p_{t_1}}(t) = \Phi^{t_0}_{p_{t_0}}(t)$ when $p_{t_1} = \Phi^{t_0}_{t_1}(p_{t_0})$, that is, $\Phi^{t_1}_t(p_{t_1}) = \Phi^{t_0}_t(p_{t_0})$ when $p_{t_1} = \Phi^{t_0}_{t_1}(p_{t_0})$, which is (5.17) for any $t = t_2$. Thus (5.20). ∴

Corollary 5.8 *A flow is compatible with the motion* $\widetilde{\Phi}$ *of an object Obj: (3.6) gives* $\Phi^{t_1}_{t_2} \circ \Phi^{t_0}_{t_1} = (\widetilde{\Phi}_{t_2} \circ (\widetilde{\Phi}_{t_1})^{-1}) \circ (\widetilde{\Phi}_{t_1} \circ (\widetilde{\Phi}_{t_0})^{-1}) = \widetilde{\Phi}_{t_2} \circ (\widetilde{\Phi}_{t_0})^{-1} = \Phi^{t_0}_{t_2}$, *that is (5.17).*

5.5.2 Stationnary case

Definition 5.9 $\vec{v}$ is a stationary vector field iff $\frac{\partial \vec{v}}{\partial t} = 0$; And then $\vec{v}(t,p) =^{\text{noted}} \vec{v}(p)$. And the associated flow Φ^{t_0} which satisfies

$$\frac{\partial \Phi^{t_0}}{\partial t}(t, p_{t_0}) = \vec{v}(p_t) \quad \text{when} \quad p_t = \Phi^{t_0}(t, p_{t_0}), \tag{5.21}$$

is said to be stationary.

Proposition 5.10 *If $\vec{v}$ is a stationary vector field then, for all t_0, t_1, h, when meaningful (h small enough and t_1 close enough to t_0),*

$$\Phi^{t_1}_{t_1+h} = \Phi^{t_0}_{t_0+h}, \quad \text{i.e.} \quad \Phi_{t_1+h;t_1} = \Phi_{t_0+h;t_0}, \tag{5.22}$$

i.e. $\Phi^{t_1}_{t_1+h}(q) = \Phi^{t_0}_{t_0+h}(q)$, *i.e.* $\Phi(t_1{+}h; t_1, q) = \Phi(t_0{+}h; t_0, q)$ *for all* $q \in \Omega_{t_0}$ *(see theorem 5.5). In other words,*

$$\Phi^{t_0+h}_{t_1+h} = \Phi^{t_0}_{t_1}, \quad \text{i.e.} \quad \Phi_{t_1+h;t_0+h} = \Phi_{t_1;t_0}, \tag{5.23}$$

i.e. $\Phi^{t_0+h}_{t_1+h}(q) = \Phi^{t_0}_{t_1}(q)$, *i.e.* $\Phi(t_1{+}h; t_0{+}h, q) = \Phi(t_1; t_0, q)$ *for all* $q \in \Omega_{t_0}$.

Proof. Let $q \in \Omega_{t_0}$, $\alpha(h) = \Phi^{t_0}_{t_0+h}(q) = \Phi^{t_0}_q(t_0{+}h)$ and $\beta(h) = \Phi^{t_1}_{t_1+h}(q) = \Phi^{t_1}_q(t_1{+}h)$.

Thus $\alpha'(h) = \frac{d\Phi^{t_0}_q}{dt}(t_0{+}h) = \vec{v}(t_0{+}h, \Phi^{t_0}_q(t_0{+}h)) = \vec{v}(\Phi^{t_0}_q(t_0{+}h)) = \vec{v}(\alpha(h))$ (stationary flow), and $\beta'(h) = \frac{d\Phi{t_1}_q}{dt}(t_1{+}h) = \vec{v}(t_1{+}h, \Phi^{t_1}_q(t_1{+}h)) = \vec{v}(\Phi^{t_1}_q(t_1{+}h)) = \vec{v}(\beta(h))$ (stationary flow).

Thus α and β satisfy the same ODE with the same initial condition $\alpha(0) = \beta(0) = q$. Thus $\alpha = \beta$. Hence (5.22). Thus, with $h = t_1{-}t_0$, i.e. with $t_1 = t_0{+}h$ and $t_0{+}h = t_1$, we get (5.23). ∴

Corollary 5.11 *If $\vec{v}$ is a stationary vector field, cf. (5.21), then*

$$d\Phi^{t_0}_t(p_{t_0}).\vec{v}(p_{t_0}) = \vec{v}(p_t) \quad \textit{when} \quad p_t = \Phi^{t_0}_t(p_{t_0}), \tag{5.24}$$

that is, if $\vec{v}$ is stationary, then $\vec{v}$ is transported (push-forwarded by $\Phi^{t_0}_t$) along itself.

Proof. (5.18), $t_2 = t_1{+}s$ and $t_1 = t_0{+}s$ give $\Phi^{t_0+s}_{t_1+s}(\Phi^{t_0}_{t_0+s}(p_{t_0})) = \Phi^{t_0}_{t_1+s}(p_{t_0})$, and $\vec{v}$ is stationary, thus $\Phi^{t_0}_{t_1}(\Phi^{t_0}_{t_0+s}(p_{t_0})) = \Phi^{t_0}_{t_1+s}(p_{t_0})$, i.e. $\Phi(t_1; t_0, \Phi_{t_0,p_{t_0}}(t_0{+}s)) = \Phi_{t_0,p_{t_0}}(t_1{+}s)$, thus ($s$ derivative)

$$d\Phi(t_1; t_0, \Phi(t_0{+}s; t_0, p_{t_0})).{\Phi_{t_0,p_{t_0}}}'(t_0{+}s) = {\Phi_{t_0,p_{t_0}}}'(t_1{+}s),$$

thus $d\Phi^{t_0}_{t_1}(\Phi(t_0{+}s; t_0, p_{t_0})).\vec{v}(t_0{+}s, \Phi_{t_0,p_{t_0}}(t_0{+}s)) = \vec{v}(t_1{+}s, \Phi_{t_0,p_{t_0}}(t_1{+}s))$. Thus with $s = 0$, and $\vec{v}$ being stationary, $d\Phi^{t_0}_{t_1}(\Phi(t_0; t_0, p_{t_0})).\vec{v}(\Phi_{t_0,p_{t_0}}(t_0)) = \vec{v}(\Phi_{t_0,p_{t_0}}(t_1))$, thus (5.24). ∴

5.6 Velocity on the trajectory traveled in the opposite direction

Let $t_0, t_1 \in \mathbb{R}$, $t_1 > t_0$, and $p_{t_0} \in \mathbb{R}^n$. Consider the trajectory $\Phi^{t_0}_{p_{t_0}} : \left\{ \begin{array}{ll} [t_0, t_1] & \to \mathbb{R}^n \\ t & \to p(t) = \Phi^{t_0}_{p_{t_0}}(t) \end{array} \right\}$. So p_{t_0} is the beginning of the trajectory, $p_{t_1} = \Phi^{t_0}_{t_1}(p_{t_0})$ at the end, $\vec{v}(t, p(t)) = \frac{d\Phi^{t_0}_{p_{t_0}}}{dt}(t)$ being the velocity.

Define the trajectory traveled in the opposite direction, i.e. define

$$\Psi^{t_1}_{p_{t_1}} : \left\{ \begin{array}{ll} [t_0, t_1] & \to \mathbb{R}^n \\ u & \to q(u) = \Psi^{t_1}_{p_{t_1}}(u) := \Phi^{t_0}_{p_{t_0}}(t_0{+}t_1{-}u) = \Phi^{t_0}_{p_{t_0}}(t) = p(t) \quad \text{when} \quad t = t_0{+}t_1{-}u. \end{array} \right. \tag{5.25}$$

In particular $q(t_0) = \Psi^{t_1}_{p_{t_1}}(t_0) = \Phi^{t_0}_{p_{t_0}}(t_1) = p(t_1)$ and $q(t_1) = \Psi^{t_1}_{p_{t_1}}(t_1) = \Phi^{t_0}_{p_{t_0}}(t_0) = p(t_0)$.

Proposition 5.12 *The velocity on the trajectory traveled in the opposite direction is the opposite of the velocity on the initial trajectory:*

$$\frac{d\Psi^{t_1}_{p_{t_1}}}{du}(u) = q'(u) = -p'(t) = -\vec{v}(t, p(t)) \quad \textit{when} \quad t = t_0{+}t_1{-}u, \tag{5.26}$$

Proof. $\Psi^{t_1}_{p_{t_1}}(u) = \Phi^{t_0}_{p_{t_0}}(t_0{+}t_1{-}u)$ gives $\frac{d\Psi^{t_1}_{p_{t_1}}}{du}(u) = -\frac{d\Phi^{t_0}_{p_{t_0}}}{dt}(t_0{+}t_1{-}u) = -\vec{v}(t_0{+}t_1{-}u, \Phi^{t_0}_{p_{t_0}}(t_0{+}t_1{-}u)) = -\vec{v}(t, \Phi^{t_0}_{p_{t_0}}(t))$ when $t = t_0{+}t_1{-}u$. ∴

5.7 Variation of the flow as a function of the initial time

5.7.1 Ambiguous and non ambiguous notations

Let $\Phi : (t, u, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \Phi(t, u, p) \in \mathbb{R}^n$ be a C^1 function. The partial derivatives are

$$\partial_1\Phi(t, u, p) := \lim_{h\to 0} \frac{\Phi(t{+}h, u, p) - \Phi(t, u, p)}{h}, \tag{5.27}$$

$$\partial_2\Phi(t, u, p) := \lim_{h\to 0} \frac{\Phi(t, u{+}h, p) - \Phi(t, u, p)}{h}, \tag{5.28}$$

and $\partial_3\Phi(t, u, p)$, defined for all $\vec{w} \in \vec{\mathbb{R}}^n$ (a vector at p) by,

$$\partial_3\Phi(t, u, p).\vec{w} := \lim_{h\to 0} \frac{\Phi(t, u, p{+}h\vec{w}) - \Phi(t, u, p)}{h} \overset{\text{noted}}{=} d\Phi(t, u, p).\vec{w}, \tag{5.29}$$

When the name of the first variable is systematically noted t, then

$$\partial_1\Phi(t, u, p) \overset{\text{noted}}{=} \frac{\partial\Phi}{\partial t}(t, u, p) \overset{\text{noted}}{=} \frac{\partial\Phi(t, u, p)}{\partial t}. \tag{5.30}$$

NB: This notation can be ambiguous: What is the meaning of $\frac{\partial\Phi}{\partial t}(t; t, p)$? In ambiguous situations, use the notation $\partial_1\Phi$, or (if no composed functions inside) use $\frac{\partial\Phi(t,u,p)}{\partial t}_{|u=t}$ (so t is the derivation variable, and after the calculation you take $u = t$).

When the name of the second variable is systematically noted u, then

$$\partial_2\Phi(t, u, p) \overset{\text{noted}}{=} \frac{\partial\Phi}{\partial u}(t, u, p) \overset{\text{noted}}{=} \frac{\partial\Phi(t, u, p)}{\partial u}. \tag{5.31}$$

NB: Idem this notation can be ambiguous: What is the meaning of $\frac{\partial\Phi}{\partial u}(u; u, p)$? In ambiguous situations, use the notation $\partial_2\Phi$, or use $\frac{\partial\Phi(t,u,p)}{\partial u}_{|t=u}$.

When the name of the third variable is systematically a space variable noted p, then

$$\partial_3\Phi(t, u, p) \overset{\text{noted}}{=} d\Phi(t, u, p) \overset{\text{noted}}{=} \frac{\partial\Phi}{\partial p}(t, u, p) \overset{\text{noted}}{=} \frac{\partial\Phi(t, u, p)}{\partial p}. \tag{5.32}$$

5.7.2 Variation of the flow as a function of the initial time

The law of composition of the flows gives (5.19) gives $\Phi(t;u,\Phi(u;t_0,p_0)) = \Phi(t;t_0,p_0)$. Thus the derivative in u gives

$$\begin{aligned} &\partial_2\Phi(t;u,\Phi(u;t_0,p_0)) + d\Phi(t;u,\Phi(u;t_0,p_0)).\partial_1\Phi(u;t_0,p_0) = 0,\\ \text{i.e.}\quad &\partial_2\Phi(t;u,p(u)) = -d\Phi(t;u,p(u)).\vec{v}(u,p(u)) \quad \text{when} \quad p(u)=\Phi(u;t_0,p_0). \end{aligned} \tag{5.33}$$

In particular $u = t_0$ gives, for all $(t,t_0,p_0) \in \mathbb{R}^2 \times \Omega_{t_0}$,

$$(\frac{\partial\Phi(t;t_0,p_0)}{\partial t_0} =) \quad \partial_2\Phi(t;t_0,p_0) = -d\Phi(t;t_0,p_0).\vec{v}(t_0,p_0). \tag{5.34}$$

In particular

$$(\frac{d\Phi(t;t_0,p_0)}{dt_0}_{|t=t_0} =) \quad \partial_2\Phi(t_0;t_0,p_0) = -\vec{v}(t_0,p_0). \tag{5.35}$$

Part II

Push-forward

6 Push-forward

The general tool to describe "transport" is "push-forward by a motion" (the "take with you" operator), cf. § 4.1 and figure 4.1. The push-forward also gives the tool needed to understand the velocity addition formula: In that case, the push-forward is the translator between observers. The push-forward can also be used to write coordinate systems. As usual, we start with qualitative results (observer independent results); Then, quantitative results are deduced.

6.1 Definition

$\mathcal{E}$ and $\mathcal{F}$ are affine spaces, E and F are the associated vector spaces equipped with norms $||.||_E$ and $||.||_F$ with $\dim E = \dim F = n$, $\mathcal{U}_\mathcal{E}$ and $\mathcal{U}_\mathcal{F}$ are open sets in the affine space $\mathcal{E}$ and $\mathcal{F}$, or possibly the vector spaces E and F, and

$$\Psi : \begin{cases} \mathcal{U}_\mathcal{E} & \to \mathcal{U}_\mathcal{F} \\ p_\mathcal{E} & \to p_\mathcal{F} = \Psi(p_\mathcal{E}) \quad \text{is a diffeomorphism} \end{cases} \tag{6.1}$$

(a C^1 invertible map which inverse is C^1), called the push-forward, and Ψ^{-1} is the pull-back (push-forward with Ψ^{-1}).

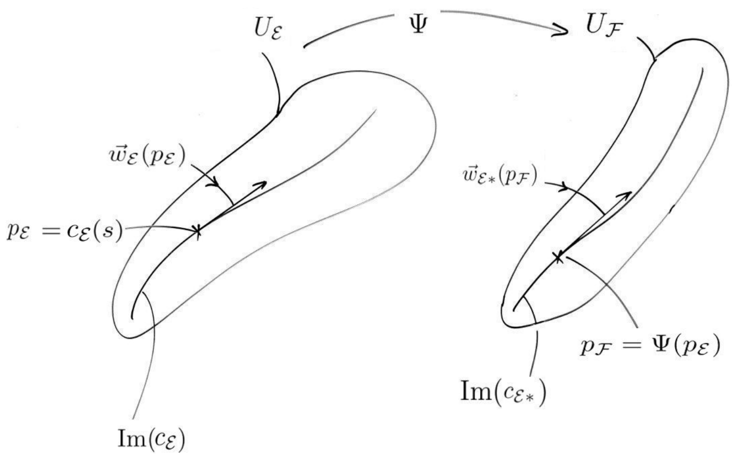
$U_\mathcal{E}$
Ψ
$U_\mathcal{F}$
$\vec{w}_\mathcal{E}(p_\mathcal{E})$
$\vec{w}_{\mathcal{E}*}(p_\mathcal{F})$
$p_\mathcal{E} = c_\mathcal{E}(s)$
$p_\mathcal{F} = \Psi(p_\mathcal{E})$
$\mathrm{Im}(c_{\mathcal{E}*})$
$\mathrm{Im}(c_\mathcal{E})$

Figure 6.1: $c_\mathcal{E} : s \to p_\mathcal{E} = c_\mathcal{E}(s)$ is a curve in $\mathcal{U}_\mathcal{E}$. Push-forwarded by Ψ it becomes the curve $c_{\mathcal{E}*} := \Psi \circ c_\mathcal{E}$ in $\mathcal{U}_\mathcal{F}$. The tangent vector at $p_\mathcal{E} = c_\mathcal{E}(s)$ is $\vec{w}_\mathcal{E}(p_\mathcal{E}) = c_\mathcal{E}'(s)$, and the tangent vector at $p_\mathcal{F} = c_\mathcal{F}(s) = \Psi(c_\mathcal{E}(s))$ is $\vec{w}_{\mathcal{E}*}(p_\mathcal{F}) = c_\mathcal{F}'(s) = d\Psi(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E})$. Other illustation: See figure 4.1.

Example: $\Psi = \Phi_t^{t_0} : \Omega_{t_0} \to \Omega_t$, the motion that transforms Ω_{t_0} into Ω_t, cf. (3.5).
Example: $\Psi : U_E \to U_F$ a coordinate system, see example 6.11.
Example: $\Psi = \Theta_t : \mathcal{R}_B \to \mathcal{R}_A$, a change of referential at t (change of observer), see § 10.

6.2 Push-forward and pull-back of points

Definition 6.1 If $p_\mathcal{E} \in \mathcal{U}_\mathcal{E}$ (a point in $\mathcal{U}_\mathcal{E}$) then its push-forward by Ψ is the point

$$p_\mathcal{F} = \boxed{\Psi_* p_\mathcal{E} := \Psi(p_\mathcal{E})} = p_{\mathcal{E}*} \in \mathcal{U}_\mathcal{F}, \tag{6.2}$$

see figure 6.1, the last notation if Ψ is implicit. And if $p_\mathcal{F} \in \mathcal{U}_\mathcal{F}$ then its pull-back by Ψ is the point

$$p_\mathcal{E} = \boxed{\Psi^* p_\mathcal{F} := \Psi^{-1}(p_\mathcal{F})} = p_\mathcal{F}{}^* \in \mathcal{U}_\mathcal{E}. \tag{6.3}$$

We immediately have $\Psi^* \circ \Psi_* = I$.

The notations $_*$ for push-forward and * for pull-back have been proposed by Spivak; Also see Abraham and Marsden [1] (second edition) who adopt this notation.

6.3 Push-forward and pull-back of curves

We push-forward (and pull-back) the points on a curve:

Definition 6.2 Let $c_{\mathcal{E}} : \left\{ \begin{array}{ll}]-\varepsilon,\varepsilon[& \to \mathcal{U}_{\mathcal{E}} \\ s & \to p_{\mathcal{E}} = c_{\mathcal{E}}(s) \end{array} \right\}$ be a curve in $\mathcal{U}_{\mathcal{E}}$. Its push-forward by Ψ is the curve

$$\Psi_* c_{\mathcal{E}} := \Psi \circ c_{\mathcal{E}} : \left\{ \begin{array}{ll}]-\varepsilon,\varepsilon[& \to \mathcal{U}_{\mathcal{F}} \\ s & \to p_{\mathcal{F}} = \Psi_* c_{\mathcal{E}}(s) := \Psi(c_{\mathcal{E}}(s)) \overset{\text{noted}}{=} c_{\mathcal{E}*}(s) \quad (= \Psi(p_{\mathcal{E}})), \end{array} \right. \tag{6.4}$$

see figure 6.1. ($\Psi_* c_{\mathcal{E}} =^{\text{noted}} c_{\mathcal{E}*}$ when Ψ is implicit.) This defines

$$\Psi_* : \left\{ \begin{array}{ll} \mathcal{F}(]-\varepsilon,\varepsilon[;\mathcal{U}_{\mathcal{E}}) & \to \mathcal{F}(]-\varepsilon,\varepsilon[;\mathcal{U}_{\mathcal{F}}) \\ c_{\mathcal{E}} & \to \Psi_*(c_{\mathcal{E}}) := \Psi \circ c_{\mathcal{E}} \overset{\text{noted}}{=} \Psi_* c_{\mathcal{E}} = c_{\mathcal{E}*}. \end{array} \right. \tag{6.5}$$

Definition 6.3 Let $c_{\mathcal{F}} : \left\{ \begin{array}{ll}]-\varepsilon,\varepsilon[& \to \mathcal{U}_{\mathcal{F}} \\ s & \to p_{\mathcal{F}} = c_{\mathcal{F}}(s) \end{array} \right\}$ is a curve in $\mathcal{U}_{\mathcal{F}}$. Its pull-back by Ψ is

$$\Psi^* c_{\mathcal{F}} := \Psi^{-1} \circ c_{\mathcal{E}} \left\{ \begin{array}{ll}]-\varepsilon,\varepsilon[& \to \mathcal{U}_{\mathcal{E}} \\ s & \to p_{\mathcal{E}} = \Psi^* c_{\mathcal{F}}(s) := \Psi^{-1}(c_{\mathcal{F}}(s)) \overset{\text{noted}}{=} c_{\mathcal{F}}{}^*(s) \quad (= \Psi^{-1}(p_{\mathcal{F}})). \end{array} \right. \tag{6.6}$$

We have thus defined

$$\Psi^* : \left\{ \begin{array}{ll} \mathcal{F}(C^1(]-\varepsilon,\varepsilon[;\mathcal{U}_{\mathcal{F}}) & \to \mathcal{F}(C^1(]-\varepsilon,\varepsilon[;\mathcal{U}_{\mathcal{E}}) \\ c_{\mathcal{F}} & \to \Psi^*(c_{\mathcal{F}}) := \Psi^{-1} \circ c_{\mathcal{F}} \overset{\text{noted}}{=} \Psi^* c_{\mathcal{F}} = c_{\mathcal{F}}{}^*. \end{array} \right. \tag{6.7}$$

6.4 Push-forward and pull-back of scalar functions

6.4.1 Definitions

Definition 6.4 Let $f_{\mathcal{E}} : \left\{ \begin{array}{ll} \mathcal{U}_{\mathcal{E}} & \to \mathbb{R} \\ p_{\mathcal{E}} & \to f_{\mathcal{E}}(p_{\mathcal{E}}) \end{array} \right\}$ (scalar valued function). Its push-forward by Ψ is the (scalar valued) function

$$\Psi_* f_{\mathcal{E}} := f_{\mathcal{E}} \circ \Psi^{-1} : \left\{ \begin{array}{ll} \mathcal{U}_{\mathcal{F}} & \to \mathbb{R} \\ p_{\mathcal{F}} & \to \Psi_* f_{\mathcal{E}}(p_{\mathcal{F}}) := f_{\mathcal{E}}(p_{\mathcal{E}}) \overset{\text{noted}}{=} f_{\mathcal{E}*}(p_{\mathcal{F}}) \quad \text{when} \quad p_{\mathcal{E}} = \Psi^{-1}(p_{\mathcal{F}}), \end{array} \right. \tag{6.8}$$

(noted $f_{\mathcal{E}*}$ when Ψ is implicit), i.e. $\Psi_* f_{\mathcal{E}}(\Psi_* p_{\mathcal{E}}) := f_{\mathcal{E}}(p_{\mathcal{E}})$, or $f_{\mathcal{E}*}(p_{\mathcal{E}*}) := f_{\mathcal{E}}(p_{\mathcal{E}})$ when $p_{\mathcal{E}*} = \Psi(p_{\mathcal{E}})$. We have thus defined

$$\Psi_* : \left\{ \begin{array}{ll} \mathcal{F}(\mathcal{U}_{\mathcal{E}};\mathbb{R}) & \to \mathcal{F}(\mathcal{U}_{\mathcal{F}};\mathbb{R}) \\ f_{\mathcal{E}} & \to f_{\mathcal{F}} := \Psi_*(f_{\mathcal{E}}) = f_{\mathcal{E}} \circ \Psi^{-1} \overset{\text{noted}}{=} \Psi_* f_{\mathcal{E}}, \end{array} \right. \tag{6.9}$$

the notation $\Psi_*(f_{\mathcal{E}}) = \Psi_* f_{\mathcal{E}}$ since Ψ_* is linear: $((f_{\mathcal{E}} + \lambda g_{\mathcal{E}}) \circ \Psi^{-1})(p_{\mathcal{F}}) = (f_{\mathcal{E}} + \lambda g_{\mathcal{E}})(p_{\mathcal{E}}) = f_{\mathcal{E}}(p_{\mathcal{E}}) + \lambda g_{\mathcal{E}}(p_{\mathcal{E}}) = (f_{\mathcal{E}} \circ \Psi^{-1})(p_{\mathcal{F}}) + \lambda(g_{\mathcal{E}} \circ \Psi^{-1})(p_{\mathcal{F}})$ gives $\Psi_*(f_{\mathcal{E}} + \lambda g_{\mathcal{E}}) = \Psi_*(f_{\mathcal{E}}) + \lambda \Psi_*(g_{\mathcal{E}})$.

Definition 6.5 Let $f_{\mathcal{F}} : \left\{ \begin{array}{ll} \mathcal{U}_{\mathcal{F}} & \to \mathbb{R} \\ p_{\mathcal{F}} & \to f_{\mathcal{F}}(p_{\mathcal{F}}) \end{array} \right\}$. Its pull-back by Ψ is the push-forward by Ψ^{-1}, i.e. is

$$\Psi^* f_{\mathcal{F}} := f_{\mathcal{F}} \circ \Psi : \left\{ \begin{array}{ll} \mathcal{U}_{\mathcal{E}} & \to \mathbb{R} \\ p_{\mathcal{E}} & \to \Psi^* f_{\mathcal{F}}(p_{\mathcal{E}}) := f_{\mathcal{F}}(p_{\mathcal{F}}) \overset{\text{noted}}{=} f_{\mathcal{F}}{}^*(p_{\mathcal{E}}) \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}), \end{array} \right. \tag{6.10}$$

i.e. $\Psi^* f_{\mathcal{F}}(\Psi^* p_{\mathcal{F}}) := f_{\mathcal{F}}(p_{\mathcal{F}})$, i.e. $f_{\mathcal{F}}{}^*(p_{\mathcal{F}}{}^*) := f_{\mathcal{F}}(p_{\mathcal{F}})$ when $p_{\mathcal{F}} = \Psi^*(p_{\mathcal{F}})$. We have thus defined

$$\Psi^* : \left\{ \begin{array}{ll} \mathcal{F}(\mathcal{U}_{\mathcal{F}};\mathbb{R}) & \to \mathcal{F}(\mathcal{U}_{\mathcal{E}};\mathbb{R}) \\ f_{\mathcal{F}} & \to \Psi^*(f_{\mathcal{F}}) = f_{\mathcal{F}}{}^* := f_{\mathcal{F}} \circ \Psi \overset{\text{noted}}{=} \Psi^* f_{\mathcal{F}}. \end{array} \right. \tag{6.11}$$

We immediately have $\Psi^* \circ \Psi_* = I$ and $\Psi_* \circ \Psi^* = I$ (the first I is the identity in $\mathcal{F}(\mathcal{U}_{\mathcal{E}};\mathbb{R})$, the second I is the identity in $\mathcal{F}(\mathcal{U}_{\mathcal{F}};\mathbb{R})$).

NB: We used the same notations Ψ_* and Ψ^* than for the push-forward and pull-backs of points: The context removes ambiguities.

6.4.2 Interpretation: Why is it useful?

E.g.: Let $\widetilde{\Phi} : \mathbb{R} \times Obj \to \mathbb{R}^n$ be a motion of an object Obj. An observer records the temperature θ at all $t \in [t_0, T]$ and all $p = \widetilde{\Phi}(t, Obj)$: He gets $\theta : \left\{ \begin{array}{ll} \mathcal{C} = \bigcup_t (\{t\} \times \Omega_t) & \to \mathbb{R} \\ (t,p) & \to \theta(t,p) \end{array} \right\}$ a Eulerian scalar valued function, cf. (2.2). Then he chooses an initial time t_0 and considers the associated motion Φ^{t_0}, cf. (3.1), and considers $\theta_{t_0} : \left\{ \begin{array}{ll} \Omega_{t_0} & \to \mathbb{R} \\ p_{t_0} & \to \theta_{t_0}(p_{t_0}) := \theta(t_0, p_{t_0}) \end{array} \right\}$ (snapshot of the temperatures at t_0 in Ω_{t_0}). The push-forward of θ_{t_0} by $\Phi^{t_0}_t$ is $(\Phi^{t_0}_t)_* \theta_{t_0} := \theta_{t_0} \circ (\Phi^{t_0}_t)^{-1}$ defines the "memory function"

$$(\Phi^{t_0}_t)_* \theta_{t_0} : \left\{ \begin{array}{ll} \Omega_t & \to \mathbb{R} \\ p_t & \to (\Phi^{t_0}_t)_* \theta_{t_0}(p_t) := \theta_{t_0}(p_{t_0}) \quad \text{when} \quad p_t = \Phi^{t_0}_t(p_{t_0}), \end{array} \right. \tag{6.12}$$

And he writes $(\Phi^{t_0}_t)_* \theta_{t_0}(p_t) =^{\text{noted}} \theta_{t_0 *}(t, p_t)$, so the memory transported is at t at p_t (along a trajectory) by

$$\theta_{t_0 *}(t, p(t)) = \theta_{t_0}(p_{t_0}). \tag{6.13}$$

Question: Why do we introduce $\theta_{t_0 *}$ since we have θ_{t_0}?

Answer: An observer does not have the gift of temporal and/or spatial ubiquity; He has to do with values at the actual time t and position p_t where he is (Newton and Einstein's point of view). So, when he was at t_0 at p_{t_0} the observer wrote the value $\theta_{t_0}(p_{t_0})$ on a piece of paper (for memory), puts the piece of paper is his pocket, then once at t at $p(t) = \Phi^{t_0}(t, p_{t_0})$, he takes the paper out of his pocket, and renames the value he reads as $\theta_{t_0 *}(t, p_t)$ because he is now at t at p_t. And, now at t at p_t, he can compare the past and present value. In particular the rate

$$\frac{\theta(t, p(t)) - \theta_{t_0 *}(t, p(t))}{t - t_0} = \frac{\text{actual}(t, p(t)) - \text{memory}_*(t, p(t))}{t - t_0} \tag{6.14}$$

is physically meaningful for one observer at t at p_t (no ubiquity gift required). For scalar value functions, we get the usual rate $\frac{\theta(t,p(t)) - \theta(t_0, p(t_0))}{t - t_0}$, but it isn't that simple for vector valued functions.

And the limit $t \to t_0$ in (6.14) defines the Lie derivative for scalar valued functions.

6.5 Push-forward and pull-back of vector fields

This is one of the most important concept for mechanical engineers.

6.5.1 A definition by approximation

Elementary introduction. Let $p_{\mathcal{E}}$ and $q_{\mathcal{E}}$ be points in $\mathcal{U}_{\mathcal{E}}$, and let $p_{\mathcal{F}} = p_{\mathcal{E}*} = \Psi(p_{\mathcal{E}})$ and $q_{\mathcal{F}} = q_{\mathcal{E}*} = \Psi(q_{\mathcal{E}})$ in $\mathcal{U}_{\mathcal{F}}$ be the push-forwards by Ψ cf. (6.1). The first order Taylor expansion gives

$$(\Psi(q_{\mathcal{E}}) - \Psi(p_{\mathcal{E}}) =) \quad q_{\mathcal{F}} - p_{\mathcal{F}} = d\Psi(p_{\mathcal{E}}).(q_{\mathcal{E}} - p_{\mathcal{E}}) + o(||q_{\mathcal{E}} - p_{\mathcal{E}}||_E), \tag{6.15}$$

thus,

$$\frac{\overrightarrow{p_{\mathcal{F}} q_{\mathcal{F}}}}{||\overrightarrow{p_{\mathcal{E}} q_{\mathcal{E}}}||_E} = d\Psi(p_{\mathcal{E}}).\frac{\overrightarrow{p_{\mathcal{E}} q_{\mathcal{E}}}}{||\overrightarrow{p_{\mathcal{E}} q_{\mathcal{E}}}||_E} + o(1). \tag{6.16}$$

And the definition of the push-forward is obtained by "neglecting" the $o(1)$ (limit as $q_{\mathcal{E}} \to p_{\mathcal{E}}$):

Definition 6.6 If $\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \in E$ is a vector at $p_{\mathcal{E}} \in U$ then its push-forward by Ψ is the vector $\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) =^{\text{noted}} \vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) =^{\text{noted}} \Psi_* \vec{w}_{\mathcal{E}}(p_{\mathcal{F}}) \in F$ defined at $p_{\mathcal{F}} = p_{\mathcal{E}*} = \Psi(p_{\mathcal{E}}) \in \mathcal{U}_{\mathcal{F}}$ by

$$\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) = \boxed{\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) := d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}})} \overset{\text{noted}}{=} \Psi_* \vec{w}_{\mathcal{E}}(p_{\mathcal{F}}). \tag{6.17}$$

6.5.2 The definition of the push-forward of a vector field

To fully grasp the definition, and to avoid making interpretation errors as in § 4.3 (the unfortunate notation $d\vec{x} = F.d\vec{X}$), we use the following definition of "a vector": It is a "tangent vector to a curve" (needed for surfaces and manifolds). Details:

• Let $c_{\mathcal E} : \left\{\begin{array}{ll}]-\varepsilon,\varepsilon[& \to \mathcal{U}_{\mathcal E} \\ s & \to p_{\mathcal E} = c_{\mathcal E}(s) \end{array}\right\}$ be a C^1 curve in $\mathcal{U}_{\mathcal E}$. Its tangent vector at $p_{\mathcal E} = c_{\mathcal E}(s)$ is

$$\vec{w}_{\mathcal E}(p_{\mathcal E}) := c_{\mathcal E}{}'(s) \quad (= \lim_{h\to 0} \frac{c_{\mathcal E}(s+h) - c_{\mathcal E}(s)}{h}), \tag{6.18}$$

see figure 6.1. This defines the function $\vec{w}_{\mathcal E} : \left\{\begin{array}{ll} \mathrm{Im}(c_{\mathcal E}) & \to E \\ p_{\mathcal E} & \to \vec{w}_{\mathcal E}(p_{\mathcal E}) \end{array}\right\}$ called a vector field along $\mathrm{Im}(c_{\mathcal E})\subset\mathcal{U}_{\mathcal E}$.

• The push-forward of $c_{\mathcal E}$ by Ψ being the image curve $c_{\mathcal E *} = \Psi \circ c_{\mathcal E}$ (the curve transformed by Ψ) cf. (6.4), its tangent vector at $p_{\mathcal F} = c_{\mathcal E *}(s)$ is

$$\vec{w}_{\mathcal E *}(p_{\mathcal F}) := c_{\mathcal E *}{}'(s) \quad \text{thus} \quad = d\Psi(p_{\mathcal E}).c_{\mathcal E}{}'(s) = d\Psi(p_{\mathcal E}).\vec{w}_{\mathcal E}(p_{\mathcal E}). \tag{6.19}$$

Thus we have defined the vector field $\vec{w}_{\mathcal E *}$ along $\mathrm{Im}(c_{\mathcal E *})$ called the push-forward of $\vec{w}_{\mathcal E}$ by Ψ.

With all the integral curves of a vector field defined in $\mathcal{U}_{\mathcal E}$, we get:

Definition 6.7 The push-forward by Ψ of a C^0 vector field $\vec{w}_{\mathcal E} : \left\{\begin{array}{ll} \mathcal{U}_{\mathcal E} & \to E \\ p_{\mathcal E} & \to \vec{w}_{\mathcal E}(p_{\mathcal E}) \end{array}\right\}$ is the vector field

$$\Psi_*\vec{w}_{\mathcal E} = \vec{w}_{\mathcal E *} : \left\{\begin{array}{ll} \mathcal{U}_{\mathcal F} & \to F \\ p_{\mathcal F} & \to \boxed{\Psi_*\vec{w}_{\mathcal E}(p_{\mathcal F}) := d\Psi(p_{\mathcal E}).\vec{w}_{\mathcal E}(p_{\mathcal E})} \overset{\text{noted}}{=} \vec{w}_{\mathcal E *}(p_{\mathcal F}) \quad \text{when} \quad p_{\mathcal F} = \Psi(p_{\mathcal E}), \end{array}\right. \tag{6.20}$$

see figure 6.1. ($\Psi_*\vec{w}_{\mathcal E} =^{\text{noted}} \vec{w}_{\mathcal E *}$ if Ψ is implicit). In other words,

$$\Psi_*\vec{w}_{\mathcal E} := (d\Psi.\vec{w}_{\mathcal E}) \circ \Psi^{-1}. \tag{6.21}$$

This defines the map $\Psi_* : \left\{\begin{array}{ll} C^\infty(\mathcal{U}_{\mathcal E}; E) & \to C^\infty(\mathcal{U}_{\mathcal F}; F) \\ \vec{w}_{\mathcal E} & \to \Psi_*(\vec{w}_{\mathcal E}) := \Psi_*\vec{w}_{\mathcal E} = \vec{w}_{\mathcal E *} \end{array}\right\}$. (We use the same notation Ψ_* as in definition 6.4 for scalar valued functions: The context removes ambiguity.)

Remark 6.8 Unlike scalar functions, cf. § 6.4.2: At t_0 at p_{t_0} you cannot just draw a vector $\vec{w}_{t_0}(p_{t_0})$ on a piece of paper, put the paper in your pocket, then let yourself be carried by the flow $\Psi = \Phi_t^{t_0}$ (push-forward), then, once arrived at t at p_t, take the paper out of your pocket and read it to get the push-forward: The direction and length of the vector $\vec{w}_{t_0 *}(t, p_t)$ are modified by the flow (a vector is not just a collection of scalar components). ∴

Exercice 6.9 Prove:

$$\vec{c_{\mathcal E}}''(s) = d\vec{w}_{\mathcal E}(p_{\mathcal E}).\vec{w}_{\mathcal E}(p_{\mathcal E}), \tag{6.22}$$

and

$$d\vec{w}_{\mathcal E *}(p_{\mathcal F}).d\Psi(p_{\mathcal E}) = d\Psi(p_{\mathcal E}).d\vec{w}_{\mathcal E}(p_{\mathcal E}) + d^2\Psi(p_{\mathcal E}).\vec{w}_{\mathcal E}(p_{\mathcal E}), \tag{6.23}$$

and

$$c_{\mathcal E *}{}''(s) = d\vec{w}_{\mathcal E *}(p_{\mathcal F}).\vec{w}_{\mathcal E *}(p_{\mathcal F}) \quad (= d\Psi(p_{\mathcal E}).\vec{c_{\mathcal E}}''(s) + d^2\Psi(p_{\mathcal E}).\vec{c_{\mathcal E}}'(s).\vec{c_{\mathcal E}}'(s)). \tag{6.24}$$

Answer. $\vec{c_{\mathcal E}}'(s) = \vec{w}_{\mathcal E}(c_{\mathcal E}(s))$ gives $\vec{c_{\mathcal E}}''(s) = d\vec{w}_{\mathcal E}(c_{\mathcal E}(s)).\vec{c_{\mathcal E}}'(s)$, hence (6.22).

$\vec{w}_{\mathcal E *}(\Psi(p_{\mathcal E})) = d\Phi(p_{\mathcal E}).\vec{w}_{\mathcal E}(p_{\mathcal E})$ by definition of $\vec{w}_{\mathcal E *}$, hence (6.23).

$c_{\mathcal F}(s) = \Psi(c_{\mathcal E}(s))$ gives $\vec{c_{\mathcal F}}'(s) = d\Psi(c_{\mathcal E}(s)).\vec{c_{\mathcal E}}'(s) = d\Psi(c_{\mathcal E}(s)).\vec{w}_{\mathcal E}(c_{\mathcal E}(s)) = \vec{w}_{\mathcal E *}(c_{\mathcal F}(s))$. Thus $\vec{c_{\mathcal F}}''(s) = (d^2\Psi(c_{\mathcal E}(s)).\vec{c_{\mathcal E}}'(s)).\vec{c_{\mathcal E}}'(s) + d\Psi(c_{\mathcal E}(s)).\vec{c_{\mathcal E}}''(s) = d\vec{w}_{\mathcal E *}(c_{\mathcal F}(s)).\vec{c_{\mathcal F}}'(s)$, hence (6.24). ∴

6.5.3 Pull-back of a vector field

Definition 6.10 If $\vec{w}_{\mathcal F} : \left\{\begin{array}{ll} \mathcal{U}_{\mathcal F} & \to F \\ p_{\mathcal F} & \to \vec{w}_{\mathcal F}(p_{\mathcal F}) \end{array}\right\}$ is a vector field on $\mathcal{U}_{\mathcal F}$, then its pull-back by Ψ is the push-forward by Ψ^{-1}, i.e. is the vector field on $\mathcal{U}_{\mathcal E}$ defined by

$$\Psi^*\vec{w}_{\mathcal F} : \left\{\begin{array}{ll} \mathcal{U}_{\mathcal E} & \to E \\ p_{\mathcal E} & \to \boxed{\Psi^*\vec{w}_{\mathcal F}(p_{\mathcal E}) := d\Psi^{-1}(p_{\mathcal F}).\vec{w}_{\mathcal F}(p_{\mathcal F})} \overset{\text{noted}}{=} \vec{w}_{\mathcal F}{}^*(p_{\mathcal E}), \quad \text{when} \quad p_{\mathcal F} = \Psi(p_{\mathcal E}). \end{array}\right. \tag{6.25}$$

In other words,

$$\Psi^*\vec{w}_{\mathcal F} := (d\Psi^{-1}.\vec{w}_{\mathcal F}) \circ \Psi \overset{\text{noted}}{=} \vec{w}_{\mathcal F}{}^*. \tag{6.26}$$

And we get

$$\Psi^* \circ \Psi_* = I \quad \text{and} \quad \Psi_* \circ \Psi^* = I. \tag{6.27}$$

Indeed, $\Psi^*(\Psi_* \vec{w}_{\mathcal{E}})(p_{\mathcal{E}}) = d\Psi^{-1}(p_{\mathcal{F}}).\Psi_*\vec{w}_{\mathcal{E}}(p_{\mathcal{F}}) = d\Psi^{-1}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) = \vec{w}_{\mathcal{E}}(p_{\mathcal{E}})$, for all $p_{\mathcal{E}}$. Idem for the second equality. ∎

6.6 Quantification with bases

6.6.1 Usual result

$(\vec{a}_i)$ is a Cartesian basis in E, $O_{\mathcal{F}}$ and $(\vec{b}_i)$ are an origin in $\mathcal{F}$ and a Cartesian basis in F, $p_{\mathcal{E}} \in \mathcal{U}_{\mathcal{E}}$,

$$p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}) = O_{\mathcal{F}} + \sum_{i=1}^{n} \psi_i(p_{\mathcal{E}})\, \vec{b}_i, \quad \text{i.e.} \quad [\overrightarrow{O_{\mathcal{F}} p_{\mathcal{F}}}]_{|\vec{b}} = \begin{pmatrix} \psi_1(p_{\mathcal{E}}) \\ \vdots \\ \psi_n(p_{\mathcal{E}}) \end{pmatrix}. \tag{6.28}$$

Then, if $\vec{w}_{\mathcal{E}}$ is a vector field in $\mathcal{U}_{\mathcal{E}}$ and $\vec{w}_{\mathcal{E}} = \sum_i w_j \vec{a}_i$, we get $\Psi_* \vec{w}_{\mathcal{E}}(p_{\mathcal{F}}) = d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) = \sum_{i=1}^{n} (d\psi_i(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}))\, \vec{b}_i = \sum_{i,j=1}^{n} w_j(p_{\mathcal{E}})(d\psi_i(p_{\mathcal{E}}).\vec{a}_j)\, \vec{b}_i = \sum_{i,j=1}^{n} \frac{\partial \psi_i}{\partial x_j}(p_{\mathcal{E}}) w_j(p_{\mathcal{E}})\, \vec{b}_i$, so

$$[\Psi_* \vec{w}_{\mathcal{E}}(p_{\mathcal{F}})]_{|\vec{b}} = [d\Psi(p_{\mathcal{E}})]_{|\vec{a},\vec{b}} . [\vec{w}_{\mathcal{E}}(p_{\mathcal{E}})]_{|\vec{a}}, \tag{6.29}$$

where $[d\Psi(p_{\mathcal{E}})]_{|\vec{a},\vec{b}} = [d\psi_i(p_{\mathcal{E}}).\vec{a}_j] =^{\text{noted}} [\frac{\partial \psi_i}{\partial x_j}(p_{\mathcal{E}})]$ is the Jacobian matrix.

6.6.2 Example: Polar coordinate system

Example 6.11 Change of coordinate system interpreted as a push-forward: Paradigmatic example of the polar coordinate system (model generalized for the parametrization of any manifold).

Parametric Cartesian vector space $\mathbb{R} \times \mathbb{R} =^{\text{noted}} \vec{\mathbb{R}}^2_p = \{\vec{q} = (r, \theta)\}$, with its canonical basis $(\vec{a}_1, \vec{a}_2)$, and $\vec{q} = r\vec{a}_1 + \theta\vec{a}_2 =^{\text{noted}} (r, \theta)$, so $[\vec{q}]_{|\vec{a}} = \begin{pmatrix} r \\ \theta \end{pmatrix}$. Geometric affine space $\mathbb{R}^2$ (of positions), $p \in \mathbb{R}^2$, associated vector space $\vec{\mathbb{R}^2}$, $O \in \mathbb{R}^2$ (origin), $\vec{x} = \overrightarrow{Op}$, and a Euclidean basis $(\vec{b}_1, \vec{b}_2)$ in $\vec{\mathbb{R}^2}$. The "polar coordinate system" is the associated map $\Psi : \left\{ \begin{matrix} \vec{\mathbb{R}}^*_+ \times \mathbb{R} \subset \vec{\mathbb{R}}^2_p & \rightarrow \vec{\mathbb{R}^2} \\ \vec{q} = (r, \theta) & \rightarrow \vec{x} = \Psi(\vec{q}) = \Psi(r, \theta), \end{matrix} \right\}$ defined by

$$\vec{x} = \Psi(\vec{q}) := r\cos\theta\, \vec{b}_1 + r\sin\theta\, \vec{b}_2, \quad \text{i.e.} \quad [\vec{x}]_{|\vec{b}} = \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}. \tag{6.30}$$

The i-th coordinate line at $\vec{q}$ in $\vec{\mathbb{R}}^2_p$ (parametric space) is the straight line $\vec{c}_{\vec{q},i} : \left\{ \begin{matrix} \mathbb{R} & \rightarrow \vec{\mathbb{R}}^2_p \\ s & \rightarrow \vec{c}_{\vec{q},i}(s) = \vec{q} + s\vec{a}_i \end{matrix} \right\}$, and its tangent vector at $\vec{c}_{\vec{q},i}(s)$ is $\vec{c}_{\vec{q},i}{}'(s) = \vec{a}_i$ for all s. This line is transformed by Ψ into the curve $\Psi_*(c_{q,i}) = \Psi \circ \vec{c}_{\vec{q},i} =^{\text{noted}} c_{\vec{x},i} : \left\{ \begin{matrix} \mathbb{R} & \rightarrow \mathbb{R}^2 \\ s & \rightarrow c_{\vec{x},i}(s) = \Psi(\vec{q} + s\vec{a}_i) \end{matrix} \right\}$ (in particular $c_{\vec{x},i}(0) = \vec{x}$). So

$$[\overrightarrow{Oc_{\vec{x},1}(s)}]_{|\vec{b}} = \begin{pmatrix} (r{+}s)\cos\theta \\ (r{+}s)\sin\theta \end{pmatrix} \quad \text{(straight line)}, \quad \text{and} \quad [\overrightarrow{Oc_{\vec{x},2}(s)}]_{|\vec{b}} = \begin{pmatrix} r\cos(\theta{+}s) \\ r\sin(\theta{+}s) \end{pmatrix} \quad \text{(circle)}. \tag{6.31}$$

And the tangent vector at $c_{\vec{x},i}(s)$ is $c_{\vec{x},i}{}'(s) =^{\text{noted}} \vec{a}_{i*}(\vec{x})$ (push-forward by Ψ), so

$$\begin{aligned} \vec{a}_{1*}(\vec{x}) &:= \Psi_* \vec{a}_1(\vec{x}) = d\Psi(\vec{q}).\vec{a}_1 = \lim_{h\to 0} \frac{\Psi(\vec{q}{+}h\vec{a}_1) - \Psi(\vec{q})}{h} = \lim_{h\to 0} \frac{\Psi(r{+}h, \theta) - \Psi(r, \theta)}{h} = \frac{\partial \Psi}{\partial r}(\vec{q}), \\ \vec{a}_{2*}(\vec{x}) &:= \Psi_* \vec{a}_2(\vec{x}) = d\Psi(\vec{q}).\vec{a}_2 = \lim_{h\to 0} \frac{\Psi(\vec{q}{+}h\vec{a}_2) - \Psi(\vec{q})}{h} = \lim_{h\to 0} \frac{\Psi(r, \theta{+}h) - \Psi(r, \theta)}{h} = \frac{\partial \Psi}{\partial \theta}(\vec{q}), \end{aligned} \tag{6.32}$$

Thus

$$\vec{a}_{1*}(\vec{x}) = \cos\theta \vec{b}_1 + \sin\theta \vec{b}_2 \quad \text{and} \quad \vec{a}_{2*}(\vec{x}) = -r\sin\theta \vec{b}_1 + r\cos\theta \vec{b}_2, \tag{6.33}$$

i.e.

$$[\vec{a}_{1*}(\vec{x})]_{|\vec{b}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad [\vec{a}_{2*}(\vec{x})]_{|\vec{b}} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}. \tag{6.34}$$

The basis $(\vec{a}_{1*}(\vec{x}), \vec{a}_{2*}(\vec{x}))$ is called the basis of the polar coordinate system at $\vec{x}$ (it is orthogonal but not orthonormal since $||\vec{a}_{2*}(\vec{x})|| = r \neq 1$ in general); And $[d\Psi(\vec{q})]_{|\vec{a},\vec{b}} = \big([\frac{\partial \Psi}{\partial r}(\vec{q})]_{|\vec{b}} \;\; [\frac{\partial \Psi}{\partial \theta}(\vec{q})]_{|\vec{b}} \big) = \big([\vec{a}_{1*}(\vec{x})]_{|\vec{b}} \;\; [\vec{a}_{2*}(\vec{x})]_{|\vec{b}} \big) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = [\frac{\partial \Psi^i}{\partial q^j}(\vec{q})]$ is the Jacobian matrix of Ψ at $\vec{q}$.

And the dual basis of the polar system basis $(\vec{a}_{1*}(\vec{x}),\vec{a}_{2*}(\vec{x}))$ is called $(dq_1(\vec{x}),dq_2(\vec{x}))$ (defined by $dq_i(\vec{x}).\vec{a}_{j*}(\vec{x})=\delta_{ij}$), so

$$dq_1(\vec{x})=\cos\theta\,dx_1+\sin\theta\,dx_2 \quad\text{and}\quad dq_2(\vec{x})=-\frac{1}{r}\sin\theta\,dx_1+\frac{1}{r}\cos\theta\,dx_2, \tag{6.35}$$

i.e. $[dq_1(\vec{x})]_{|\vec{b}}=(\cos\theta \quad \sin\theta)$ and $[dq_2(\vec{x})]_{|\vec{b}}=-\frac{1}{r}(\sin\theta \quad \cos\theta)$ (row matrices) when $\vec{x}=\Psi(\vec{q})$. ∴

Remark 6.12 The components $\gamma_{ij}^k(\vec{x})$ of the vector $d\vec{a}_{j*}(\vec{x}).\vec{a}_{i*}(\vec{x})\in\vec{\mathbb{R}}^2$ in the basis $(\vec{a}_{i*}(\vec{x}))$ are the Christoffel symbols of the polar coordinate system (with duality notations as it is usually presented):

$$d\vec{a}_{j*}(\vec{x}).\vec{a}_{i*}(\vec{x})=\sum_{k=1}^{n}\gamma_{ij}^k(\vec{x})\vec{a}_{k*}(\vec{x}). \tag{6.36}$$

At $\vec{x}=\Psi(\vec{q})$, with $\vec{a}_{j*}(\vec{x})=d\Psi(\vec{q}).\vec{a}_j$, i.e. $(\vec{a}_{j*}\circ\Psi)(\vec{q})=\frac{\partial\Psi}{\partial q^j}$, we get

$$d\vec{a}_{j*}(\vec{x}).\vec{a}_{i*}(\vec{x})=\frac{\partial^2\Psi}{\partial q^i\partial q^j}(\vec{q})=d\vec{a}_{i*}(\vec{x}).\vec{a}_{j*}(\vec{x}), \quad\text{so}\quad \gamma_{ij}^k=\gamma_{ji}^k \tag{6.37}$$

for all i,j (symmetry of the bottom indices as soon as Ψ is C^2).

Here for the polar coordinates, $\frac{\partial\Psi}{\partial r}(\vec{q})=\cos\theta\vec{b}_1+\sin\theta\vec{b}_2$ gives $\frac{\partial^2\Psi}{\partial r^2}(\vec{q})=\vec{0}$, thus $\gamma_{11}^1=\gamma_{11}^2=0$, and $\frac{\partial^2\Psi}{\partial\theta\partial r}(\vec{q})=-\sin\theta\vec{b}_1+\cos\theta\vec{b}_2=\frac{1}{r}\vec{a}_{2*}(\vec{x})$, thus $\gamma_{12}^1=0=\gamma_{21}^1$ and $\gamma_{12}^2=\frac{1}{r}=\gamma_{21}^2$. And $\frac{\partial\Psi}{\partial\theta}(\vec{q})=-r\sin\theta\vec{b}_1+r\cos\theta\vec{b}_2$ gives $\frac{\partial^2\Psi}{\partial\theta^2}(\vec{q})=-r\cos\theta\vec{b}_1-r\sin\theta\vec{b}_2=-r\vec{a}_{1*}(\vec{x})$, thus $\gamma_{22}^1=-r$ and $\gamma_{22}^2=0$. ∴

Remark 6.13 The (widely used) normalized polar coordinate basis $(\vec{n}_1(\vec{x}),\vec{n}_2(\vec{x}))=(\vec{a}_{1*}(\vec{x}),\frac{1}{r}\vec{a}_{2*}(\vec{x}))$ is not holonomic, i.e. is not the basis of a coordinate system (and its use makes higher derivation formulas complicated). Indeed $\vec{n}_2(\vec{x})=\frac{1}{r}\vec{a}_{2*}(\vec{x})$ gives $d\vec{n}_2(\vec{x}).\vec{n}_1(\vec{x})=(d(\frac{1}{r})(\vec{x}).\vec{n}_1(\vec{x}))\vec{a}_{2*}(\vec{x})+\frac{1}{r}d\vec{a}_{2*}(\vec{x}).\vec{n}_1(\vec{x})$, and $\vec{n}_1(\vec{x})=\vec{a}_{1*}(\vec{x})$ gives $d\vec{n}_1(\vec{x}).\vec{n}_2(\vec{x})=d\vec{a}_{1*}(\vec{x}).(\frac{1}{r}\vec{a}_{2*})$, thus $d\vec{n}_2(\vec{x}).\vec{n}_1(\vec{x})-d\vec{n}_1(\vec{x}).\vec{n}_2(\vec{x})=(d(\frac{1}{r})(\vec{x}).\vec{n}_1(\vec{x}))\vec{a}_{2*}(\vec{x})\neq\vec{0}$, since $\frac{1}{r}=(x^2+y^2)^{-\frac{1}{2}}$ gives $d(\frac{1}{r})(\vec{x}).\vec{n}_1(\vec{x})=(-x(x^2+y^2)^{-\frac{3}{2}} \quad -y(x^2+y^2)^{-\frac{3}{2}}).\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}=\frac{1}{r^3}(-r\cos^2\theta-r\sin^2\theta)=\frac{-1}{r^2}\neq 0$. ∴

Remark 6.14 (Pay attention to the notations.) Let $f:\vec{q}\in\vec{\mathbb{R}}^2_p\to f(\vec{q})\in\mathbb{R}$ be C^2. Call g its push-forward by Ψ, i.e. $g:\vec{x}\in\mathbb{R}^2\to g(\vec{x})=f(\vec{q})\in\mathbb{R}$ when $\vec{x}=\Psi(\vec{q})$. So $f(\vec{q})=(g\circ\Psi)(\vec{q})$and

$$df(\vec{q}).\vec{a}_j=dg(\Psi(\vec{q})).d\Psi(\vec{q}).\vec{a}_j=dg(\vec{x}).\vec{a}_{j*}(\vec{x}). \tag{6.38}$$

With $df(\vec{q}).\vec{a}_j=^{\text{noted}}\frac{\partial f}{\partial q^j}(\vec{q})$ and $dg(\vec{x}).\vec{b}_j=^{\text{noted}}\frac{\partial g}{\partial x^j}(\vec{x})$ and $\vec{a}_{j*}(\vec{x})=d\Psi(\vec{q}).\vec{a}_j=\sum_i\frac{\partial\Psi^i}{\partial q^j}(\vec{q})\vec{a}_j$, we get

$$\frac{\partial f}{\partial q^j}(\vec{q})=\sum_i\frac{\partial g}{\partial x^i}(\vec{x})\frac{\partial\Psi^i}{\partial q^j}(\vec{q})\overset{\text{noted}}{=}\frac{\partial g}{\partial q^j}(\vec{x}) \;\text{... (!!)} \tag{6.39}$$

Mind this notation!! g is a function of $\vec{x}$, not of $\vec{q}$, so $\frac{\partial g}{\partial q^i}(\vec{x})\overset{\text{means}}{=}\frac{\partial f}{\partial q^i}(\vec{q})$, i.e. $\frac{\partial g}{\partial q^i}(\vec{x})\overset{\text{means}}{=}\frac{\partial(g\circ\Psi)}{\partial q^i}(\vec{q})$... which is $[df(\vec{q})]=[dg(\vec{x})].[d\Psi(\vec{q})$...

Then (with f and Ψ C^2)

$$\begin{aligned}\frac{\partial\frac{\partial g}{\partial q^i}}{\partial q^j}(\vec{x})\overset{\text{means}}{=}\frac{\partial\frac{\partial(g\circ\Psi)}{\partial q^i}}{\partial q^j}(\vec{q})&=d(dg.\vec{a}_{i*})(\vec{x}).d\Psi(\vec{q}).\vec{a}_j=d(dg.\vec{a}_{i*})(\vec{x}).\vec{a}_{j^*}(\vec{x})\\ &=d((dg(\vec{x}).\vec{a}_{j^*}(\vec{x})).\vec{a}_{i*}(\vec{x})+dg(\vec{x}).(d\vec{a}_{i*}(\vec{x}).\vec{a}_j(\vec{x}))\overset{\text{noted}}{=}\frac{\partial^2 g}{\partial q^i\partial q^j}(\vec{x}).\end{aligned} \tag{6.40}$$

So

$$\frac{\partial^2 g}{\partial q^i\partial q^j}(\vec{x})\overset{\text{means}}{=}d^2g(\vec{x})(\vec{a}_{i*}(\vec{x}),\vec{a}_{j*}(\vec{x}))+\sum_{k=1}^{n}\frac{\partial g}{\partial x^k}(\vec{x})\gamma_{ij}^k(\vec{x})\vec{a}_k(\vec{x}), \tag{6.41}$$

and $\frac{\partial^2 g}{\partial q^i\partial q^j}(\vec{x})$ is ***not*** reduced to $d^2g(\vec{x})(\vec{a}_{i*}(\vec{x}),\vec{a}_{j*}(\vec{x}))$ (the Christoffel symbols have appeared): First order derivatives $\frac{\partial g}{\partial x^k}$ are still alive. (Contrary to $\frac{\partial^2 g}{\partial x^i\partial x^j}(\vec{x})=d^2g(\vec{x})(\vec{b}_i,\vec{b}_j)$ with a Cartesian basis $(\vec{b}_i)$.)

NB: The independent variables r and θ don't have the same dimension (a length and an angle): There is no physical meaningful inner dot product in the parameter space $\vec{\mathbb{R}}^2_p=\mathbb{R}\times\mathbb{R}=\{(r,\theta)\}$, but this space is very useful... (As in thermodynamics: No meaningful inner dot product in the (T,P) space.) ∴

7 Push-forward and pull-back of differential forms

7.1 Definition

Setting of § 6.1. Consider a differential form $\alpha_\mathcal{E} : \left\{\begin{array}{ll} \mathcal{U}_\mathcal{E} & \to E^* = \mathcal{L}(E;\mathbb{R}) \\ p_\mathcal{E} & \to \alpha_\mathcal{E}(p_\mathcal{E}) \end{array}\right\}$ on $\mathcal{U}_\mathcal{E}$ (a field of linear forms), and a vector field $\vec{w}_\mathcal{E} : \left\{\begin{array}{ll} \mathcal{U}_\mathcal{E} & \to E \\ p_\mathcal{E} & \to \vec{w}_\mathcal{E}(p_\mathcal{E}) \end{array}\right\}$. Hence

$$f_\mathcal{E} = \alpha_\mathcal{E}.\vec{w}_\mathcal{E} : \left\{\begin{array}{ll} \mathcal{U}_\mathcal{E} & \to \mathbb{R} \\ p_\mathcal{E} & \to f_\mathcal{E}(p_\mathcal{E}) = (\alpha.\vec{w}_\mathcal{E})(p_\mathcal{E}) = \alpha_\mathcal{E}(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E}) \end{array}\right.$$

is a scalar valued function (value of $\vec{w}_\mathcal{E}$ given by $\alpha_\mathcal{E}$). And (6.8) gives (push-forward $f_\mathcal{E} = \alpha_\mathcal{E}.\vec{w}_\mathcal{E}$ by Ψ)

$$\Psi_*(\alpha_\mathcal{E}.\vec{w}_\mathcal{E})(p_\mathcal{F}) = (\alpha_\mathcal{E}.\vec{w}_\mathcal{E})(p_\mathcal{E}) = \alpha_\mathcal{E}(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E}) \quad \text{when} \quad p_\mathcal{F} = \Psi(p_\mathcal{E}). \tag{7.1}$$

With $\vec{w}_{\mathcal{E}*}(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E})$ cf. (6.20) (push-forward of $\vec{w}_\mathcal{E}$), we get

$$\Psi_*(\alpha_\mathcal{E}.\vec{w}_\mathcal{E})(p_\mathcal{F}) = \underbrace{\alpha_\mathcal{E}(p_\mathcal{E}).d\Psi(p_\mathcal{E})^{-1}}_{=\text{noted } \alpha_{\mathcal{E}*}(p_\mathcal{F})}.\vec{w}_\mathcal{F}(p_\mathcal{F}) \quad \text{when} \quad p_\mathcal{F} = \Psi(p_\mathcal{E}) \ : \tag{7.2}$$

Definition 7.1 The push-forward of a differential form $\alpha_\mathcal{E} \in \Omega^1(\mathcal{U}_\mathcal{E})$ is the differential form $\in \Omega^1(\mathcal{U}_\mathcal{F})$ given by

$$\Psi_*\alpha_\mathcal{E} : \left\{\begin{array}{ll} \mathcal{U}_\mathcal{F} & \to F^* = \mathcal{L}(F;\mathbb{R}) \\ p_\mathcal{F} & \to \boxed{\Psi_*\alpha_\mathcal{E}(p_\mathcal{F}) := \alpha_\mathcal{E}(p_\mathcal{E}).d\Psi(p_\mathcal{E})^{-1}} \overset{\text{noted}}{=} \alpha_{\mathcal{E}*}(p_\mathcal{F}) \quad \text{when} \quad p_\mathcal{F} = \Psi(p_\mathcal{E}), \end{array}\right. \tag{7.3}$$

the last notation when Ψ is implicit. In other words, $\Psi_*\alpha_\mathcal{E}(p_\mathcal{F}) = \alpha_\mathcal{E}(\Psi^{-1}(p_\mathcal{F})).d\Psi^{-1}(p_\mathcal{F})$, i.e.

$$\Psi_*\alpha_\mathcal{E} := (\alpha_\mathcal{E} \circ \Psi^{-1}).d\Psi^{-1}. \tag{7.4}$$

(Once again, we used the same notation Ψ_* than for the push-forward of vector fields and functions: The context removes any ambiguities.)

Remark 7.2 We cannot always see a vector field (e.g. we can't see an internal force field): To know it we need to measure it with a well defined tool, the tool being here a differential form; And the definition 7.1 is a compatbility definition so that we can recover the push-forward of the vector field. ⁂

Definition 7.3 The pull-forward of a a differential form $\alpha_\mathcal{F} \in \Omega^1(\mathcal{U}_\mathcal{F})$ is the differential form

$$\Psi^*\alpha_\mathcal{F} : \left\{\begin{array}{ll} \mathcal{U}_\mathcal{E} & \to \mathcal{L}(E;\mathbb{R}) \\ p_\mathcal{E} & \to \Psi^*\alpha_\mathcal{F}(p_\mathcal{E}) := \alpha_\mathcal{F}(p_\mathcal{F}).d\Psi(p_\mathcal{E}) \overset{\text{noted}}{=} {\alpha_\mathcal{F}}^*(p_\mathcal{E}) \quad \text{when} \quad p_\mathcal{F} = \Psi(p_\mathcal{E}), \end{array}\right. \tag{7.5}$$

In other words,

$$\Psi^*\alpha_\mathcal{F} := (\alpha_\mathcal{F} \circ \Psi).d\Psi. \tag{7.6}$$

(For an alternative definition, see remark 7.5.)

Proposition 7.4 *For all $\alpha_\mathcal{E} \in \Omega^1(\mathcal{U}_\mathcal{E})$ and $\alpha_\mathcal{F} \in \Omega^1(\mathcal{U}_\mathcal{F})$ (differential forms), and $\vec{w}_\mathcal{E} \in \Gamma(\mathcal{U}_\mathcal{E})$ and $\vec{w}_\mathcal{F} \in \Gamma(\mathcal{U}_\mathcal{F})$ (vector fields), we have (objectivity result)*

$$(\Psi_*\alpha_\mathcal{E})(p_\mathcal{F}).\vec{w}_\mathcal{F}(p_\mathcal{F}) = \alpha_\mathcal{E}(p_\mathcal{E}).(\Psi_*\vec{w}_\mathcal{F})(p_\mathcal{E}) \quad \text{when} \quad p_\mathcal{F} = \Psi(p_\mathcal{E}), \tag{7.7}$$

i.e. $\alpha_{\mathcal{E}*}(p_\mathcal{F}).\vec{w}_\mathcal{F}(p_\mathcal{F}) = \alpha_\mathcal{E}(p_\mathcal{E}).{\vec{w}_\mathcal{F}}^*(p_\mathcal{E})$. In particular with $\alpha_\mathcal{E} = df$ (exact differential form) where $f \in C^1(\mathcal{U}_\mathcal{E};\mathbb{R})$,

$$d(\Psi_*f) = \Psi_*(df). \tag{7.8}$$

(This commutativity result is very particular to the case $\alpha = df$: In general $d(\Psi_*T) \neq \Psi_*(dT)$ for a tensor of order ≥ 2, see e.g. (8.19)).

Proof. $\alpha_{\mathcal{E}*}(p_{\mathcal{F}}).\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) = (\alpha_{\mathcal{E}}(p_{\mathcal{E}}).d\Psi^{-1}(p_{\mathcal{F}})).\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) = \alpha_{\mathcal{E}}(p_{\mathcal{E}}).(d\Psi^{-1}(p_{\mathcal{F}}).\vec{w}_{\mathcal{F}}(p_{\mathcal{F}})) = \alpha_{\mathcal{E}}(p_{\mathcal{E}}).\vec{w}_{\mathcal{F}}^*(p_{\mathcal{E}})$, for all $p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}) \in \mathcal{U}_{\mathcal{F}}$.

And $\Psi_* f(p_{\mathcal{F}}) := f(p_{\mathcal{E}}) = f(\Psi^{-1}(p_{\mathcal{F}}))$, thus $d(\Psi_* f)(p_{\mathcal{F}}) = df(p_{\mathcal{E}}).d\Psi^{-1}(p_{\mathcal{F}}) = \Psi_*(df)(p_{\mathcal{F}})$. ∎

And we have

$$\Psi^* \circ \Psi_* = I \quad \text{and} \quad \Psi_* \circ \Psi^* = I. \tag{7.9}$$

Indeed $\Psi^*(\Psi_*\alpha_{\mathcal{E}})(p_{\mathcal{E}}) = \Psi_*\alpha_{\mathcal{E}}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}) = \alpha_{\mathcal{E}}(p_{\mathcal{E}}).d\Psi^{-1}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}) = \alpha_{\mathcal{E}}(p_{\mathcal{E}})$. Idem for $\Psi_* \circ \Psi^* = I$.

Remark 7.5 The pull-back $\alpha_{\mathcal{F}}{}^*$ can also be defined thanks to the natural canonical isomorphism $\left\{\begin{array}{ll} \mathcal{L}(E;F) & \to \mathcal{L}(F^*;E^*) \\ L & \to L^* \end{array}\right\}$ given by $L^*(\ell_F).\vec{u}_E = \ell_F.(L.\vec{u}_E)$ for all $(\vec{u}_E, \ell_F) \in E \times F^*$, and $L^*(\ell_F) = \ell_F.L$ is called the pull-back of ℓ_F by L. In particular with $\ell_F = \alpha_{\mathcal{F}}(p_{\mathcal{F}})$ and $L = d\Psi(p_{\mathcal{E}})$ we get $d\Psi(p_{\mathcal{E}})^*(\alpha_{\mathcal{F}}(p_{\mathcal{F}})) = \alpha_{\mathcal{F}}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}})$, i.e. (7.5). ∎

7.2 Incompatibility: Riesz representation and push-forward

A push-forward is independent of any inner dot product: It is objective.

But here we introduce inner dot products $(\cdot,\cdot)_g$ in E and $(\cdot,\cdot)_h$ in F, e.g. Euclidean dot products in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$ (observer dependent therefore subjective), because some mechanical engineers can't begin with their beloved Euclidean dot products.

Let $\alpha_{\mathcal{E}} \in \Omega^1(\mathcal{U}_{\mathcal{E}})$ and call $\beta_{\mathcal{F}} := \Psi_*\alpha_{\mathcal{E}}$ its push-forward by Ψ, i.e.

$$\beta_{\mathcal{F}}(p_{\mathcal{F}}) := \alpha_{\mathcal{E}}(p_{\mathcal{E}}).d\Psi(p_{\mathcal{E}})^{-1} \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}). \tag{7.10}$$

Then call $\vec{a}_g(p_{\mathcal{E}}) \in E$ and $\vec{b}_h(p_{\mathcal{F}}) \in F$ the $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$-Riesz representation vectors of $\alpha_{\mathcal{E}}$ and $\beta_{\mathcal{F}}$, so, for all $\vec{u}_{\mathcal{E}} \in \Gamma(\mathcal{U}_{\mathcal{E}})$ and all $\vec{w}_{\mathcal{F}} \in \Gamma(\mathcal{U}_{\mathcal{F}})$, in short,

$$\alpha_{\mathcal{E}}.\vec{u}_{\mathcal{E}} = (\vec{a}_g, \vec{u}_{\mathcal{E}})_g, \quad \text{and} \quad \beta_{\mathcal{F}}.\vec{w}_{\mathcal{F}} = (\vec{b}_h, \vec{w}_{\mathcal{F}})_h, \tag{7.11}$$

which means $\alpha_{\mathcal{E}}(p_{\mathcal{E}}).\vec{u}_{\mathcal{E}}(p_{\mathcal{E}}) = (\vec{a}_g(p_{\mathcal{E}}), \vec{u}_{\mathcal{E}}(p_{\mathcal{E}}))_g$ and $\beta_{\mathcal{F}}(p_{\mathcal{F}}).\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) = (\vec{b}_h(p_{\mathcal{F}}), \vec{w}_{\mathcal{F}}(p_{\mathcal{F}}))_h$, for all $p_{\mathcal{E}} \in \mathcal{U}_{\mathcal{E}}$ and $p_{\mathcal{F}} \in \mathcal{U}_{\mathcal{F}}$. This defines the vector fields $\vec{a}_g \in \Gamma(\mathcal{U}_{\mathcal{E}})$ and $\vec{b}_h \in \Gamma(\mathcal{U}_{\mathcal{F}})$.

Proposition 7.6 *$\vec{b}_h \neq \Psi_*\vec{a}_g$ in general (although $\beta_{\mathcal{F}} = \Psi_*\alpha_{\mathcal{E}}$), because*

$$\begin{aligned} \vec{b}_h(p_{\mathcal{F}}) &= d\Psi(p_{\mathcal{E}})^{-T}.\vec{a}_g(p_{\mathcal{E}}) \\ &\neq d\Psi(p_{\mathcal{E}}).\vec{a}_g(p_{\mathcal{E}}) \ \text{ in general} \end{aligned} \tag{7.12}$$

(unless $d\Psi(p_{\mathcal{E}})^{-T} = d\Psi(p_{\mathcal{E}})$, i.e. $d\Psi(p_{\mathcal{E}})^T.d\Psi(p_{\mathcal{E}})^{-1} = I$, as a rigid body motion).

So the Riesz representation vector of the push-forwarded linear form is not the push-forwarded representation vector of the linear form push-forwarded.

This is not a surprise: A push-forward is independent of any inner dot product, while a Riesz representation vector depends on a chosen inner dot product (E.g. Euclidean foot? metre?).

So, as long as possible (not before you need to quantify), you should avoid using a Riesz representation vector, i.e. you should use the original (the qualitative differential form) as long as possible, and delay the use of a representative (quantification with which dot product?) as late as possible.

Proof. Recall: The transposed relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ of the linear map $d\Psi(p_{\mathcal{E}}) \in \mathcal{L}(E;F)$ is the linear map $d\Psi(p_{\mathcal{E}})^T_{gh} \stackrel{\text{noted}}{=} d\Psi(p_{\mathcal{E}})^T \in \mathcal{L}(F;E)$ defined by, for all $\vec{u}_{\mathcal{E}} \in E$ and $\vec{w}_{\mathcal{F}} \in F$ vectors at $p_{\mathcal{E}}$ and $p_{\mathcal{F}}$, cf. (A.68),

$$(d\Psi(p_{\mathcal{E}})^T.\vec{w}_{\mathcal{F}}, \vec{u}_{\mathcal{E}})_g = (\vec{w}_{\mathcal{F}}, d\Psi(p_{\mathcal{E}}).\vec{u}_{\mathcal{E}})_h. \tag{7.13}$$

(7.11) gives, with $p_{\mathcal{F}} = \Psi(p_{\mathcal{E}})$,

$$\begin{aligned} (\vec{a}_g(p_{\mathcal{E}}), \vec{u}_{\mathcal{E}})_g &= \alpha_{\mathcal{E}}(p_{\mathcal{E}}).\vec{u}_{\mathcal{E}} = \big(\beta_{\mathcal{F}}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}})\big).\vec{u}_{\mathcal{E}} = \beta_{\mathcal{F}}(p_{\mathcal{F}}).\big(d\Psi(p_{\mathcal{E}}).\vec{u}_{\mathcal{E}}\big) \\ &= (\vec{b}_h(p_{\mathcal{F}}), d\Psi(p_{\mathcal{E}}).\vec{u}_{\mathcal{E}})_h = (d\Psi(p_{\mathcal{E}})^T.\vec{b}_h(p_{\mathcal{F}}), \vec{u}_{\mathcal{E}})_g, \end{aligned} \tag{7.14}$$

true for all $\vec{u}_{\mathcal{E}}$, thus $\vec{a}_g(p_{\mathcal{E}}) = d\Psi(p_{\mathcal{E}})^T.\vec{b}_h(p_{\mathcal{F}})$, thus (7.12). ∎

8 Push-forward and pull-back of tensors

To lighten the presentation, we only deal with order 1 and 2 tensors. Similar approach for any tensor.

8.1 Push-forward and pull-back of order 1 tensors

Proposition 8.1 *If T is either a vector field or a differential form, then its push-forward satisfies, for all ξ vector field or differential form (when required) in $\mathcal{U}_\mathcal{F}$,*

$$\textit{in short:}\quad (\Psi_*T)(\xi)=T(\Psi^*\xi),\quad \textit{written}\quad \Psi_*T(.)=T(\Psi^*.),\tag{8.1}$$

i.e. $(\Psi_*T)(p_\mathcal{F}).\xi(p_\mathcal{F})=T(p_\mathcal{E}).\Psi^*\xi(p_\mathcal{E})$ *when* $p_\mathcal{F}=\Psi(p_\mathcal{E})$*. Similarly:*

$$\textit{in short:}\quad (\Psi^*T)(\xi)=T(\Psi_*\xi),\quad \textit{written}\quad \Psi^*T(.)=T(\Psi_*.),\tag{8.2}$$

i.e. $(\Psi^*T)(p_\mathcal{E}).\xi(p_\mathcal{E})=T(p_\mathcal{F}).\Psi_*\xi(p_\mathcal{F})$ *when* $p_\mathcal{F}=\Psi(p_\mathcal{E})$*.*

Proof. • Case $T=\alpha_\mathcal{E}\in\Omega^1(\mathcal{U}_\mathcal{E})$ (differential form = a $\binom{0}{1}$ tensor), then here $\xi=\vec{w}_\mathcal{F}\in\Gamma(\mathcal{U}_\mathcal{F})$ and we have to check: $(\Psi_*\alpha_\mathcal{E})(p_\mathcal{F}).\vec{w}_\mathcal{F}(p_\mathcal{F})=\alpha_\mathcal{E}(p_\mathcal{E}).\Psi^*\vec{w}_\mathcal{F}(p_\mathcal{E})$, i.e. $(\alpha_\mathcal{E}(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{E})).\vec{w}_\mathcal{F}(p_\mathcal{F})=\alpha_\mathcal{E}(p_\mathcal{E}).(d\Psi^{-1}(p_\mathcal{E}).\vec{w}_\mathcal{F}(p_\mathcal{F}))$: True.

• Case $T=\vec{w}_\mathcal{E}\in\Gamma(\mathcal{U}_\mathcal{E})$ (vector field $\simeq$ a $\binom{1}{0}$ tensor), then here $\xi=\alpha_\mathcal{F}\in\Omega^1(\mathcal{U}_\mathcal{F})$ we have to check: $(\Psi_*\vec{w}_\mathcal{E})(p_\mathcal{F}).\alpha_\mathcal{F}(p_\mathcal{F})=\vec{w}_\mathcal{E}(p_\mathcal{E}).\Psi^*(\alpha_\mathcal{F})(p_\mathcal{E})$, where we implicitly use to the natural canonical isomorphism $\mathcal{J}:\left\{\begin{matrix}E & \to E^{**}\\ \vec{w} & \to w\overset{\text{noted}}{=}\vec{w}\end{matrix}\right\}$ defined by $w(\ell)=\ell.\vec{w}$ for all $\ell\in E^*$. So we have to check: $\alpha_\mathcal{F}(p_\mathcal{F}).(\Psi_*\vec{w}_\mathcal{E})(p_\mathcal{F})=\Psi^*(\alpha_\mathcal{F})(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E})$, i.e. $\alpha_\mathcal{F}(p_\mathcal{F}).(d\Psi(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E}))=(\alpha_\mathcal{F}(p_\mathcal{F}).d\Psi(p_\mathcal{E})^{-1}).\vec{w}_\mathcal{E})(p_\mathcal{E})$: True.

For (8.2), use Ψ^{-1} instead of Ψ. ∎

8.2 Push-forward and pull-back of order 2 tensors

Definition 8.2 Let T be an order 2 tensor in $\mathcal{U}_\mathcal{E}$. Its push-forward by Ψ is the order 2 tensor Ψ_*T in $\mathcal{U}_\mathcal{F}$ defined by, for all ξ_1,ξ_2 vector field or differential form (when required) in $\mathcal{U}_\mathcal{F}$,

$$\text{in short:}\quad \Psi_*T(\xi_1,\xi_2):=T(\Psi^*\xi_1,\Psi^*\xi_2)\quad\text{written}\quad \Psi_*T(\cdot,\cdot):=T(\Psi^*\cdot,\Psi^*\cdot),\tag{8.3}$$

i.e. $\Psi_*T(p_\mathcal{F})(\xi_1(p_\mathcal{F}),\xi_2(p_\mathcal{F})):=T(p_\mathcal{E})(\Psi^*\xi_1(p_\mathcal{E}),\Psi^*\xi_2(p_\mathcal{E}))$ when $p_\mathcal{F}=\Psi(p_\mathcal{E})$.

Let T be an order 2 tensor in $\mathcal{U}_\mathcal{F}$. Its pull-back by Ψ is the order 2 tensor Ψ^*T in $\mathcal{U}_\mathcal{E}$ defined by, for all ξ_1,ξ_2 vector field or differential form (when required) in $\mathcal{U}_\mathcal{E}$,

$$\text{in short:}\quad \Psi^*T(\xi_1,\xi_2):=T(\Psi_*\xi_1,\Psi_*\xi_2)\quad\text{written}\quad \Psi^*T(\cdot,\cdot):=T(\Psi_*\cdot,\Psi_*\cdot),\tag{8.4}$$

i.e., $\Psi^*T(p_\mathcal{E})(\xi_1(p_\mathcal{E}),\xi_2(p_\mathcal{E})):=T(p_\mathcal{F})(\Psi_*\xi_1(p_\mathcal{F}),\Psi_*\xi_2(p_\mathcal{F}))$ when $p_\mathcal{F}=\Psi(p_\mathcal{E})$.

Example 8.3 If $T\in T^0_2(\mathcal{U}_\mathcal{E})$ (e.g., a metric) then, for all vector fields $\vec{w}_1,\vec{w}_2$ in $\mathcal{U}_\mathcal{F}$,

$$T_*(\vec{w}_1,\vec{w}_2)\overset{(8.3)}{=}T(\vec{w}_1{}^*,\vec{w}_2{}^*)=T(d\Psi^{-1}.\vec{w}_1,d\Psi^{-1}.\vec{w}_2),\tag{8.5}$$

i.e., $T_*(p_\mathcal{F})(\vec{w}_1(p_\mathcal{F}),\vec{w}_2(p_\mathcal{F}))=T(p_\mathcal{E})(d\Psi^{-1}(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F}),d\Psi^{-1}(p_\mathcal{F}).\vec{w}_2(p_\mathcal{F}))$ when $p_\mathcal{F}=\Psi(p_\mathcal{E})$.

Expression with bases $(\vec{a}_i)$ in E and $(\vec{b}_i)$ in F: In short we have $(T_*)_{ij}=T_*(\vec{b}_i,\vec{b}_j)=T(\vec{b}_i{}^*,\vec{b}_j{}^*)=[\vec{b}^*_i]^T_{|\vec{a}}.[T]_{|\vec{a}}.[\vec{b}^*_j]_{|\vec{a}}=([\vec{b}_i]^T_{|\vec{b}}.[d\Psi]^{-T}_{|\vec{a},\vec{b}}).[T]_{|\vec{a}}.([d\Psi]^{-1}_{|\vec{a},\vec{b}}.[\vec{b}_j]_{|\vec{b}})=([d\Psi]^{-T}_{|\vec{a},\vec{b}}.[T]_{|\vec{a}}.[d\Psi]^{-1}_{|\vec{a},\vec{b}})_{ij}$, thus

$$[T_*]_{|\vec{b}}=[d\Psi]^{-T}_{|\vec{a},\vec{b}}.[T]_{|\vec{a}}.[d\Psi]^{-1}_{|\vec{a},\vec{b}},\tag{8.6}$$

which means $[(\Psi_*T)(p_\mathcal{F})]_{|\vec{b}}=([d\Psi(p_\mathcal{E})]_{|\vec{a},\vec{b}})^{-T}.[T(p_\mathcal{E})]_{|\vec{a}}.([d\Psi(p_\mathcal{E})]_{|\vec{a},\vec{b}})^{-1}$ when $p_\mathcal{F}=\Psi(p_\mathcal{E})$.

Particular case of an elementary tensor $T=\alpha_1\otimes\alpha_2\in T^0_2(\mathcal{U}_\mathcal{E})$, where $\alpha_1,\alpha_2\in\Omega^1(\mathcal{U}_\mathcal{E})$, so $T(\vec{u}_1,\vec{u}_2)=(\alpha_1\otimes\alpha_2)(\vec{u}_1,\vec{u}_2)=(\alpha_1.\vec{u}_1)(\alpha_2.\vec{u}_2)$: For all $\vec{w}_1,\vec{w}_2\in\Gamma(\mathcal{U}_\mathcal{F})$,

$$(\alpha_1\otimes\alpha_2)_*(\vec{w}_1,\vec{w}_2)\overset{(8.3)}{=}(\alpha_1\otimes\alpha_2)(\vec{w}^*_1,\vec{w}^*_2)=(\alpha_1.\vec{w}^*_1)(\alpha_2.\vec{w}^*_2)\overset{(7.7)}{=}(\alpha_{1*}.\vec{w}_1)(\alpha_{2*}.\vec{w}_2),\tag{8.7}$$

thus

$$(\alpha_1\otimes\alpha_2)_*=\alpha_{1*}\otimes\alpha_{2*}.\tag{8.8}$$

(And any tensor is a finite sum of elementary tensors.)

And for the pull-back: For all vector fields $\vec{u}_1, \vec{u}_2$ in $\mathcal{U}_\mathcal{E}$,

$$T^*(\vec{u}_1, \vec{u}_2) \overset{(8.3)}{=} T(\vec{u}_{1*}, \vec{u}_{2*}) = T(d\Psi.\vec{u}_1, d\Psi.\vec{u}_2). \tag{8.9}$$

⁂

Example 8.4 If $T \in T^1_1(\mathcal{U}_\mathcal{E})$ then for all vector fields $\vec{w} \in \Gamma(\mathcal{U}_\mathcal{F})$ and differential forms $\beta \in \Omega^1(\mathcal{U}_\mathcal{F})$,

$$T_*(\beta, \vec{w}) = T(\beta^*, \vec{w}^*) = T(\beta.d\Psi, d\Psi^{-1}.\vec{w}), \tag{8.10}$$

i.e., $T_*(p_\mathcal{F})(\beta(p_\mathcal{F}), \vec{w}(p_\mathcal{F})) = T(p_\mathcal{E})(\beta(p_\mathcal{F}).d\Psi(p_\mathcal{E}), d\Psi^{-1}(p_\mathcal{F}).\vec{w}(p_\mathcal{F}))$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

For the elementary tensor $T = \vec{u} \otimes \alpha \in T^1_1(\mathcal{U}_\mathcal{E})$, made of the vector field $\vec{u} \in \Gamma(\mathcal{U}_\mathcal{E})$ and of the differential form $\alpha \in \Omega^1(\mathcal{U}_\mathcal{E})$: For all $\beta, \vec{w} \in \Omega^1(\mathcal{U}_\mathcal{F}) \times \Gamma(\mathcal{U}_\mathcal{F})$, in short,

$$(\vec{u} \otimes \alpha)_*(\beta, \vec{w}) \overset{(8.3)}{=} (\vec{u} \otimes \alpha)(\beta^*, \vec{w}^*) = (\vec{u}.\beta^*)(\alpha.\vec{w}^*) \overset{(7.7)}{=} (\vec{u}_*.\beta)(\alpha_*.\vec{w}) = (\vec{u}_* \otimes \alpha_*)(\beta, \vec{w}), \tag{8.11}$$

thus

$$(\vec{u} \otimes \alpha)_* = \vec{u}_* \otimes \alpha_*. \tag{8.12}$$

Expression with bases $(\vec{a}_i)$ in E and $(\vec{b}_i)$ in F: In short we have $(T_*)_{ij} = T_*(b^i, \vec{b}_j) = T(\Psi^*(b^i), \Psi^*(\vec{b}_j)) = [\Psi^*(b^i)].[T].[\Psi^*(\vec{b}_j)] = [b^i].[d\Psi].[T].[d\Psi^{-1}].[\vec{b}_j] = ([d\Psi].[T].[d\Psi^{-1}])_{ij}$, thus

$$[T_*]_{|\vec{b}} = [d\Psi]_{|\vec{a},\vec{b}}.[T]_{|\vec{a}}.[d\Psi]^{-1}_{|\vec{a},\vec{b}}, \tag{8.13}$$

which means $[(\Psi_*T)(p_\mathcal{F})]_{|\vec{b}} = [d\Psi(p_\mathcal{E})]_{|\vec{a},\vec{b}}.[T(p_\mathcal{E})]_{|\vec{a}}.[d\Psi(p_\mathcal{E})]^{-1}_{|\vec{a},\vec{b}}$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$. ⁂

8.3 Push-forward and pull-back of endomorphisms

We have the natural canonical isomorphism

$$\mathcal{J}_2 : \begin{cases} \mathcal{L}(E;E) & \to \mathcal{L}(E^*, E; \mathbb{R}) \\ \qquad L & \to T_L = \mathcal{J}_2(L) \quad \text{where} \quad T_L(\alpha, \vec{u}) := \alpha.L.\vec{u}, \quad \forall (\alpha, \vec{u}) \in E^* \times E. \end{cases} \tag{8.14}$$

Thus $\Psi_*T_L(m, \vec{w}) = T_L(\Psi^*m, \Psi^*\vec{w}) = (\Psi^*m).L.(\Psi^*\vec{w}) = m.d\Psi.L.d\Psi^{-1}.\vec{w}$, thus:

Definition 8.5 The push-forward by Ψ of a field of endomorphisms L on $\mathcal{U}_\mathcal{E}$ is the field of endomorphisms $\Psi_*L = L_*$ on $\mathcal{U}_\mathcal{F}$ defined by

$$\text{in short:} \quad \Psi_*L = \boxed{L_* = d\Psi.L.d\Psi^{-1}}, \tag{8.15}$$

i.e., $L_*(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).L(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F})$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

Thus with bases we get $[L_*]_{|\vec{b}} = [d\Psi]_{|\vec{a},\vec{b}}.[L]_{|\vec{a}}.[d\Psi]^{-1}_{|\vec{a},\vec{b}}$, “as in (8.13)”.

Example 8.6 Elementary field of endomorphisms $L = (\mathcal{J}_2)^{-1}(\vec{u} \otimes \alpha)$, where $\vec{u} \in \Gamma(E)$ and $\alpha \in \Omega^1(E)$: So $T_L = \vec{u} \otimes \alpha$ and $L.\vec{u}_2 = (\alpha.\vec{u}_2)\vec{u}$ for all $\vec{u}_2 \in \Gamma(\mathcal{U}_\mathcal{E})$). Thus $L_*.\vec{w}_2 = d\Psi.L.d\Psi^{-1}.\vec{w}_2 = d\Psi.L.\vec{w}_2{}^* = (\alpha.\vec{w}_2{}^*)d\Psi.\vec{u} = (\alpha_*.\vec{w}_2)\vec{u}_*$ for all $\vec{w}_2 \in \Gamma(E)$, thus $(T_L)_* = \vec{u}_* \otimes \alpha_*$. ⁂

Definition 8.7 Let L be a field of endomorphisms on $\mathcal{U}_\mathcal{F}$. Its pull-back by Ψ is the field of endomorphisms $\Psi^*L = L^*$ on $\mathcal{U}_\mathcal{E}$ defined by

$$\text{in short:} \quad \Psi^*L = \boxed{L^* = d\Psi^{-1}.L.d\Psi}, \tag{8.16}$$

i.e., $L^*(p_\mathcal{E}) = d\Psi^{-1}(p_\mathcal{F}).L(p_\mathcal{F}).d\Psi(p_\mathcal{E})$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

8.4 Application to derivatives of vector fields

$\vec{u} \in \Gamma(\mathcal{U}_\mathcal{E})$ is a C^1 vector field in $\mathcal{U}_\mathcal{E}$), $p_\mathcal{E} \in \mathcal{U}_\mathcal{E}$, so $d\vec{u} : \mathcal{U}_\mathcal{E} \to \mathcal{L}(E;E)$ (given by $d\vec{u}(p_\mathcal{E}).\vec{w}(p_\mathcal{E}) = \lim_{h \to 0} \frac{\vec{u}(p_\mathcal{E} + h\vec{w}(p_\mathcal{E})) - \vec{u}(p_\mathcal{E})}{h}$ for all $\vec{w} \in \Gamma(\mathcal{U}_\mathcal{E})$). Thus its push-forward:

$$((d\vec{u})_* =) \quad \Psi_*(d\vec{u}) = d\Psi.d\vec{u}.d\Psi^{-1} \tag{8.17}$$

i.e. $(d\vec{u})_*(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).d\vec{u}(p_\mathcal{E}).d\Psi(p_\mathcal{E})^{-1}$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

8.5 $\Psi_*(d\vec{u})$ versus $d(\Psi_*\vec{u})$: No commutativity

Here Ψ is C^2, $\vec{u}\in\Gamma(\mathcal{U}_{\mathcal{E}})$, $p_{\mathcal{E}}\in\mathcal{U}_{\mathcal{E}}$, $p_{\mathcal{F}}=\Psi(p_{\mathcal{E}})$, so $\Psi_*\vec{u}(p_{\mathcal{F}})=d\Psi(p_{\mathcal{E}}).\vec{u}(p_{\mathcal{E}})=(d\Psi(\Psi^{-1}(p_{\mathcal{F}})).(\vec{u}(\Psi^{-1}(p_{\mathcal{F}}))$, and, for all $\vec{w}\in\Gamma(\mathcal{U}_{\mathcal{F}})$,

$$d(\Psi_*\vec{u})(p_{\mathcal{F}}).\vec{w}(p_{\mathcal{F}})=(d^2\Psi(p_{\mathcal{E}}).(d\Psi^{-1}(p_{\mathcal{F}}).\vec{w}(p_{\mathcal{F}}))).\vec{u}(p_{\mathcal{E}})+d\Psi(p_{\mathcal{E}}).d\vec{u}(p_{\mathcal{E}}).d\Psi^{-1}(p_{\mathcal{F}}).\vec{w}(p_{\mathcal{F}}),\tag{8.18}$$

with $\Psi_*(d\vec{u})(p_{\mathcal{F}})=d\Psi(p_{\mathcal{E}}).d\vec{u}(p_{\mathcal{E}}).d\Psi^{-1}(p_{\mathcal{F}})$, thus, in short,

$$d(\Psi_*\vec{u}).\vec{w}=\Psi_*(d\vec{u}).\vec{w}+d^2\Psi(\Psi^*\vec{w},\vec{u})\neq\Psi_*(d\vec{u})\quad\text{in general.}\tag{8.19}$$

So the differentiation d and the push-forward $_*$ do not commute ($d(\Psi_*\vec{u})=\Psi_*(d\vec{u})$ iff Ψ is affine).

8.6 Application to derivative of differential forms

Let $\alpha\in\Omega^1(\mathcal{U}_{\mathcal{E}})$ (a differential form on $\mathcal{U}_{\mathcal{E}}$). Its derivative $d\alpha:\mathcal{U}_{\mathcal{E}}\to\mathcal{L}(E;E^*)$ is given by $d\alpha(p_{\mathcal{E}}).\vec{u}(p_{\mathcal{E}})=\lim_{h\to0}\frac{\alpha(p_{\mathcal{E}}+h\vec{u}(p_{\mathcal{E}}))-\alpha(p_{\mathcal{E}})}{h}\in E^*$, for all $\vec{u}\in\Gamma(\mathcal{U}_{\mathcal{E}})$, i.e., for all $\vec{u}_1,\vec{u}_2\in\Gamma(\mathcal{U}_{\mathcal{E}})$,

$$(d\alpha(p_{\mathcal{E}}).\vec{u}_1(p_{\mathcal{E}})).\vec{u}_2(p_{\mathcal{E}})=\lim_{h\to0}\frac{\alpha(p_{\mathcal{E}}+h\vec{u}_1(p_{\mathcal{E}})).\vec{u}_2(p_{\mathcal{E}})-(\alpha(p_{\mathcal{E}}).\vec{u}_1(p_{\mathcal{E}})).\vec{u}_2(p_{\mathcal{E}})}{h}\quad\in\mathbb{R}.\tag{8.20}$$

With the natural canonical isomorphism $\mathcal{L}(E;E^*)\simeq\mathcal{L}(E,E;\mathbb{R})$, cf. (T.16) with $E^{**}\simeq E$, we can write $d\alpha(p_{\mathcal{E}})(\vec{u}_1(p_{\mathcal{E}})).\vec{u}_2(p_{\mathcal{E}})=d\alpha(p_{\mathcal{E}})(\vec{u}_1(p_{\mathcal{E}}),\vec{u}_2(p_{\mathcal{E}}))$, i.e.

$$d\alpha(\vec{u}_1).\vec{u}_2=d\alpha(\vec{u}_1,\vec{u}_2).\tag{8.21}$$

Thus the push-forward $\Psi_*(d\alpha)=^{\text{noted}}(d\alpha)_*$ of $d\alpha$, is given by, for all $\vec{w}_1,\vec{w}_2\in\Gamma(\mathcal{U}_{\mathcal{F}})$, in short,

$$(d\alpha)_*(\vec{w}_1,\vec{w}_2)=d\alpha(\vec{w}_1^*,\vec{w}_2^*),\tag{8.22}$$

i.e., with $p_{\mathcal{F}}=\Psi(p_{\mathcal{E}})$, $(d\alpha)_*(p_{\mathcal{F}}).\vec{w}_1(p_{\mathcal{F}})).\vec{w}_2(p_{\mathcal{F}})=(d\alpha(p_{\mathcal{E}}).d\Psi^{-1}(p_{\mathcal{F}}).\vec{w}_1(p_{\mathcal{F}})).d\Psi^{-1}(p_{\mathcal{F}}).\vec{w}_2(p_{\mathcal{F}})$.

In particular, $(d^2f)_*(\vec{w}_1,\vec{w}_2)=d^2f(d\Psi^{-1}.\vec{w}_1,d\Psi^{-1}.\vec{w}_2)$ $(=d^2f(\vec{w}_1^*,\vec{w}_2^*))$.

8.7 $\Psi_*(d\alpha)$ versus $d(\Psi_*\alpha)$: No commutativity

Here Ψ is C^2, $\vec{u}\in\Gamma(\mathcal{U}_{\mathcal{E}})$, $p_{\mathcal{E}}\in\mathcal{U}_{\mathcal{E}}$ and $p_{\mathcal{F}}=\Psi(p_{\mathcal{E}})$. We have $\Psi_*\alpha(p_{\mathcal{F}})=\alpha(p_{\mathcal{E}}).d\Psi^{-1}(p_{\mathcal{F}})=\alpha(\Psi^{-1}(p_{\mathcal{F}})).d\Psi^{-1}(p_{\mathcal{F}})$, thus, for all $\vec{w}_1\in\Gamma(\mathcal{U}_{\mathcal{F}})$,

$$d(\psi_*\alpha)(p_{\mathcal{F}}).\vec{w}_1(p_{\mathcal{F}})=(d\alpha(p_{\mathcal{E}}).d\Psi^{-1}(p_{\mathcal{F}}).\vec{w}_1(p_{\mathcal{F}})).d\Psi^{-1}(p_{\mathcal{F}})+\alpha(p_{\mathcal{E}}).d^2\Psi^{-1}(p_{\mathcal{F}}).\vec{w}_1(p_{\mathcal{F}})\ \in F^*,\tag{8.23}$$

thus, for all $\vec{w}_1,\vec{w}_2\in\Gamma(\mathcal{U}_{\mathcal{F}})$, in short

$$d(\psi_*\alpha)(\vec{w}_1,\vec{w}_2)=d\alpha(d\Psi^{-1}.\vec{w}_1,d\Psi^{-1}.\vec{w}_2)+\alpha.d^2\Psi^{-1}(\vec{w}_1,\vec{w}_2)\neq d\alpha(\vec{w}_1^*,\vec{w}_2^*)\quad\text{in general.}\tag{8.24}$$

So the differentiation d and the push-forward $_*$ do not commute ($d(\Psi_*\alpha)=\Psi_*(d\alpha)$ iff Ψ is affine).

Part III

Lie derivative

9 Lie derivative

9.0 Purpose and first results

9.0.1 Purpose?

Cauchy's approach may be insufficient, e.g.:

1. - Cauchy's approach aims to compare two vectors deformed by a motion, thanks to a Euclidean dot product and the deformation gradient F, with the deformation tensor C defined by $(C.\vec{W}_1) \bullet \vec{W}_2 := (F.\vec{W}_1) \bullet (F.\vec{W}_2)$. It is a quantitative approach (needs a chosen Euclidean dot product: foot? metre?).

   - Cauchy's approach is a first order method (dedicated to linear material): Only the first order Taylor expansion of the motion is used: Only $d\Phi = F$ is used (the "slope"), not $d^2\Phi = dF$ (the "curvature") or higher derivatives.

2. - Lie's approach aims to build qualitative "covariant objective constitutive laws" (some will be discredited **afterward**, because of invariance or thermodynamical requirements).

   - Lie's approach "naturally" applies to non-linear materials thanks to second order Lie derivatives which uses the second order Taylor expansion of the motion.

   - In a non planar surface S, you need the Lie derivative if you want to derive along a trajectory.

   - In a Galilean Euclidean framework (quantification), the first order Lie derivatives approach give the same results than Cauchy's approach.

   (Cauchy died in 1857, and Lie was born in 1842: Unfortunately Cauchy could not use the Lie derivative.)

9.0.2 Basic results

The Eulerian velocity of the motion is $\vec{v}$. With the material derivative is $\frac{D\mathcal{E}ul}{Dt} := \frac{\partial \mathcal{E}ul}{\partial t} + d\mathcal{E}ul.\vec{v}$:

1. The Lie derivative $\mathcal{L}_{\vec{v}} f$ of a Eulerian scalar valued function f is the material derivative

$$\mathcal{L}_{\vec{v}} f = \frac{Df}{Dt}. \tag{9.1}$$

2. The Lie derivative $\mathcal{L}_{\vec{v}} \vec{w}$ of a (Eulerian) vector field $\vec{w}$ is more than just the material derivative $\frac{D\vec{w}}{Dt}$:

$$\mathcal{L}_{\vec{v}} \vec{w} = \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}. \tag{9.2}$$

   $\mathcal{L}_{\vec{v}} \vec{w}$ gives the rate of stress on $\vec{w}$ due to a flow, and in particular the $-d\vec{v}.\vec{w}$ term in $\mathcal{L}_{\vec{v}} \vec{w}$ tells that the spatial variations of $\vec{v}$ (variations of the flow) act on the evolution of the stress (anticipated).

3. (9.1)-(9.2) enable to define the Lie derivatives of tensors of any order.

9.1 Definition

9.1.1 Issue (ubiquity gift)...

$\widetilde{\Phi}$ is supposed to be regular. $\vec{v}(t, p(t)) = \frac{\partial \widetilde{\Phi}}{\partial t}(t, P_{Obj})$ is the Eulerian velocity at t at $p(t) = \widetilde{\Phi}(t, P_{Obj})$. Recall: If $\mathcal{E}ul$ is a Eulerian function then its material time derivative is

$$\frac{D\mathcal{E}ul}{Dt}(t, p(t)) = \lim_{h \to 0} \frac{\mathcal{E}ul(t{+}h, p(t{+}h)) - \mathcal{E}ul(t, p(t))}{h}. \tag{9.3}$$

Issue: The rate $\frac{\mathcal{E}ul(t+h, p(t+h)) - \mathcal{E}ul(t, p(t))}{h}$ raises questions:

1- The difference $\mathcal{E}ul(t{+}h, p(t{+}h)) - \mathcal{E}ul(t, p(t))$ requires the time and space ubiquity gift to be calculated by an observer, since it mixes two distinct times, t and $t{+}h$, and two distinct locations, $p(t)$ and $p(t{+}h)$.

2- The difference $\mathcal{E}ul(t{+}h, p(t{+}h)) - \mathcal{E}ul(t, p(t))$ can be impossible: E.g. if $\mathcal{E}ul = \vec{w}$ is a vector field in a "non planar surface considered on its own" (manifold) then $\mathcal{E}ul(t{+}h, p(t{+}h))$ and $\mathcal{E}ul(t, p(t))$ don't belong to the same (tangent) vector space (so the difference $\vec{w}(t{+}h, p(t{+}h)) - \vec{w}(t, p(t))$ is meaningless).

9.1.2 ...Toward a solution (without ubiquity gift)...

To compare $\mathcal{E}ul(t{+}h,p(t{+}h))$ and $\mathcal{E}ul(t,p(t))$ (to get the evolution of $\mathcal{E}ul$ along a trajectory), you need the duration h to get from t to $t{+}h$ and to move from $p(t)$ to $p(t{+}h)$. So, you must:

- take the value $\mathcal{E}ul(t,p_t))$ with you (for memory),
- move along the considered trajectory, and doing so, the value $\mathcal{E}ul(t,p_t)$ has possibly changed to, with $\tau = t{+}h$,

$$((\Phi^t_\tau)_*\mathcal{E}ul_t)(p_\tau) \overset{\text{noted}}{=} \mathcal{E}ul_{t*}(\tau,p_\tau) \quad \text{(push-forward)}; \tag{9.4}$$

- And now, at (τ,p_τ) where you are, you can compare the actual value $\mathcal{E}ul(\tau,p_\tau)$ with the value $\mathcal{E}ul_{t*}(\tau,p_\tau)$ you arrived with (the transported memory), thus the difference

$$\mathcal{E}ul(\tau,p_\tau) - \mathcal{E}ul_{t*}(\tau,p_\tau) \tag{9.5}$$

is meaningful for a human being since it is computed at a unique time τ and at a unique point p_τ (no gift of ubiquity required).

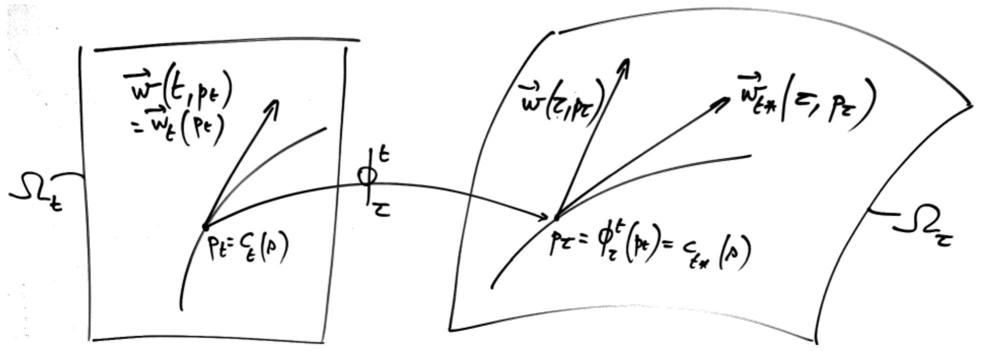

Figure 9.1: To compute (9.5) with $\mathcal{E}ul = \vec{w}$ a (Eulerian) vector field: At t define the vector field $\vec{w}_t$ in Ω_t by $\vec{w}_t(p_t) := \vec{w}(t,p_t)$. The (spatial) curve $c_t : s \to p_t = c_t(s)$ in Ω_t is an integral curve of $\vec{w}_t$, i.e. satisfies $c_t{}'(s) = \vec{w}_t(c_t(s))$. c_t is transformed by Φ^t_τ into the (spatial) curve $c_\tau = \Phi^t_\tau{\circ}c_t : s \to p_\tau = c_\tau(s){=}\Phi^t_\tau(c_t(s))$ in Ω_τ; Hence $c_\tau{}'(s) = d\Phi^t_\tau(p_t).c'(s) = d\Phi^t_\tau(p_t).\vec{w}_t(p_t) =^{\text{noted}} \vec{w}_{t*}(\tau,p_\tau)$ is the tangent vector at c_τ at p_τ (push-forward). Thus the difference $\vec{w}(\tau,p_\tau) - \vec{w}_{t*}(\tau,p_\tau)$ can be computed by a human being, i.e. without ubiquity gift.

9.1.3 ... The Lie derivative, first definition

Definition 9.1 The Lie derivative $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ along $\vec{v}$ of an Eulerian function $\mathcal{E}ul$ is the Eulerian function $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ defined by, at t at $p_t = \widetilde{\Phi}(t,P_{Obj})$,

$$\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t) := \lim_{h\to 0} \frac{\mathcal{E}ul(t{+}h,p(t{+}h)) - (\Phi^t_{t+h})_*\mathcal{E}ul_t(p(t{+}h))}{h}. \tag{9.6}$$

Interpretation: $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ measures the rate of change of $\mathcal{E}ul$ along a trajectory:

- $\mathcal{E}ul(t{+}h,p(t{+}h))$ is the value of $\mathcal{E}ul$ at $t{+}h$ at $p(t{+}h)$, see figure 9.1.
- $\mathcal{E}ul_{t*}(t{+}h,p(t{+}h)) = ((\Phi^t_{t+h})_*\mathcal{E}ul_t)(t{+}h,p(t{+}h))$ is exclusively strain related: It is the memory transported along a flow, i.e. the value $\mathcal{E}ul(t,p_t)$ distorted by the flow.

So, with g defined by $g(\tau) = ((\Phi^t_\tau)_*\mathcal{E}ul_t)(p_\tau)$ (in particular $g(t) = \mathcal{E}ul_t(p_t)$):

$$\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t) := g'(t) = \lim_{\tau\to t} \frac{g(\tau)-g(t)}{\tau - t} \overset{\text{also written}}{=} \frac{d((\Phi^t_{t+h})_*\mathcal{E}ul_t)(p(t{+}h))}{dt}_{|\tau=t}. \tag{9.7}$$

9.1.4 A more general definition

The rate in (9.6) has to be slightly modified to be adequate in all situations: $\mathcal{E}ul(t{+}h,p(t{+}h)) - \mathcal{E}ul_*(t{+}h,p(t{+}h))$ is computed at $(t{+}h,p(t{+}h))$ which moves as $h \to 0$, and on a “non-planar manifold” this is problematic. The “natural” definition is to arrive with the memory:

Definition 9.2 The Lie derivative $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ along $\vec{v}$ of an Eulerian function $\mathcal{E}ul$ is the Eulerian function $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ defined by, at t at $p_t = \widetilde{\Phi}_{P_{Obj}}(t)$,

$$\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t) := \lim_{h\to 0} \frac{\mathcal{E}ul(t,p_t) - (\Phi_t^{t-h})_*\mathcal{E}ul_{t-h}(p_t)}{h}. \tag{9.8}$$

I.e. with $\widetilde{g}$ defined by $\widetilde{g}(\tau) = ((\Phi_t^\tau)_*\mathcal{E}ul_\tau)(p_t)$ (in particular $\widetilde{g}(t) = \mathcal{E}ul(t,p_t)$):

$$\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t) := \widetilde{g}'(t) = \lim_{\tau\to t} \frac{\widetilde{g}(t)-\widetilde{g}(\tau)}{t-\tau} = \lim_{\tau\to t} \frac{\widetilde{g}(\tau)-\widetilde{g}(t)}{\tau-t} \overset{\text{also written}}{=} \frac{d((\Phi_t^\tau)_*\mathcal{E}ul_\tau)(p_t)}{d\tau}_{|\tau=t}. \tag{9.9}$$

Here the observer must:

- At $t-h$ at $p(t-h) = \widetilde{\Phi}_{P_{Obj}}(t-h)$, take the value $\mathcal{E}ul(t-h, p(t-h))$,
- travel along the trajectory $\widetilde{\Phi}_{P_{Obj}}$,
- once at t at $p_t = \widetilde{\Phi}_{P_{Obj}}(t)$, this value has become $((\Phi_{t-h}^t)_*\mathcal{E}ul_{t-h})(p_t)$ (transported memory),
- and then the comparison with $\mathcal{E}ul(t,p_t)$ can be done in Ω_t (no ubiquity gift required).

Exercice 9.3 Prove: (9.6) and (9.8) are equivalent.

Answer. With $(\Phi_{t+h}^t)^*.(\Phi_{t+h}^t)_* = I$, (9.6) gives $\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t) = \lim_{h\to 0} \frac{(\Phi_{t+h}^t)^*\mathcal{E}ul(t,p_t) - \mathcal{E}ul_t(t,p_t)}{h} = \lim_{h\to 0} \frac{(\Phi_{t-h}^t)^*\mathcal{E}ul_{t-h})(p_t) - \mathcal{E}ul_t(p_t)}{-h} = \lim_{h\to 0} \frac{\mathcal{E}ul_t(p_t) - ((\Phi_{t-h}^t)^*\mathcal{E}ul_{t-h})(p_t)}{h}$, and $(\Phi_{t-h}^t)^* = (\Phi_t^{t-h})_*$. ⁂

9.1.5 Equivalent definition (differential geometry)

Definition 9.4 The Lie derivative of a Eulerian function $\mathcal{E}ul$ along a flow of Eulerian velocity $\vec{v}$ is the Eulerian function $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ defined at (t,p_t) by

$$\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t) := \lim_{h\to 0} \frac{((\Phi_{t+h}^t)^*\mathcal{E}ul_{t+h})(p_t) - \mathcal{E}ul(t,p_t)}{h}, \quad \text{rate in } \Omega_t. \tag{9.10}$$

In other words, if $\hat{g}$ is defined by $\hat{g}(\tau) = ((\Phi_\tau^t)^*\mathcal{E}ul_\tau)(p_t)$ (in particular $\hat{g}(t) = \mathcal{E}ul(t,p_t)$), then

$$\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t) := \hat{g}'(t) = \lim_{\tau\to t} \frac{\hat{g}(\tau)-\hat{g}(t)}{\tau-t} \overset{\text{also written}}{=} \frac{d((\Phi_\tau^t)^*\mathcal{E}ul_\tau)(p_t)}{d\tau}_{|\tau=t}. \tag{9.11}$$

Exercice 9.5 Prove: (9.8) and (9.10) are equivalent.

Answer. (9.10) also reads $\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t) = \lim_{h\to 0} \frac{((\Phi_{t-h}^t)^*\mathcal{E}ul_{t-h})(p_t) - \mathcal{E}ul_t(p_t)}{-h}$, and $(\Phi_{t-h}^t)^*.(\Phi_t^{t-h})_* = I$. ⁂

Remark 9.6 More precise definition, as in (2.3): E.g. with (9.10), the Lie derivative $\widetilde{\mathcal{L}}_{\vec{v}}\mathcal{E}ul$ of a Eulerian function $\widetilde{\mathcal{E}ul}$ along a flow of Eulerian velocity $\vec{v}$ is the Eulerian function defined by, at t at $p_t = \widetilde{\Phi}(t, P_{Obj})$,

$$\widetilde{\mathcal{L}}_{\vec{v}}\mathcal{E}ul(t,p_t) := ((t,p_t), \mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t) \quad \text{(pointed function at } (t,p_t)), \tag{9.12}$$

And, to lighten the notation, $\widetilde{\mathcal{L}}_{\vec{v}}\mathcal{E}ul(t,p_t) =^{\text{noted}} \mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t)$ (second component of $\widetilde{\mathcal{L}}_{\vec{v}}\mathcal{E}ul(t,p_t)$). ⁂

9.2 Lie derivative of a scalar function

Let f be a C^1 Eulerian scalar valued function. With $(\Phi_t^{t-h})_*f_{t-h}(p_t) = f_{t-h}(p(t-h))$, cf. (6.10), we get

$$\mathcal{L}_{\vec{v}}f(t,p_t) \overset{(9.8)}{=} \lim_{h\to 0} \frac{f(t,p_t) - f(t-h,p(t-h))}{h}, \quad \text{i.e.} \quad \boxed{\mathcal{L}_{\vec{v}}f = \frac{Df}{Dt}} = \frac{\partial f}{\partial t} + df.\vec{v}. \tag{9.13}$$

So, for scalar functions, the Lie derivative is the material derivative.

Interpretation: $\mathcal{L}_{\vec{v}}f$ measures the rate of change of f along a trajectory.

Proposition 9.7 $\mathcal{L}_{\vec{v}}f = 0$ *iff f is constant along any trajectory (the real value is the memory value):*

$$\mathcal{L}_{\vec{v}}f = 0 \iff \forall t,\tau \in [t_0,T],\ (\Phi_{\tau*}^t)f_t(p_\tau) = f(t,p(t)) \ \textit{when}\ p_\tau = \Phi_\tau^t(p_t), \tag{9.14}$$

i.e. iff $f(t,p(t)) = f(t_0,p_{t_0})$ when $p(t) = \Phi^{t_0}(t,p_{t_0})$, i.e. iff f let itself be carried by the flow (unchanged).

Proof. Let $p(t) = \widetilde{\Phi}(t, P_{Obj}) = p_t$ for all t, so $p(\tau) = \widetilde{\Phi}(\tau, P_{Obj}) = p_\tau = \Phi^t_{t+h}(p_t) = \Phi^t(\tau, p_t)$.

$\Leftarrow$: If $f_\tau = (\Phi^t_{t+h})_* f_t$, then $f_\tau(p_\tau) = f_t(p_t)$, thus $\lim_{\tau\to t} \frac{f(\tau,p(\tau))-f(t,p(t))}{\tau-t} = 0$, that is, $\frac{Df}{Dt} = 0$.

$\Rightarrow$: If $\frac{Df}{Dt} = 0$ then $f(t,p(t))$ is a constant function on the trajectory $t \to \widetilde{\Phi}(t, P_{Obj})$, for any particle P_{Obj}, so $f(\tau,p(\tau)) = f(t,p_t)$ when $p(\tau) = \Phi^t_{t+h}(p_t)$, that is, $f(\tau,p_\tau) = (\Phi^t_{t+h})_* f_t(p_\tau)$. ∴

Exercice 9.8 Prove: $\mathcal{L}_{\vec v}(\mathcal{L}_{\vec v} f) = \frac{D^2 f}{Dt^2} = \frac{\partial^2 f}{\partial t^2} + 2d(\frac{\partial f}{\partial t}).\vec v + d^2 f(\vec v, \vec v) + df.(\frac{\partial \vec v}{\partial t} + d\vec v)$.

Answer. See (2.23). ∴

9.3 Lie derivative of a vector field

9.3.1 Formula

Let $\vec w$ be a C^1 (Eulerian) vector field (interpreted as an “internal force field” in the following).

Proposition 9.9

$$\boxed{\mathcal{L}_{\vec v}\vec w = \frac{D\vec w}{Dt} - d\vec v.\vec w} = \frac{\partial \vec w}{\partial t} + d\vec w.\vec v - d\vec v.\vec w. \tag{9.15}$$

So the Lie derivative is not reduced to the material derivative $\frac{D\vec w}{Dt}$ *(unless* $d\vec v = 0$*, i.e. unless* $\vec v$ *is uniform): The spatial variations* $d\vec v$ *of* $\vec v$ *influences the rate of stress:* $\vec v$ *tries to bend* $\vec w$ *(which is expected).*

Proof. Let $\vec g : \tau \to \vec g(\tau) = (\Phi^{t*}_\tau \vec w)(t, p(t)) = d\Phi^t_\tau(p_t)^{-1}.\vec w(\tau, p(\tau))$ when $p(\tau) = \Phi^t(\tau, p_t)$, so (9.10) reads $\mathcal{L}_{\vec v}\vec w(t,p_t) = \vec g\,'(t)$. With $\vec z(\tau) := \vec w(\tau, p(\tau)) = d\Phi^t(\tau,p_t).\vec g(\tau)$,

$$\begin{aligned}\vec z\,'(\tau) = \frac{D\vec w}{D\tau}(\tau,p(\tau)) &= \frac{\partial(d\Phi^t)}{\partial\tau}(\tau,p_t).\vec g(\tau) + d\Phi^t(\tau,p_t).\vec g\,'(\tau)\\ &= (d\vec v(\tau,p(\tau)).F^t(\tau,p_t)).(F^t(\tau,p_t)^{-1}.\vec w(\tau,p(\tau))) + F^t_\tau(p_t).\vec g\,'(\tau)\\ &= d\vec v(\tau,p(\tau)).\vec w(\tau,p(\tau)) + F^t_\tau(p_t).\vec g\,'(\tau).\end{aligned} \tag{9.16}$$

Thus $\frac{D\vec w}{Dt}(t,p_t) = d\vec v(t,p_t).\vec w(t,p_t) + I.\vec g\,'(t)$, thus $\vec g\,'(t) = \frac{D\vec w}{Dt}(t,p_t) - d\vec v(t,p_t).\vec w(t,p_t)$. ∴

Quantification: Basis $(\vec e_i)$, $\vec v = \sum_i v_i \vec e_i$, $\vec w = \sum_i w_i \vec e_i$, $d\vec v.\vec e_j = \sum_{ij} v_{i|j}\vec e_i$, $d\vec w.\vec e_j = \sum_{ij} w_{i|j}\vec e_i$; Then

$$\mathcal{L}_{\vec v}\vec w = \sum_{i=1}^n \frac{\partial w_i}{\partial t}\vec e_i + \sum_{i,j=1}^n w_{i|j} v_j \vec e_i - \sum_{i,j=1}^n v_{i|j} w_j \vec e_i. \tag{9.17}$$

So (column matrix), with $[\cdot] := [\cdot]_{|\vec e}$,

$$\boxed{[\mathcal{L}_{\vec v}\vec w] = [\frac{D\vec w}{Dt}] - [d\vec v].[\vec w]} \quad (= [\frac{\partial \vec w}{\partial t}] + [d\vec w.\vec v] - [d\vec v].[\vec w]). \tag{9.18}$$

(And $[d\vec w.\vec v] = [d\vec w].[\vec v]$.) Duality notations: $\mathcal{L}_{\vec v}\vec w = \sum_i \frac{\partial w^i}{\partial t}\vec e_i + \sum_{ij} w^i_{|j} v^j \vec e_i - \sum_{ij} v^i_{|j} w^j \vec e_i$.

9.3.2 Interpretation: Flow resistance measurement

Proposition 9.10 Φ^{t_0} *is supposed to be a* C^2 *motion and a* C^1 *diffeomorphism in space, and* $\vec w$ *is a vector field.*

$$\mathcal{L}_{\vec v}\vec w = 0 \iff \forall t \in [t_0, T],\ \ \vec w_t = (\Phi^{t_0}_t)_* \vec w_{t_0}. \tag{9.19}$$

i.e., $\frac{D\vec w}{Dt} = d\vec v.\vec w \Leftrightarrow$ *the actual vector* $\vec w(t,p(t))$ *is equal to* $F^{t_0}_t(p_{t_0}).\vec w_{t_0}(p_{t_0}) = \vec w_{t_0 *}(t,p(t))$ *the deformed vector by the flow, see figure 9.1. So: The Lie derivative* $\mathcal{L}_{\vec v}\vec w$ *vanishes iff* $\vec w$ *does not resist the flow (let itself be deformed by the flow), i.e. iff* $\vec w(t,p_t) = \vec w_{t_0 *}(t,p_t)$*.*

Proof. We have $\mathcal{L}_{\vec v}\vec w = \frac{D\vec w}{Dt} - d\vec v.\vec w$ and $\frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}) = d\vec v(t,p(t)).F^{t_0}_t(p_{t_0})$, cf. (3.33).

$\Leftarrow$ (derivation): Suppose $\vec w(t,p(t)) = F^{t_0}(t,p_{t_0}).\vec w(t_0,p_{t_0})$ when $p(t) = \Phi^{t_0}_t(p_{t_0})$. Then $\frac{D\vec w}{Dt}(t,p(t)) = \frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}).\vec w(t_0,p_{t_0}) = (d\vec v(t,p(t)).F^{t_0}_t(p_{t_0})).(F^{t_0}_t(p_{t_0})^{-1}.\vec w(t,p(t))) = d\vec v(t,p(t)).\vec w(t,p(t))$, thus $\frac{D\vec w}{Dt} - d\vec v.\vec w = 0$. (See proposition 3.14.)

$\Rightarrow$ (integration): Suppose $\frac{D\vec w}{Dt} = d\vec v.\vec w$. Let $\vec f(t) = (F^{t_0}_t(p_{t_0}))^{-1}.\vec w(t,p(t))$ (= pull-back $(\Phi^{t_0}_t)^*\vec w(t_0,p_{t_0})$) when $p(t) = \Phi^{t_0}(t,p_{t_0})$; So $\vec w(t,p(t)) = F^{t_0}(t,p_{t_0}).\vec f(t)$ and $\frac{D\vec w}{Dt}(t,p(t)) = \frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}).\vec f(t) + F^{t_0}_t(p_{t_0}).\vec f\,'(t) = d\vec v(t,p(t)).F^{t_0}_t(p_{t_0}).\vec f(t) + F^{t_0}_t(p_{t_0}).\vec f\,'(t) = d\vec v(t,p(t)).\vec w(t,p(t)) + F^{t_0}_t(p_{t_0}).\vec f\,'(t) =^{hyp.} \frac{D\vec w}{Dt}(t,p(t)) + F^{t_0}_t(p_{t_0}).\vec f\,'(t)$ for all t; Thus $F^{t_0}_t(p_{t_0}).\vec f\,'(t) = \vec 0$, thus $\vec f\,'(t) = \vec 0$ (because $\Phi^{t_0}_t$ is a diffeomorphism), thus $\vec f(t) = \vec f(t_0)$, i.e. $\vec w_t = (\Phi^{t_0}_t)_*\vec w_{t_0}$, for all t. ∴

9.3.3 Autonomous Lie derivative and Lie bracket

The Lie bracket of two vector fields $\vec{v}$ and $\vec{w}$ is

$$[\vec{v},\vec{w}] := d\vec{w}.\vec{v} - d\vec{v}.\vec{w} \overset{\text{noted}}{=} \mathcal{L}^0_{\vec{v}}\vec{w}. \tag{9.20}$$

And $\mathcal{L}^0_{\vec{v}}\vec{w} = [\vec{v},\vec{w}]$ is called the autonomous Lie derivative of $\vec{w}$ along $\vec{v}$. Thus

$$\mathcal{L}_{\vec{v}}\vec{w} = \frac{\partial\vec{w}}{\partial t} + [\vec{v},\vec{w}] = \frac{\partial\vec{w}}{\partial t} + \mathcal{L}^0_{\vec{v}}\vec{w}. \tag{9.21}$$

NB: $\mathcal{L}^0_{\vec{v}}\vec{w}$ is used when $\vec{v}$ et $\vec{w}$ are stationary vector fields, thus does not concern objectivity: A stationary vector field in a referential is not necessary stationary in another (moving) referential.

9.4 Examples

9.4.1 Lie Derivative of a vector field along itself

(9.15) with $\vec{w} = \vec{v}$ gives $\mathcal{L}_{\vec{v}}\vec{v} = \frac{\partial\vec{v}}{\partial t}$. In particular, if $\vec{v}$ is a stationary vector field then $\mathcal{L}_{\vec{v}}\vec{v} = \vec{0}$ $(= [\vec{v},\vec{v}])$.

9.4.2 Lie derivative along a uniform flow

Here $d\vec{v} = 0$, thus

$$\mathcal{L}_{\vec{v}}\vec{w} = \frac{D\vec{w}}{Dt} = \frac{\partial\vec{w}}{\partial t} + d\vec{w}.\vec{v} \quad (\text{when } d\vec{v} = 0). \tag{9.22}$$

Here the flow is rectilinear ($d\vec{v} = 0$): there is no curvature (of the flow) to influence the stress on $\vec{w}$.

Moreover, if $\vec{w}$ is stationary, that is $\frac{\partial\vec{w}}{\partial t} = 0$, then $\mathcal{L}_{\vec{v}}\vec{w} = d\vec{w}.\vec{v} =$ the directional derivative $\frac{\partial\vec{w}}{\partial\vec{v}}$ of the vector field $\vec{w}$ in the direction $\vec{v}$.

9.4.3 Lie derivative of a uniform vector field

Here $d\vec{w}(t,p) = 0$, thus

$$\mathcal{L}_{\vec{v}}\vec{w} = \frac{\partial\vec{w}}{\partial t} - d\vec{v}.\vec{w} \quad (\text{when } d\vec{w} = 0), \tag{9.23}$$

thus the stress on $\vec{w}$ is due to the space variations of $\vec{v}$. Moreover, is $\vec{w}$ is stationary then $\mathcal{L}_{\vec{v}}\vec{w} = -d\vec{v}.\vec{w}$.

9.4.4 Uniaxial stretch of an elastic material

- Strain. With $[\overrightarrow{OP}]_{|\vec{e}} = [\vec{X}]_{|\vec{e}} = \begin{pmatrix} X \\ Y \end{pmatrix}$, with $\xi > 0$, $t \ge t_0$, $p(t) = \Phi^{t_0}(t,P)$ and $[\vec{x}]_{|\vec{e}} = [\overrightarrow{Op(t)}]_{|\vec{e}}$:

$$[\vec{x}]_{|\vec{e}} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} + \xi(t{-}t_0)\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} X(1+\xi(t{-}t_0)) \\ Y \end{pmatrix}. \tag{9.24}$$

- Eulerian velocity $\vec{v}(t,p) = \begin{pmatrix} \xi X \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\xi}{1+\xi(t-t_0)}x \\ 0 \end{pmatrix}$, $d\vec{v}(t,p) = \begin{pmatrix} \frac{\xi}{1+\xi(t-t_0)} & 0 \\ 0 & 0 \end{pmatrix}$ (independent of p).
- Deformation gradient (independent of P), with $\kappa_t = \xi(t{-}t_0)$:

$$F_t = d\Phi^{t_0}_t(P) = \begin{pmatrix} 1+\kappa_t & 0 \\ 0 & 1 \end{pmatrix} = I + \kappa_t\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{9.25}$$

Infinitesimal strain tensor, with $F_t^T = F_t$ here:

$$\underline{\underline{\varepsilon}}^{t_0}_t(P) = F_t - I = \kappa_t\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \underline{\underline{\varepsilon}}_t. \tag{9.26}$$

- Stress. Constitutive law = Linear isotropic elasticity:

$$\underline{\underline{\sigma}}_t(p_t) = \lambda\mathrm{Tr}(\underline{\underline{\varepsilon}}_t)I + 2\mu\underline{\underline{\varepsilon}}_t = \kappa_t\begin{pmatrix} \lambda{+}2\mu & 0 \\ 0 & \lambda \end{pmatrix} = \underline{\underline{\sigma}}_t. \tag{9.27}$$

Cauchy stress vector $\vec{T}$ on a surface at p with normal $\vec{n}_t(p) = \begin{pmatrix} n^1 \\ n^2 \end{pmatrix} = \vec{n}$:

$$\vec{T}_t(p_t) = \underline{\underline{\sigma}}_t.\vec{n} = \kappa_t\begin{pmatrix} (\lambda{+}2\mu)n_1 \\ \lambda n_2 \end{pmatrix} = \xi(t{-}t_0)\begin{pmatrix} (\lambda{+}2\mu)n_1 \\ \lambda n_2 \end{pmatrix} = \vec{T}_t. \tag{9.28}$$

- Push-forwards: $\vec{T}_{t_0}(p_{t_0}) = 0$, thus $F^{t_0}_{t_0+h}(p_{t_0}).\vec{T}_{t_0}(p_{t_0}) = \vec{0}$.

- Lie derivative:

$$\mathcal{L}_{\vec v}\vec T(t_0,p_{t_0}) = \lim_{t\to t_0}\frac{\vec T_t(p_t) - F_t^{t_0}(p_{t_0}).\vec T_{t_0}(p_{t_0})}{t-t_0} = \xi\begin{pmatrix}(\lambda{+}2\mu)n_1\\ \lambda n_2\end{pmatrix} \quad \text{(rate of stress at } (t_0,p_{t_0})\text{)}. \tag{9.29}$$

- Generic computation with $\mathcal{L}_{\vec v}\vec T = \frac{\partial \vec T}{\partial t} + d\vec T.\vec v - d\vec v.\vec T$: (9.28) gives $\frac{\partial \vec T}{\partial t} = \xi\begin{pmatrix}(\lambda{+}2\mu)\,n^1\\ \lambda\,n^2\end{pmatrix}$ and $d\vec T = 0$ and $d\vec v_t.\vec T_t = \begin{pmatrix}\frac{\xi}{1+\xi(t-t_0)} & 0\\ 0 & 0\end{pmatrix}.\xi(t{-}t_0)\begin{pmatrix}(\lambda{+}2\mu)\,n^1\\ \lambda\,n^2\end{pmatrix} = \frac{\xi^2(t-t_0)}{1+\xi(t-t_0)}\begin{pmatrix}(\lambda{+}2\mu)\,n^1\\ 0\end{pmatrix}$. In particular, $d\vec v(t_0,p_{t_0}).\vec T(t_0,p_{t_0}) = \vec 0$. Thus $\mathcal{L}_{\vec v}\vec T(t_0,p_{t_0}) = \xi\begin{pmatrix}(\lambda{+}2\mu)\,n^1\\ \lambda\,n^2\end{pmatrix}$ = rate of stress at the initial (t_0,p_{t_0}).

9.4.5 Simple shear of an elastic material

Euclidean basis $(\vec e_1,\vec e_2)$ in $\mathbb{R}^2$, the same basis at any time. Initial configuration $\Omega_{t_0} = [0,L_1]\otimes[0,L_2]$. Initial position $[\overrightarrow{OP}]_{\vec e} = [\overrightarrow{Op_{t_0}}]_{\vec e} = [\vec X]_{\vec e} = \begin{pmatrix}X\\ Y\end{pmatrix}$. Let $\xi\in\mathbb{R}^*$, $p_t = \Phi_t^{t_0}(p_{t_0})$, $[\vec x]_{|\vec e} = [\overrightarrow{Op(t)}]_{|\vec e}$, and

$$[\vec x]_{\vec e} = \begin{pmatrix}x = \varphi^1(t,X,Y) = X\\ y = \varphi^2(t,X,Y)\end{pmatrix} = \begin{pmatrix}X + \xi(t{-}t_0)Y\\ Y\end{pmatrix}. \tag{9.30}$$

- Eulerian velocity $\vec v_t(p_t) = \begin{pmatrix}\xi Y\\ 0\end{pmatrix} = \begin{pmatrix}\xi y\\ 0\end{pmatrix}$, thus $d\vec v_t(p_t) = \begin{pmatrix}0 & \xi\\ 0 & 0\end{pmatrix}$.
- Strain. With $\kappa_t = \xi(t{-}t_0)$, deformation gradient (independent of P):

$$d\Phi_t^{t_0}(P) = \begin{pmatrix}1 & \kappa_t\\ 0 & 1\end{pmatrix} = F_t^{t_0}, \quad \text{thus} \quad F_t^{t_0} - I = \kappa_t\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}. \tag{9.31}$$

- Infinitesimal strain tensor:

$$\underline{\underline{\varepsilon}}_t^{t_0}(P) = \frac{F_t^{t_0}(P){-}I + (F_t^{t_0}(P){-}I)^T}{2} = \frac{\kappa_t}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \underline{\underline{\varepsilon}}_t. \tag{9.32}$$

- Stress. Constitutive law, usual linear isotropic elasticity (requires a Euclidean dot product):

$$\underline{\underline{\sigma}}(t,p_t) = \lambda\mathrm{Tr}(\underline{\underline{\varepsilon}}_t)I + 2\mu\underline{\underline{\varepsilon}}_t = \mu\kappa_t\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \underline{\underline{\sigma}}_t. \tag{9.33}$$

Cauchy stress vector $\vec T(t,p_t)$ (at t at p_t) on a surface at p with normal $\vec n_t(p) = \begin{pmatrix}n^1\\ n^2\end{pmatrix} = \vec n$:

$$\vec T_t = \underline{\underline{\sigma}}_t.\vec n = \mu\kappa_t\begin{pmatrix}n^2\\ n^1\end{pmatrix} = \mu\xi(t{-}t_0)\begin{pmatrix}n^2\\ n^1\end{pmatrix} = \vec T(t) \quad \text{(stress independent of } p_t\text{)}. \tag{9.34}$$

- Lie derivative, with $\vec T_{t_0} = \vec 0$:

$$\mathcal{L}_{\vec v}\vec T(t_0,p_{t_0}) = \lim_{t\to t_0}\frac{\vec T_t(p_t) - F_t^{t_0}(p_{t_0}).\vec T_{t_0}(p_{t_0})}{t-t_0} = \mu\xi\begin{pmatrix}n^2\\ n^1\end{pmatrix} \quad \text{(rate of stress at } (t_0,p_{t_0})\text{)}. \tag{9.35}$$

- Generic computation: $\mathcal{L}_{\vec v}\vec T = \frac{\partial \vec T}{\partial t} + d\vec T.\vec v - d\vec v.\vec T$. (9.34) gives $\frac{\partial \vec T}{\partial t}(t,p) = \mu\xi\begin{pmatrix}n^2\\ n^1\end{pmatrix}$ and $d\vec T = 0$. With $d\vec v_{t_0}.\vec T_{t_0} = \vec 0$. Thus $\mathcal{L}_{\vec v}\vec T(t_0,p_{t_0}) = \mu\xi\begin{pmatrix}n^2\\ n^1\end{pmatrix}$.

9.4.6 Shear flow

Stationary shear field, see (5.11) with $\alpha = 0$ and $t_0 = 0$:

$$\vec v(x,y) = \begin{cases} v^1(x,y) = \lambda y,\\ v^2(x,y) = 0,\end{cases} \qquad d\vec v(x,y) = \begin{pmatrix}0 & \lambda\\ 0 & 0\end{pmatrix}. \tag{9.36}$$

Let $\vec w(t,p) = \begin{pmatrix}0\\ b\end{pmatrix} = \vec w(t_0,p_{t_0})$ (constant in time and uniform in space). Then $\mathcal{L}_{\vec v}\vec w = -d\vec v.\vec w = \begin{pmatrix}-\lambda b\\ 0\end{pmatrix}$ measures "the resistance to deformation due to the flow". See figure 9.2, the virtual vector $\vec w_*(t,p) = d\Phi(t_0,p_{t_0}).\vec w(t_0,p_{t_0})$ being the vector that would have let itself be carried by the flow (the push-forward).

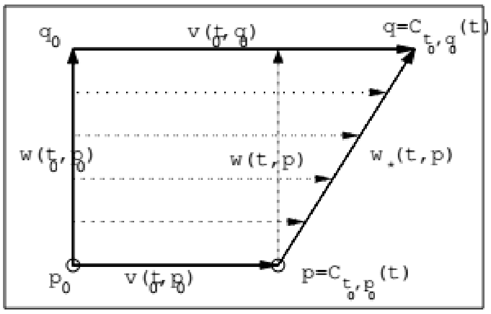

Figure 9.2: Shear flow, cf. (9.36), with $\vec w$ constant and uniform. $\mathcal{L}_{\vec v}\vec w$ measures the resistance to the deformation.

9.4.7 Spin

Rotating flow: Continuing (5.14):

$$\vec v(x,y)=\omega\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix},\quad d\vec v(x,y)=\omega\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\omega\,Rot(\pi/2). \tag{9.37}$$

In particular $d^2\vec v=0$. With $\vec w=\vec w_0$ constant and uniform we get

$$\mathcal{L}_{\vec v}\vec w_0=-d\vec v(p).\vec w_0=-\omega\,Rot(\pi/2).\vec w_0 \qquad (\perp\begin{pmatrix}a\\b\end{pmatrix}=\vec w_0). \tag{9.38}$$

gives “the force at which $\vec w$ refuses to turn with the flow”.

9.4.8 Second order Lie derivative

Exercice 9.11 Let $\vec v,\vec w$ be C^2. Prove:

$$\begin{aligned}\mathcal{L}_{\vec v}(\mathcal{L}_{\vec v}\vec w)&=\frac{D^2\vec w}{Dt^2}-2d\vec v.\frac{D\vec w}{Dt}-\frac{D(d\vec v)}{Dt}.\vec w+d\vec v.d\vec v.\vec w,\\&=\frac{\partial^2\vec w}{\partial t^2}+2d\frac{\partial\vec w}{\partial t}.\vec v-2d\vec v.\frac{\partial\vec w}{\partial t}+d\vec w.\frac{\partial\vec v}{\partial t}-d\frac{\partial\vec v}{\partial t}.\vec w\\&\qquad+(d^2\vec w.\vec v).\vec v+d\vec w.d\vec v.\vec v-2d\vec v.d\vec w.\vec v-(d^2\vec v.\vec v).\vec w+d\vec v.d\vec v.\vec w.\end{aligned} \tag{9.39}$$

Answer.

$$\begin{aligned}\mathcal{L}_{\vec v}(\mathcal{L}_{\vec v}\vec w)&=\frac{D(\mathcal{L}_{\vec v}\vec w)}{Dt}-d\vec v.(\mathcal{L}_{\vec v}\vec w)=\frac{D(\frac{D\vec w}{Dt}-d\vec v.\vec w)}{Dt}-d\vec v.(\frac{D\vec w}{Dt}-d\vec v.\vec w)\\&=\frac{D^2\vec w}{Dt^2}-\frac{D(d\vec v)}{Dt}.\vec w-d\vec v.\frac{D\vec w}{Dt}-d\vec v.\frac{D\vec w}{Dt}+d\vec v.d\vec v.\vec w,\end{aligned}$$

thus $(9.39)_1$, thus $(9.39)_2$. ∴

9.5 Lie derivative of a differential form

When the Lie derivative of a vector field $\vec w$ cannot be obtained by direct measurements, you need to use a “measuring device” (Germain: To know the weight of a suitcase you have to lift it: You use work).

Here we consider a measuring device which is a differential form α. So, if $\vec w$ is a vector field then $f=\alpha.\vec v$ is a scalar function, and (9.13) gives $\mathcal{L}_{\vec v}(\alpha.\vec w)=\frac{D(\alpha.\vec w)}{Dt}=\frac{D\alpha}{Dt}.\vec w+\alpha.\frac{D\vec w}{Dt}$, thus

$$\mathcal{L}_{\vec v}(\alpha.\vec w)=\underbrace{\frac{D\alpha}{Dt}.\vec w+\alpha.d\vec v.\vec w}_{\to(\mathcal{L}_{\vec v}\alpha).\vec w}+\underbrace{\alpha.\frac{D\vec w}{Dt}-\alpha.d\vec v.\vec w}_{=\alpha.\mathcal{L}_{\vec v}\vec w}: \tag{9.40}$$

Definition 9.12 Let α be a differential form. The Lie derivative of α along $\vec v$ is the differential form

$$\boxed{\mathcal{L}_{\vec v}\alpha:=\frac{D\alpha}{Dt}+\alpha.d\vec v}=\frac{\partial\alpha}{\partial t}+d\alpha.\vec v+\alpha.d\vec v. \tag{9.41}$$

(An equivalent definition is given at (9.47).) I.e., for all vector field $\vec w$,

$$\mathcal{L}_{\vec v}\alpha.\vec w:=\frac{D\alpha}{Dt}.\vec w+\alpha.d\vec v.\vec w\quad(=\frac{\partial\alpha}{\partial t}.\vec w+(d\alpha.\vec v).\vec w+\alpha.d\vec v.\vec w). \tag{9.42}$$

The definition of $\mathcal{L}_{\vec{v}}\alpha$, cf. (9.41), immediately gives the "derivation property"

$$\mathcal{L}_{\vec{v}}(\alpha.\vec{w}) = (\mathcal{L}_{\vec{v}}\alpha).\vec{w} + \alpha.(\mathcal{L}_{\vec{v}}\vec{w}) \quad \text{(i.e. } \mathcal{L}_{\vec{v}} \text{ is a derivation).} \tag{9.43}$$

Quantification: Relative to a basis $(\vec{e}_i)$ and with $[\cdot] := [\cdot]_{|\vec{e}}$,

$$\boxed{[\mathcal{L}_{\vec{v}}\alpha] = [\frac{D\alpha}{Dt}] + [\alpha].[d\vec{v}]} \quad \text{(row matrix)} \;= [\frac{\partial\alpha}{\partial t}] + [d\alpha.\vec{v}] + [\alpha].[d\vec{v}]. \tag{9.44}$$

Thus

$$[\mathcal{L}_{\vec{v}}\alpha.\vec{w}] = [\mathcal{L}_{\vec{v}}\alpha].[\vec{w}] = [\frac{\partial\alpha}{\partial t}].[\vec{w}] + [d\alpha.\vec{v}].[\vec{w}] + [\alpha].[d\vec{v}].[\vec{w}]. \tag{9.45}$$

Exercice 9.13 Prove (9.44) with components. And prove $[d\alpha.\vec{v}] = [\vec{v}]^T.[d\alpha]^T$ (row matrix), thus $[d\alpha.\vec{v}].[\vec{w}] = [\vec{v}]^T.[d\alpha]^T.[\vec{w}] = [\vec{w}]^T.[d\alpha].[\vec{v}]$.

Answer. Basis $(\vec{e}_i)$, dual basis (π_{ei}), thus (9.41) gives $[\mathcal{L}_{\vec{v}}\alpha] = [\frac{D\alpha}{Dt}] + [\alpha.d\vec{v}]$. Let $\alpha = \sum_i \alpha_i\pi_{ei}$, $\vec{v} = \sum_i v_i\vec{e}_i$, $d\vec{v} = \sum_{ij} v_{i|j}\vec{e}_i \otimes \pi_{ej}$ (tensorial writing convenient for calculations), i.e. $[d\vec{v}]_{|\vec{e}} = [v_{i|j}]$, thus $\alpha.d\vec{v} = \sum_{ij} \alpha_i v_{i|j}\pi_{ej}$, thus $[\alpha.d\vec{v}]_{|\pi_e} = [\alpha]_{|\pi_e}.[d\vec{v}]_{|\vec{e}}$ (row matrix). And $d\alpha = \sum_{ij} \alpha_{i|j}\pi_{ei} \otimes \pi_{ej}$, i.e. $[d\alpha]_{|\pi_e} = [\alpha_{i|j}]$, gives $d\alpha.\vec{v} = \sum_{ij} \alpha_{i|j}v_j\pi_{ei} = \sum_{ij} v_i\alpha_{j|i}\pi_{ej}$, and $[d\alpha.\vec{v}]_{|\pi_e}$ is a row matrix ($d\alpha.\vec{v}$ is a differential form), thus $[d\alpha.\vec{v}]_{|\pi_e} = [\vec{v}]^T_{|\vec{e}}.[d\alpha]^T_{|\pi_e}$. (Or compute $(d\alpha.\vec{v}).\vec{w} = \sum_{ij} \alpha_{i|j}v_j w_i = [\vec{w}]^T_{|\vec{e}}.[d\alpha]_{|\vec{e}}.[\vec{v}]_{|\vec{e}} = [\vec{v}]^T_{|\vec{e}}.[d\alpha]^T_{|\pi_e}.[\vec{w}]_{|\vec{e}}$.) ∎

Exercice 9.14 Let α be a differential form, and let $\alpha_t(p) := \alpha(t,p)$. Prove, when $\Phi_t^{t_0}$ is a diffeomorphism,

$$\mathcal{L}_{\vec{v}}\alpha = 0 \quad \Longleftrightarrow \quad \forall t \in [t_0, T],\ \alpha_t = (\Phi_t^{t_0})_*\alpha_{t_0}. \tag{9.46}$$

I.e.: $\frac{D\alpha}{Dt} = -\alpha.d\vec{v} \Longleftrightarrow \alpha_t(p_t) = \alpha_{t_0}(p_{t_0}).F_t^{t_0}(p_{t_0})^{-1}$ for all t, when $p_t = \Phi_t^{t_0}(p_{t_0})$.

Answer. $\Leftarrow$: If $\alpha_t(p(t)) = \alpha_{t_0}(p_{t_0}).F_t^{t_0}(p_{t_0})^{-1}$, then $\alpha(t,p(t)).F^{t_0}(t,p_{t_0}) = \alpha_{t_0}(p_{t_0})$, thus $\frac{D\alpha}{Dt}(t,p_t).F_t^{t_0}(p_{t_0}) + \alpha_t(p_t).\frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}) = 0$, thus $\frac{D\alpha}{Dt}(t,p(t)).F_t^{t_0}(p_{t_0}) + \alpha_t(p_t).d\vec{v}(t,p_t).F_t^{t_0}(p_{t_0}) = 0$, thus $\mathcal{L}_{\vec{v}}\alpha = 0$, since $\Phi_t^{t_0}$ is a diffeomorphism.

$\Rightarrow$: If $\beta(t) := (\Phi_t^{t_0})_*\alpha_{t_0}(p_{t_0}) = \alpha_t(p(t)).F_t^{t_0}(p_{t_0})$ (pull-back at (t_0,p_{t_0})), then $\beta(t) = \alpha(t,p(t)).F^{t_0}(t,p_{t_0})$, thus $\beta'(t) = \frac{D\alpha}{Dt}(t,p_t).F_t^{t_0}(p_{t_0}) + \alpha(t,p_t).d\vec{v}(t,p_t).F_t^{t_0}(p_{t_0}) = 0$ (hypothesis $\mathcal{L}_{\vec{v}}\alpha = 0$), thus $\beta(t) = \beta(t_0) = \alpha_{t_0}(p_{t_0})$. ∎

Remark 9.15 A definition equivalent to (9.41) is, cf. (9.10),

$$\begin{aligned}\mathcal{L}_{\vec{v}}\alpha(t,p_t) &:= \lim_{\tau\to t}\frac{(\Phi_\tau^t)^*\alpha_\tau(p_t) - \alpha_t(p_t)}{\tau - t} \quad (= \lim_{\tau\to t}\frac{\alpha_\tau(p_\tau).d\Phi_\tau^t(p_t) - \alpha_t(p_t)}{\tau - t}) \\ &\overset{\text{noted}}{=} \frac{D(\Phi_\tau^{t*}\alpha_\tau(p_t))}{D\tau}_{|\tau=t} \overset{\text{noted}}{=} \frac{D(\alpha_\tau^*(p_t))}{D\tau}_{|\tau=t} \quad (= \frac{D(\alpha_\tau(p_\tau).d\Phi_\tau^t(p_t))}{D\tau}_{|\tau=t}).\end{aligned} \tag{9.47}$$

Indeed, if $\beta(\tau) = (\Phi_\tau^t)^*\alpha_\tau(p_t) = \alpha_\tau(p_\tau).d\Phi_\tau^t(p_t)$, then $\beta'(\tau)$ and then $\tau = t$ give (9.41). ∎

Exercice 9.16 $\vec{v}$ and α being C^2, prove:

$$\begin{aligned}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\alpha) = &\frac{\partial^2\alpha}{\partial t^2} + 2d\frac{\partial\alpha}{\partial t}.\vec{v} + 2\frac{\partial\alpha}{\partial t}.d\vec{v} + d\alpha.\frac{\partial\vec{v}}{\partial t} + \alpha.\frac{\partial d\vec{v}}{\partial t} \\ &+ (d^2\alpha.\vec{v}).\vec{v} + d\alpha.(d\vec{v}.\vec{v}) + 2(d\alpha.\vec{v}).d\vec{v} + \alpha.(d^2\vec{v}.\vec{v}) + (\alpha.d\vec{v}).d\vec{v}.\end{aligned} \tag{9.48}$$

Answer. (9.41) gives

$$\begin{aligned}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\alpha) &= \mathcal{L}_{\vec{v}}(\frac{\partial\alpha}{\partial t}) + \mathcal{L}_{\vec{v}}(d\alpha.\vec{v}) + \mathcal{L}_{\vec{v}}(\alpha.d\vec{v}) \\ &= \frac{\partial^2\alpha}{\partial t^2} + d\frac{\partial\alpha}{\partial t}.\vec{v} + \frac{\partial\alpha}{\partial t}.d\vec{v} + \frac{\partial(d\alpha.\vec{v})}{\partial t} + d(d\alpha.\vec{v}).\vec{v} + (d\alpha.\vec{v}).d\vec{v} + \frac{\partial(\alpha.d\vec{v})}{\partial t} + d(\alpha.d\vec{v}).\vec{v} + (\alpha.d\vec{v}).d\vec{v} \\ &= \frac{\partial^2\alpha}{\partial t^2} + d\frac{\partial\alpha}{\partial t}.\vec{v} + \frac{\partial\alpha}{\partial t}.d\vec{v} + \frac{\partial d\alpha}{\partial t}.\vec{v} + d\alpha.\frac{\partial\vec{v}}{\partial t} + (d^2\alpha.\vec{v}).\vec{v} + d\alpha.(d\vec{v}.\vec{v}) + (d\alpha.\vec{v}).d\vec{v} \\ &\quad + \frac{\partial\alpha}{\partial t}.d\vec{v} + \alpha.\frac{\partial d\vec{v}}{\partial t} + (d\alpha.\vec{v}).d\vec{v} + \alpha.d^2\vec{v}.\vec{v} + (\alpha.d\vec{v}).d\vec{v} \\ &= \frac{\partial^2\alpha}{\partial t^2} + 2d\frac{\partial\alpha}{\partial t}.\vec{v} + 2\frac{\partial\alpha}{\partial t}.d\vec{v} + d\alpha.\frac{\partial\vec{v}}{\partial t} + (d^2\alpha.\vec{v}).\vec{v} + d\alpha.(d\vec{v}.\vec{v}) + 2(d\alpha.\vec{v}).d\vec{v} + \alpha.\frac{\partial d\vec{v}}{\partial t} \\ &\quad + \alpha.(d^2\vec{v}.\vec{v}) + (\alpha.d\vec{v}).d\vec{v}.\end{aligned}$$

∎

9.6 Incompatibility with Riesz representation vectors

The Lie derivative has nothing to do with any inner dot product (the Lie derivative does not compare two vectors, contrary to a Cauchy type approach).

Here we introduce a Euclidean dot product $(\cdot,\cdot)_g$ and show that the Lie derivative of a linear form α is not trivially deduced from the Lie derivative of a Riesz representation vector of α (which one?). (Same issue as at § 7.2.)

Let α be a Eulerian differential form; Then let $\vec{a}_g(t,p) \in \vec{\mathbb{R}}^n$ be the $(\cdot,\cdot)_g$-Riesz representation vector of the linear form $\alpha(t,p) \in \mathbb{R}^{n*}$: So, for all Eulerian vector field $\vec{w}$,

$$\alpha.\vec{w} = (\vec{a}_g, \vec{w})_g \qquad (= \vec{a}_g \bullet_g \vec{w}), \tag{9.49}$$

which means $\alpha(t,p).\vec{w}(t,p) = (\vec{a}_g(t,p), \vec{w}(t,p))_g$ at all admissible (t,p). This defines the Eulerian vector field $\vec{a}_g$ (not intrinsic to α: $\vec{a}_g$ depends on the choice of $(\cdot,\cdot)_g$, cf. (F.12)).

Proposition 9.17 *For all* $\vec{v}, \vec{w} \in \vec{\mathbb{R}}^n$,

$$\frac{\partial\alpha}{\partial t}.\vec{w} = (\frac{\partial\vec{a}_g}{\partial t}, \vec{w})_g, \quad (d\alpha.\vec{v}).\vec{w} = (d\vec{a}_g.\vec{v}, \vec{w})_g, \quad \frac{D\alpha}{Dt}.\vec{w} = (\frac{D\vec{a}_g}{Dt}, \vec{w})_g. \tag{9.50}$$

Thus

$$\mathcal{L}_{\vec{v}}\alpha.\vec{w} = (\mathcal{L}_{\vec{v}}\vec{a}_g, \vec{w})_g + (\vec{a}_g, (d\vec{v}{+}d\vec{v}^T).\vec{w})_g, \quad \textit{and} \quad \boxed{\mathcal{L}_{\vec{v}}\alpha.\vec{w} \neq (\mathcal{L}_{\vec{v}}\vec{a}_g, \vec{w})_g} \quad \textit{in general.} \tag{9.51}$$

*So $\mathcal{L}_{\vec{v}}\vec{a}_g$ is **not** the Riesz representation vector of $\mathcal{L}_{\vec{v}}\alpha$ (but for solid body motions). (Expected: A Lie derivative is covariant objective, see § 11.4, and the use of an inner dot product ruins this objectivity.)*

Proof. A Euclidean dot product $g(\cdot,\cdot)$ is bilinear constant and uniform, thus:

$\alpha.\vec{w} = (\vec{a}_g, \vec{w})_g$ gives $\frac{\partial\alpha}{\partial t}.\vec{w} + \alpha.\frac{\partial\vec{w}}{\partial t} = (\frac{\partial\vec{a}_g}{\partial t}, \vec{w})_g + (\vec{a}_g, \frac{\partial\vec{w}}{\partial t})_g$, with $\alpha.\frac{\partial\vec{w}}{\partial t} = (\vec{a}_g, \frac{\partial\vec{w}}{\partial t})_g$, thus we are left with $\frac{\partial\alpha}{\partial t}.\vec{w} = (\frac{\partial\vec{a}_g}{\partial t}, \vec{w})_g$, for all $\vec{w}$.

$\alpha.\vec{w} = (\vec{a}_g, \vec{w})_g$ gives $d(\alpha.\vec{w}).\vec{v} = d(\vec{a}_g, \vec{w})_g.\vec{v}$ for all $\vec{v}, \vec{w}$, thus $(d\alpha.\vec{v}).\vec{w} + \alpha.(d\vec{w}.\vec{v}) = (d\vec{a}_g.\vec{v}, \vec{w})_g + (\vec{a}_g, d\vec{w}.\vec{v})_g$, with $\alpha.(d\vec{w}.\vec{v}) = (\vec{a}_g, d\vec{w}.\vec{v})_g$, thus we are left with $(d\alpha.\vec{v}).\vec{w} = (d\vec{a}_g.\vec{v}, \vec{w})_g$.

Thus $\frac{D\alpha}{Dt}.\vec{w} = (\frac{D\vec{a}_g}{Dt}, \vec{w})_g$.

Thus $(\mathcal{L}_{\vec{v}}\alpha).\vec{w} = \frac{D\alpha}{Dt}.\vec{w} + \alpha.d\vec{v}.\vec{w} = (\frac{D\vec{a}_g}{Dt}, \vec{w})_g + (\vec{a}_g, d\vec{v}.\vec{w})_g = (\mathcal{L}_{\vec{v}}\vec{a}_g + d\vec{v}.\vec{a}_g, \vec{w})_g + (d\vec{v}_g^T.\vec{a}_g, \vec{w})_g$. ▪

Remark 9.18 Chorus: a "differential form" (measuring instrument, covariant) should not be confused with a "vector field" (object to be measured, contravariant); Thus, the use of a dot product (which one?) and the Riesz representation theorem should be restricted for computational purposes, after an objective equation has been established. See also remark F.13. ▪

9.7 Lie derivative of a tensor

The Lie derivative of any tensor of order ≥ 2 is defined thanks to

$$\mathcal{L}_{\vec{v}}(T \otimes S) = (\mathcal{L}_{\vec{v}}T) \otimes S + T \otimes (\mathcal{L}_{\vec{v}}S) \quad \text{(derivation formula).} \tag{9.52}$$

(Or direct definition: $\mathcal{L}_{\vec{v}}T(t_0, p_{t_0}) = \frac{D((\Phi_t^{t_0})^* T_t)(p_{t_0})}{Dt}_{|t=t_0})$.

9.7.1 Lie derivative of a mixed tensor

Let $T_m \in T_1^1(\Omega)$, and T_m is called a mixed tensor; Its Lie derivative, called the Jaumann derivative, is given by

$$\boxed{\mathcal{L}_{\vec{v}}T_m = \frac{DT_m}{Dt} - d\vec{v}.T_m + T_m.d\vec{v}} = \frac{\partial T_m}{\partial t} + dT_m.\vec{v} - d\vec{v}.T_m + T_m.d\vec{v}. \tag{9.53}$$

Can be checked with an elementary tensor $T = \vec{w} \otimes \alpha$: we have $d(\vec{w} \otimes \alpha).\vec{v} = (d\vec{w}.\vec{v}) \otimes \alpha + \vec{w} \otimes (d\alpha.\vec{v})$ and $(d\vec{v}.\vec{w}) \otimes \alpha = d\vec{v}.(\vec{w} \otimes \alpha)$, and $\vec{w} \otimes (\alpha.d\vec{v}) = (\vec{w} \otimes \alpha).d\vec{v}$, thus (9.52) gives $\mathcal{L}_{\vec{v}}(\vec{w} \otimes \alpha) = (\mathcal{L}_{\vec{v}}\vec{w}) \otimes \alpha + \vec{w} \otimes (\mathcal{L}_{\vec{v}}\alpha)$
$= \frac{\partial\vec{w}}{\partial t} \otimes \alpha + (d\vec{w}.\vec{v}) \otimes \alpha - (d\vec{v}.\vec{w}) \otimes \alpha + \vec{w} \otimes \frac{\partial\alpha}{\partial t} + \vec{w} \otimes (d\alpha.\vec{v}) + \vec{w} \otimes (\alpha.d\vec{v})$
$= \frac{\partial\vec{w}\otimes\alpha}{\partial t} + d(\vec{w} \otimes \alpha).\vec{v} - d\vec{v}.(\vec{w} \otimes \alpha) + (\vec{w} \otimes \alpha).d\vec{v}$.

Quantification. Relative to a basis $(\vec{e}_i)$:

$$[\mathcal{L}_{\vec{v}}T_m] = [\frac{DT_m}{Dt}] - [d\vec{v}].[T_m] + [T_m].[d\vec{v}] \tag{9.54}$$

(the signs $\mp$ are mixed). "Mixed" also refers to positions of indices (up and down with duality notations): $T_m = \sum_{i,j=1}^n T^i{}_j \vec{e}_i \otimes e^j$ with the dual basis (e^i), i.e. $[T_m]_{|\vec{e}} = [T^i{}_j]$.

Exercice 9.19 With components, prove (9.54).

Answer. $\frac{\partial T_m}{\partial t} = \sum_{ij} \frac{\partial T^i{}_j}{\partial t} \vec{e}_i \otimes e^j$, $dT_m = \sum_{ijk} T^i{}_{j|k} \vec{e}_i \otimes e^j \otimes e^k$, $\vec{v} = \sum_i v^i \vec{e}_i$, $d\vec{v} = \sum_{ij} v^i_{|j} \vec{e}_i \otimes e^j$, thus $dT_m.\vec{v} = \sum_{ijk} T^i{}_{j|k} v^k \vec{e}_i \otimes e^j$, $d\vec{v}.T_m = \sum_{ijk} v^i_{|k} T^k{}_j \vec{e}_i \otimes e^j$, $T_m.d\vec{v} = \sum_{ijk} T^i{}_k v^k_{|j} \vec{e}_i \otimes e^j$. ∴

9.7.2 Lie derivative of a up-tensor

Recall: If $L \in \mathcal{L}(E;F)$ (a linear map) then its adjoint $L^* \in \mathcal{L}(F^*;E^*)$ is defined by, cf. § A.12,

$$\forall m \in F^*, \quad \boxed{L^*.m := m.L}, \quad \text{i.e.,} \quad \forall m, \vec{u} \in (F^* \times E), \quad (L^*.m).\vec{u} = m.L.\vec{u}. \tag{9.55}$$

(There is no inner dot product involved here.) In particular, $d\vec{v}^*.m := m.d\vec{v}$ for all $m \in \vec{\mathbb{R}}^{n*}_t$, i.e. $(d\vec{v}^*.m).\vec{u} = (m.d\vec{v}).\vec{u} = m.(d\vec{v}.\vec{u})$ for all $m \in \vec{\mathbb{R}}^{n*}_t$ and all $\vec{u} \in \vec{\mathbb{R}}^n_t$.

Let $T_u \in T^2_0(\Omega)$, and T_u is called a up tensor; Its Lie derivative is called the upper-convected (Maxwell) derivative or the Oldroyd derivative and is given by

$$\boxed{\mathcal{L}_{\vec{v}}T_u = \frac{DT_u}{Dt} - d\vec{v}.T_u - T_u.d\vec{v}^*} = \frac{\partial T_u}{\partial t} + dT_u.\vec{v} - d\vec{v}.T_u - T_u.d\vec{v}^*. \tag{9.56}$$

Can be checked with an elementary tensor $T = \vec{u} \otimes \vec{w}$ and $\mathcal{L}_{\vec{v}}(\vec{u} \otimes \vec{w}) = (\mathcal{L}_{\vec{v}}\vec{u}) \otimes \vec{w} + \vec{u} \otimes (\mathcal{L}_{\vec{v}}\vec{w})$.

Quantification. Relative to a basis $(\vec{e}_i)$:

$$[\mathcal{L}_{\vec{v}}T_u] = [\frac{DT_u}{Dt}] - [d\vec{v}].[T_u] - [T_u].[d\vec{v}]^T. \tag{9.57}$$

"up" also refers to positions of indices (with duality notations): $T_u = \sum_{i,j=1}^n T^{ij} \vec{e}_i \otimes \vec{e}_j$ with the dual basis (e^i), i.e. $[T_u]_{|\vec{e}} = [T^{ij}]$.

Exercice 9.20 With components, prove (9.56).

Answer. $\frac{\partial T_u}{\partial t} = \sum_{ij} \frac{\partial T^{ij}}{\partial t} \vec{e}_i \otimes \vec{e}_j$, $dT_u = \sum_{ijk} T^{ij}_{|k} \vec{e}_i \otimes \vec{e}_j \otimes e^k$, $\vec{v} = \sum_i v^i \vec{e}_i$, $d\vec{v} = \sum_{ij} v^i_{|j} \vec{e}_i \otimes e^j$, $d\vec{v}^* = \sum_{ij} v^j_{|i} e^i \otimes \vec{e}_j$, thus $dT_u.\vec{v} = \sum_{ijk} T^{ij}_{|k} v^k \vec{e}_i \otimes e^j$, $d\vec{v}.T_u = \sum_{ijk} v^i_{|k} T^{kj} \vec{e}_i \otimes \vec{e}_j$, $T_u.d\vec{v}^* = \sum_{ijk} T^{ik} v^j_{|k} e^i \otimes \vec{e}_j$. ∴

9.7.3 Lie derivative of a down-tensor

Let $T_d \in T^0_2(\Omega)$, and T_d is called a down tensor; The Lie derivative is called the lower-convected Maxwell derivative and is given by

$$\boxed{\mathcal{L}_{\vec{v}}T_d = \frac{DT_d}{Dt} + T_d.d\vec{v} + d\vec{v}^*.T_d} = \frac{\partial T_d}{\partial t} + dT_d.\vec{v} + T_d.d\vec{v} + d\vec{v}^*.T_d. \tag{9.58}$$

Can be checked with an elementary tensor $T = \ell \otimes m$ and $\mathcal{L}_{\vec{v}}(\ell \otimes m) = (\mathcal{L}_{\vec{v}}\ell) \otimes m + \ell \otimes (\mathcal{L}_{\vec{v}}m)$.

Quantification. Relative to a basis $(\vec{e}_i)$:

$$[\mathcal{L}_{\vec{v}}T_d] = [\frac{DT_d}{Dt}] + [T_d].[d\vec{v}] + [d\vec{v}]^T.[T_d]. \tag{9.59}$$

"down" also refers to positions of indices (with duality notations): $T_d = \sum_{i,j=1}^n T_{ij} e^i \otimes e^j$ with the dual basis (e^i), i.e. $[T_d]_{|\vec{e}} = [T_{ij}]$.

Exercice 9.21 With components, prove (9.59).

Answer. $\frac{\partial T_d}{\partial t} = \sum_{ij} \frac{\partial T_{ij}}{\partial t} e^i \otimes e^j$, $dT_d = \sum_{ijk} T_{ij|k} e^i \otimes e^j \otimes e^k$, $\vec{v} = \sum_i v^i \vec{e}_i$, $d\vec{v} = \sum_{ij} v^i_{|j} \vec{e}_i \otimes e^j$, $d\vec{v}^* = \sum_{ij} v^j_{|i} e^i \otimes \vec{e}_j$, thus $dT_d.\vec{v} = \sum_{ijk} T_{ij|k} v^k e^i \otimes e^j$, $T_d.d\vec{v} = \sum_{ijk} T_{ik} v^k_{|j} e^i \otimes \vec{e}_j$, $d\vec{v}^*.T_d = \sum_{ijk} v^k_{|i} T_{kj} e^i \otimes \vec{e}_j$. ∴

Example 9.22 Let $g = (\cdot,\cdot)_g \in T^0_2(\Omega)$ be a constant and uniform metric (a unique inner dot product for all t, p, e.g., a Euclidean dot product at all t). Then $\frac{Dg}{Dt} = 0$, thus $\mathcal{L}_{\vec{v}}g = 0 + g.d\vec{v} + d\vec{v}^*.g$, thus $[\mathcal{L}_{\vec{v}}g] = [g].[d\vec{v}] + [d\vec{v}]^T.[g]$. ∴

Part IV
Velocity-addition formula

10 Change of referential and velocity-addition formula

10.0 Issue and result (summary)

The velocity-addition formula is (in classical mechanics)

$$\vec v_A = \vec v_B + \vec v_D, \tag{10.1}$$

where $\vec v_A$, $\vec v_B$ and $\vec v_D$ are the absolute, relative and drive velocity, $\vec v_A$ and $\vec v_D$ being velocities described by an observer A with his referential $\mathcal{R}_A = (O_A, (\vec A_i))$ and $\vec v_B$ being a velocity described by an observer B with his referential $\mathcal{R}_B = (O_B, (\vec B_i))$. But (10.1) is problematic (inconsistent):

- The velocities $\vec v_A$ and $\vec v_D$ are quantified in $\mathcal{R}_A$, e.g. expressed in foot/s by the absolute observer,
- The velocity $\vec v_B$ is a quantified in $\mathcal{R}_B$, e.g. expressed in metre/s by the relative observer,

Thus (10.1) with $\vec v_B + \vec v_D$ tells that you add metre/s and foot/s... absurd. So:

Question: What are we missing (and what does (10.1) really mean)?
Answer: We miss a functional link: The translator between A and B. Summary:

Call $\widetilde\Phi$ the motion of a observed object *Obj*; $\widetilde\Phi$ is *quantified* by A in his referential $\mathcal{R}_A = (O_A, (\vec A_i))$ as the "motion" $\vec\varphi_A = [\overrightarrow{O_A\widetilde\Phi}]_{|\vec A}$, and is *quantified* by B in his referential $\mathcal{R}_B = (O_B, (\vec B_i))$ as the "motion" $\vec\varphi_B = [\overrightarrow{O_B\widetilde\Phi}]_{|\vec B}$. At t, the translator Θ connects these numerical values: $\vec\varphi_A(t, P_{Obj}) = \Theta(t, \vec\varphi_B(t, P_{Obj}))$. Thus $\frac{\partial\vec\varphi_A}{\partial t}(t, P_{Obj}) = \frac{\partial\Theta}{\partial t}(t, \vec x_{Bt}) + d\Theta(t, \vec x_{Bt}).\frac{\partial\vec\varphi_B}{\partial t}(t, P_{Obj})$, i.e. $\vec v_A(t, \vec x_{At}) = \frac{\partial\Theta}{\partial t}(t, \vec x_{Bt}) + d\Theta(t, \vec x_{Bt}).\vec v_B(t, \vec x_{Bt})$ where $\vec x_{At} = \vec\varphi_A(t, P_{Obj})$ and $\vec x_{Bt} = \vec\varphi_B(t, P_{Obj})$. Then call

$$d\Theta_t(\vec x_{Bt}).\vec v_{Bt}(\vec x_{Bt}) = \vec v_{Bt*}(\vec x_{At}) = \text{“the translated relative velocity at } t \text{ from B to A”}, \tag{10.2}$$

thus, with $\frac{\partial\Theta}{\partial t}(t, \vec x_{Bt}) = \vec v_D(t, \vec x_{At})$ the drive velocity, which gives $\vec v_A(t, \vec x_{At}) = \vec v_{B*}(t, \vec x_{At}) + \vec v_D(t, \vec x_{At})$: so

$$\vec v_A = \vec v_{B*} + \vec v_D \quad = \quad \text{the velocity addition formula in } \mathcal{R}_A, \tag{10.3}$$

i.e.: (Absolute velocity) = (Translated relative velocity) + (Drive velocity).

In other words, with $\vec v$ the velocity of *Obj* and with $\vec v_{\mathcal{R}_B}$ the velocity of $\mathcal{R}_B$ in $\mathcal{R}_A$: For all $p_t = \widetilde\Phi(t, P_{Obj})$,

$$[\vec v_t(p_t)]_{|\vec A} = d\Theta_t.[\vec v_t(p_t)]_{\vec B} + [\vec v_{\mathcal{R}_{Bt}}(p_t)]_{|\vec A}, \tag{10.4}$$

relation between the numerical values of the velocities stored by A and B.

Example 10.1 Translation motion of $\mathcal{R}_B$ in $\mathcal{R}_A$, so $[\vec v_{\mathcal{R}_{Bt}}(p_t)]_{|\vec A} = [\vec v_{\mathcal{R}_{Bt}}]_{|\vec A}$ is independent of p_t; And, e.g. with $(\vec B_{it}) = \lambda(\vec A_{it})$ (e.g. $\vec A_i$ in foot and $\vec B_i$ in meter give $\lambda \simeq 3.28$), $d\Theta_t = \lambda I$, hence $[\vec v_t(p_t)]_{|\vec A} = \lambda[\vec v_t(p_t)]_{\vec B} + [\vec v_{\mathcal{R}_{Bt}}]_{|\vec A}$, which is the expected relation ("sum of the velocities with the good units"). ⁂

Example 10.2 Motion of the Earth around the Sun: See § 10.11. ⁂

10.1 Referentials and "matrix motions"

10.1.1 Absolute and relative referentials

Classical mechanics framework: Time and space are decoupled, all the observers share the same time unit (e.g. the second) and live in "our" Universe modeled as $\mathbb{R}^3$ (affine space) with its usual associated vector space $\vec{\mathbb{R}}^3$. In the following, the affine space is $\mathbb{R}^n$ associated to the vector space $\vec{\mathbb{R}}^n$, $n \in \{1, 2, 3\}$.

An observer A, which we will call the absolute observer, chooses a (rigid body) object $Obj\mathcal{R}_A$ in the Universe, chooses one particle in $Obj\mathcal{R}_A$, calls O_{At} its position at t, and chooses three more particles in $Obj\mathcal{R}_A$, calls P_{Ati} their positions at t (in the Universe), such that the bi-point vectors $\vec A_{it} := \overrightarrow{O_{At}P_{Ati}}$ make a basis in $\vec{\mathbb{R}}^n$. He has thus built his (Cartesian) referential $\mathcal{R}_{At} = (O_{At}, (\vec A_{it}))$, called the absolute referential, and written $\mathcal{R}_A = (O_A, (\vec A_i))$ when used by A. E.g. $Obj\mathcal{R}_A$ is the "Sun extended to infinity", and at t, O_{At} is the position of the center of the Sun in the Universe, $(\vec A_{it})$ is a Euclidean basis in foot fixed relative to stars.

An observer B, which we will call the relative observer, proceeds similarly: He chooses a (rigid body) object $\mathit{Obj}\mathcal{R}_B$ in the Universe, builds his Cartesian referential $\mathcal{R}_{Bt} = (O_{Bt}, (\vec{B}_{it}))$, called the relative referential, written $\mathcal{R}_B = (O_B, (\vec{B}_i))$ when used by B. E.g. $\mathit{Obj}\mathcal{R}_B$ is the "Earth extended to infinity", and at t, O_{Bt} is the position of the center of the Earth and $(\vec{B}_{it})$ is a Euclidean basis in metre fixed relative to the Earth.

$\mathcal{M}_{n1}$ is the vectorial space of $n*1$ matrices (column matrices). A and B call $\mathcal{M}_{n1}$(A) and $\mathcal{M}_{n1}$(B) the affine spaces of $n*1$ matrices made of the "matrix positions" $[\overrightarrow{O_{At}p_t}]_{|\vec{A}}$ and $[\overrightarrow{O_{Bt}p_t}]_{|\vec{B}}$ where p_t is the position at t of a particle in the Universe.

If a function φ is given as $\varphi(t,x)$, then $\varphi_t(x) := \varphi(t,x)$, and conversely.

10.1.2 Motion of a material object *Obj*

An object *Obj* is considered by all observers. Its motion in the Universe is

$$\widetilde{\Phi} : \begin{cases} [t_1,t_2] \times \mathit{Obj} & \rightarrow \mathbb{R}^n \\ (t, P_{\mathit{Obj}}) & \rightarrow p_t = \widetilde{\Phi}(t, P_{\mathit{Obj}}) = \text{position of the particle } P_{\mathit{Obj}} \text{ at t in the Universe.} \end{cases} \tag{10.5}$$

At t at $p_t = \widetilde{\Phi}(t, P_{\mathit{Obj}})$, the Eulerian velocities and accelerations of P_{Obj} are

$$\vec{v}(t,p_t) = \frac{\partial \widetilde{\Phi}}{\partial t}(t, P_{\mathit{Obj}}) \quad \text{and} \quad \vec{\gamma}(t,p_t) = \frac{\partial^2 \widetilde{\Phi}}{\partial^2 t}(t, P_{\mathit{Obj}}) \quad (\in \vec{\mathbb{R}}^n). \tag{10.6}$$

10.1.3 Quantification: Absolute and relative "motion" of *Obj*

At t, the position $p_t = \widetilde{\Phi}(t, P_{\mathit{Obj}})$ of a particle $P_{\mathit{Obj}} \in \mathit{Obj}$ is spotted by A, resp. B, with the bi-point vectors $\overrightarrow{O_{At}p_t}$, resp. $\overrightarrow{O_{Bt}p_t}$ in $\vec{\mathbb{R}}^n$, which components is stored by A, resp. B, in his referentials: With

$$\overrightarrow{O_{At}p_t} = \sum_{i=1}^n x_{Ati}\vec{A}_{it} \quad \text{and} \quad \overrightarrow{O_{Bt}p_t} = \sum_{i=1}^n x_{Bti}\vec{B}_{it}, \tag{10.7}$$

and with $(\vec{E}_i)$ the canonical basis in $\mathcal{M}_{n1}$, the $n*1$ matrices

$$\vec{x}_{At} := [\overrightarrow{O_A p_t}]_{|\vec{A}} = \begin{pmatrix} x_{At1} \\ \vdots \\ x_{Atn} \end{pmatrix} = \sum_{i=1}^n x_{Ati}\vec{E}_i, \quad \text{and} \quad \vec{x}_{Bt} := [\overrightarrow{O_B p_t}]_{|\vec{B}} = \begin{pmatrix} x_{Bt1} \\ \vdots \\ x_{Btn} \end{pmatrix} = \sum_{i=1}^n x_{Bti}\vec{E}_i, \tag{10.8}$$

are stored by A and B. (Initial notation: $\vec{x}_{At} := [\overrightarrow{O_{At}p_t}]_{|(\vec{A}_{it})}$, but here O_{At} and $(\vec{A}_{it})$ are fixed in $\mathcal{R}_A$, idem for B.)

Mind the notations: p_t is a point, $\overrightarrow{O_{At}p_t}$ is a vector, $\vec{x}_{At}$ is a column matrix (components).

This defines the "absolute motion" $\vec{\varphi}_A$ and "relative motion" $\vec{\varphi}_B$ of *Obj* (matrix valued):

$$\vec{\varphi}_A : \begin{cases} [t_1,t_2] \times \mathit{Obj} & \rightarrow \mathcal{M}_{n1}(\text{A}) \\ (t, P_{\mathit{Obj}}) & \rightarrow \boxed{\vec{\varphi}_A(t, P_{\mathit{Obj}}) := [\overrightarrow{O_A\widetilde{\Phi}(t, P_{\mathit{Obj}})}]_{|\vec{A}}} = \sum_{i=1}^n x_{Ai}(t)\vec{E}_i \overset{\text{noted}}{=} \vec{x}_A(t) = [\overrightarrow{O_A p(t)}]_{|\vec{A}}, \end{cases} \tag{10.9}$$

$$\vec{\varphi}_B : \begin{cases} [t_1,t_2] \times \mathit{Obj} & \rightarrow \mathcal{M}_{n1}(\text{B}) \\ (t, P_{\mathit{Obj}}) & \rightarrow \boxed{\vec{\varphi}_B(t, P_{\mathit{Obj}}) := [\overrightarrow{O_B\widetilde{\Phi}(t, P_{\mathit{Obj}})}]_{|\vec{B}}} = \sum_{i=1}^n x_{Bi}(t)\vec{E}_i \overset{\text{noted}}{=} \vec{x}_B(t) = [\overrightarrow{O_B p(t)}]_{|\vec{B}}. \end{cases} \tag{10.10}$$

And the "absolute" and "relative" velocities and accelerations of P_{Obj} are (matrix valued in $\mathcal{M}_{n1}$):

$$\boxed{\vec{v}_A(t, \vec{x}_{At}) := [\vec{v}(t,p_t)]_{|\vec{A}}} \quad \text{and} \quad \vec{\gamma}_A(t, \vec{x}_{At}) := [\vec{\gamma}(t,p_t)]_{|\vec{A}}, \quad \text{when} \quad \vec{x}_{At} := [\overrightarrow{O_A p_t}]_{|\vec{A}}, \tag{10.11}$$

$$\boxed{\vec{v}_B(t, \vec{x}_{Bt}) := [\vec{v}(t,p_t)]_{|\vec{B}}} \quad \text{and} \quad \vec{\gamma}_B(t, \vec{x}_{Bt}) := [\vec{\gamma}(t,p_t)]_{|\vec{B}}, \quad \text{when} \quad \vec{x}_{Bt} := [\overrightarrow{O_B p_t}]_{|\vec{B}}. \tag{10.12}$$

Exercice 10.3 Prove: $\vec v_A(t,\vec x_{At}) = \frac{\partial\vec\varphi_A}{\partial t}(t,P_{\mathit{Obj}})$.

Answer. $\overrightarrow{O_{At}\widetilde\Phi_{P_{\mathit{Obj}}}(t)} = \sum_i x_{Ai}(t)\vec A_i(t)$ gives $\vec v(t,p_t) = \sum_i x_{Ai}{}'(t)\vec A_i(t) + x_{Ai}(t)\vec A_i{}'(t)$, thus $[\vec v(t,p_t)]_{|\vec A} = \sum_i x_{Ai}{}'(t)[\vec A_i(t)]_{|\vec A} + x_{Ai}(t)[\vec A_i{}'(t)]_{|\vec A}$ (since $\mathcal{M}_{n1}$ is a vector space) $= \sum_i x_{Ai}{}'(t)\vec E_i + [\vec 0]$ (the $\vec A_i(t)$ are static in $\mathcal{R}_A$: $[\vec A_i{}'(t)]_{|\vec A} = [\lim_{h\to0}\frac{\vec A_i(t+h)-\vec A_i(t)}{h}]_{|\vec A} = \lim_{h\to0}\frac{[\vec A_i(t+h)]_{|\vec A}-[\vec A_i(t)]_{|\vec A}}{h} = \lim_{h\to0}\frac{\vec E_i-\vec E_i}{h} = 0$). And $\vec\varphi_{AP_{\mathit{Obj}}}(t) = [\overrightarrow{O_A\widetilde\Phi_{P_{\mathit{Obj}}}(t)}]_{|\vec A} = \sum_i x_{Ai}(t)[\vec A_i(t)]_{|\vec A} = \sum_i x_{Ai}(t)\vec E_i$, thus $\vec\varphi_{AP_{\mathit{Obj}}}{}'(t) = \sum_i x_{Ai}{}'(t)\vec E_i$.

Exercice 10.4 $\vec u$ is a C^1 vector field, p is a point, $\vec x_A := [\overrightarrow{O_Ap}]_{|\vec A}$ and $\vec u_A(\vec x_A) := [\vec u(p)]_{|\vec A}$ (matrices). Prove: $d\vec u_A(\vec x_A) = [d\vec u(p)]_{|\vec A}$ (endomorphism in $\mathcal{M}_{n1}$), i.e. $d\vec u_A(\vec x_A).[\vec w]_{|\vec A} = [d\vec u(p).\vec w]_{|\vec A}$ for all $\vec w \in \vec{\mathbb{R}}^n$.

Answer. The point $p{+}h\vec w \in \mathbb{R}^n$ is referenced by A as $[\overrightarrow{O_Ap} + h\vec w]_{|\vec A} = [\overrightarrow{O_Ap}]_{|\vec A} + h[\vec w]_{|\vec A} = \vec x_A + h[\vec w]_{|\vec A}$. Thus $d\vec u_A(\vec x_A).[\vec w]_{|\vec A} = \lim_{h\to0}\frac{\vec u_A(\vec x_A+h[\vec w]_{|\vec A})-\vec u_A(\vec x_A)}{h} = \lim_{h\to0}\frac{[\vec u(p+h\vec w)]_{|\vec A}-[\vec u(p)]_{|\vec A}}{h} = \lim_{h\to0}\frac{[\vec u(p+h\vec w)-\vec w(p)]_{|\vec A}}{h} = [\lim_{h\to0}\frac{\vec u(p+h\vec w)-\vec w(p)}{h}]_{|\vec A} = [d\vec u(p).\vec w]_{|\vec A} = [d\vec u(p)]_{|\vec A}.[\vec w]_{|\vec A}$, true for all $\vec w$.

Exercice 10.5 Call Q_t the transition matrix from $(\vec A_{it})$ to $(\vec B_{it})$ at t. Prove $\vec x_{At} = [\overrightarrow{O_AO_{Bt}}]_{|\vec A} + Q_t.\vec x_{Bt}$.

Answer. $\vec x_{At} = [\overrightarrow{O_Ap_t}]_{|\vec A} = [\overrightarrow{O_AO_{Bt}} + \overrightarrow{O_{Bt}p_t}]_{|\vec A} = [\overrightarrow{O_AO_{Bt}}]_{|\vec A} + [\overrightarrow{O_{Bt}p_t}]_{|\vec A}$, and the change of basis formula gives $[\overrightarrow{O_{Bt}p_t}]_{|\vec B} = Q_t^{-1}.[\overrightarrow{O_{Bt}p_t}]_{|\vec A}$.

10.1.4 Motion of $\mathcal{R}_B$

Particular case $\mathit{Obj} = \mathit{Obj}\mathcal{R}_B$: Its motion in the Universe, also called the motion of $\mathcal{R}_B$, is noted

$$\widetilde\Phi_{\mathcal{R}_B} : \begin{cases} [t_1,t_2]\times \mathit{Obj}\mathcal{R}_B & \to \mathbb{R}^n \\ \quad (t,Q_{\mathcal{R}_B}) & \to q_t = \widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B}). \end{cases} \tag{10.13}$$

At t at $q_t = \widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})$, the Eulerian velocities and accelerations of $Q_{\mathcal{R}_B}$ are

$$\vec v_{\mathcal{R}_B}(t,q_t) = \frac{\partial\widetilde\Phi_{\mathcal{R}_B}}{\partial t}(t,Q_{\mathcal{R}_B}) \quad\text{and}\quad \vec\gamma_{\mathcal{R}_B}(t,q_t) = \frac{\partial^2\widetilde\Phi_{\mathcal{R}_B}}{\partial^2 t}(t,Q_{\mathcal{R}_B}). \tag{10.14}$$

10.1.5 Quantification: Drive and static “motion” of $\mathcal{R}_B$

The “drive motion” $\vec\varphi_D$, also called the motion of $\mathcal{R}_B$ in $\mathcal{R}_A$, and the “static motion” $\vec\varphi_S$ is the quantification of $\widetilde\Phi_{\mathcal{R}_B}$ by A and by B:

$$\vec\varphi_D : \begin{cases} [t_1,t_2]\times \mathit{Obj}\mathcal{R}_B & \to \mathcal{M}_{n1}(\text{A}) \\ \quad (t,Q_{\mathcal{R}_B}) & \to \vec\varphi_D(t,Q_{\mathcal{R}_B}) := [\overrightarrow{O_A\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec A} \overset{\text{noted}}{=} \vec y_D(t) = [\overrightarrow{O_Aq(t)}]_{|\vec A}, \end{cases} \tag{10.15}$$

$$\vec\varphi_S : \begin{cases} \mathit{Obj}\mathcal{R}_B & \to \mathcal{M}_{n1}(\text{B}) \\ \quad Q_{\mathcal{R}_B} & \to \vec\varphi_S(Q_{\mathcal{R}_B}) := [\overrightarrow{O_B\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec B} \overset{\text{noted}}{=} \vec y_S = [\overrightarrow{O_Bq(t)}]_{|\vec B}. \end{cases} \tag{10.16}$$

($\vec\varphi_S$ is independent of t since $\mathit{Obj}\mathcal{R}_B$ is fixed in $\mathcal{R}_B$.)

The drive velocity, also called the velocity of $\mathcal{R}_B$ in $\mathcal{R}_A$, and static velocity of $Q_{\mathcal{R}_B}$ are

$$\vec v_D(t,\vec y_{Dt}) := [\vec v_{\mathcal{R}_B}(t,q_t)]_{|\vec A} \quad\text{when}\quad \vec y_{Dt} := [\overrightarrow{O_Aq_t}]_{|\vec A}, \tag{10.17}$$

$$\vec v_S(t,\vec y_S) := [\vec v_{\mathcal{R}_B}(t,q_t)]_{|\vec B} = [\vec 0] \overset{\text{noted}}{=} \vec 0 \text{ (null matrix)}. \tag{10.18}$$

And the drive and static accelerations are $\vec\gamma_D(t,\vec y_{Dt}) = [\vec\gamma_{\mathcal{R}_B}(t,q_t)]_{|\vec A}$ and $\vec\gamma_S(t,\vec y_S) = \vec 0$.

Exercice 10.6 Why introduce $\vec\varphi_S$ (static)?

Answer. You can’t confuse a particle $Q_{\mathcal{R}_B}$ with its stored positions $\vec y_S$ or $\vec y_{Dt}$ at t. And see (10.19).

10.2 The translator Θ_t

10.2.1 Definition of Θ_t

Definition 10.7 At t, the translator $\Theta_t : \mathcal{M}_{n1}(\mathrm{B}) \to \mathcal{M}_{n1}(\mathrm{A})$ is defined with (10.15)-(10.16) by:

$$\left.\begin{cases} [\overrightarrow{O_Bq(t)}]_{|\vec B} = \vec\varphi_S(Q_{\mathcal{R}_B}) = \vec y_S \text{ position of } Q_{\mathcal{R}_B} \text{ in } \mathcal{R}_B \text{ (static), and} \\ [\overrightarrow{O_Aq(t)}]_{|\vec A} = \vec\varphi_{Dt}(Q_{\mathcal{R}_B}) = \vec y_{Dt} \text{ position of } Q_{\mathcal{R}_B} \text{ at } t \text{ in } \mathcal{R}_A \text{ (moving)} \end{cases}\right\} \Longrightarrow \vec y_{Dt} = \Theta_t(\vec y_S), \tag{10.19}$$

i.e. Θ_t is the "inter-referential function at t" which translates the "matrix position" $\vec y_S = \vec\varphi_S(Q_{\mathcal{R}_B}) = [\overrightarrow{O_B\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec B} \in \mathcal{M}_{n1}(\mathrm{B})$ (position of $Q_{\mathcal{R}_B}$ as stored by B) to the "matrix position" $\vec y_{Dt} = \vec\varphi_D(t,Q_{\mathcal{R}_B}) = [\overrightarrow{O_A\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec A} \in \mathcal{M}_{n1}(\mathrm{A})$ (position of $Q_{\mathcal{R}_B}$ as stored by A). So Θ_t is defined by

$$\boxed{\vec\varphi_{Dt} = \Theta_t \circ \vec\varphi_S} : \begin{cases} Obj\mathcal{R}_B & \to \mathcal{M}_{n1}(\mathrm{A}) \\ Q_{\mathcal{R}_B} & \to \vec\varphi_{Dt}(Q_{\mathcal{R}_B}) := \Theta_t(\vec\varphi_S(Q_{\mathcal{R}_B})), \end{cases} \tag{10.20}$$

i.e. defined by

$$\Theta_t := \vec\varphi_{Dt} \circ \vec\varphi_S^{-1} : \begin{cases} \mathcal{M}_{n1}(\mathrm{B}) & \to \mathcal{M}_{n1}(\mathrm{A}) \\ \vec y_S & \to \vec y_{Dt} = \Theta_t(\vec y_S) := \vec\varphi_{Dt}(\vec\varphi_S^{-1}(\vec y_S)) \end{cases} \tag{10.21}$$

(stored position by B to stored position by A).

E.g., for Q_{O_B} the particle in $Obj\mathcal{R}_B$ at t at O_{Bt} (chosen by B to locate its origin), (10.20) gives, with $\vec 0$ the null matrix in $\mathcal{M}_{n1}$,

$$[\overrightarrow{O_AO_{Bt}}]_{|\vec A} = \Theta_t(\vec 0). \tag{10.22}$$

So, Θ_t is defined such that the following diagram commutes:

$$\begin{tikzcd} & \vec y_S = \vec\varphi_S(Q_{\mathcal{R}_B}) = \text{localization of } Q_{\mathcal{R}_B} \text{ by B} \arrow[dd, "\Theta_t"] \\ Q_{\mathcal{R}_B} \in Obj\mathcal{R}_B \arrow[ur, "\vec\varphi_S"] \arrow[dr, "\vec\varphi_{Dt}"'] & \\ & \vec y_{Dt} = \vec\varphi_{Dt}(Q_{\mathcal{R}_B}) = \Theta_t(\vec y_S) = \text{localization at } t \text{ of } Q_{\mathcal{R}_B} \text{ by A.} \end{tikzcd} \tag{10.23}$$

10.2.2 Translation at t for the motion $\widetilde\Phi$

t is fixed, the position $p_t = \widetilde\Phi(t, P_{Obj})$ of a particle $P_{Obj} \in Obj$ is also the position $q_t = \widetilde\Phi_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})$ of a particle $Q_{\mathcal{R}_B} \in Obj\mathcal{R}_B$, so $\vec\varphi_{At}(P_{Obj}) = [\overrightarrow{O_Ap_t}]_{|\vec A} = [\overrightarrow{O_Aq_t}]_{|\vec A} = \vec\varphi_{Dt}(Q_{\mathcal{R}_B})$, and $\vec\varphi_{Bt}(P_{Obj}) = [\overrightarrow{O_Bp_t}]_{|\vec B} = [\overrightarrow{O_Bq_t}]_{|\vec B} = \vec\varphi_S(Q_{\mathcal{R}_B})$, thus (10.20) gives

$$\boxed{\vec\varphi_{At} = \Theta_t \circ \vec\varphi_{Bt}}. \tag{10.24}$$

10.3 $d\Theta_t$

10.3.1 Push-forward

If $\vec y_S \in \mathcal{M}_{n1}(\mathrm{B})$, $\vec w_S \in \mathcal{M}_{n1}$ and $\vec y_{Dt} = \Theta_t(\vec y_S)$, then

$$\vec w_{St*}(\vec y_{Dt}) := d\Theta_t(\vec y_S).\vec w_S \quad (= \lim_{h\to 0} \frac{\Theta_t(\vec y_S + h\vec w_B) - \Theta_t(\vec y_S)}{h}) \tag{10.25}$$

is the push-forward of the matrix $\vec w_S \in \mathcal{M}_{n1}$ by Θ_t.

So, $\vec w_{St*}([\overrightarrow{O_Aq_t}]_{|\vec A}) = d\Theta_t([\overrightarrow{O_Bq_t}]_{|\vec B}).[\vec w_t(q_t)]_{|\vec B}$ for all $q_t \in \mathbb{R}^n$ and all $\vec w_t : \mathbb{R}^n \to \vec{\mathbb{R}}^n$ (vector field).

10.3.2 Θ_t is affine in classical mechanics

Proposition 10.8 *In $\mathbb{R}^3$ (in classical mechanics), Θ_t is affine: For all $Q_{B0}, Q_{B1} \in \mathit{Obj}\mathcal{R}_B$ and all $t, u \in \mathbb{R}$, with $q_{t_i} = \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{Bi}) \in \mathbb{R}^n$ (positions at t in our Universe),*

$$\Theta_t([\overrightarrow{O_B q_{t_0}}]_{|\vec{B}} + u\,[\overrightarrow{q_{t_0} q_{t_1}}]_{|\vec{B}}) = [\overrightarrow{O_A q_{t_0}}]_{|\vec{A}} + u\,[\overrightarrow{q_{t_0} q_{t_1}}]_{|\vec{A}}, \quad \text{and} \quad [\overrightarrow{q_{t_0} q_{t_1}}]_{|\vec{A}} = d\Theta_t.[\overrightarrow{q_{t_0} q_{t_1}}]_{|\vec{B}} \tag{10.26}$$

the differential $d\Theta_t(\vec{y}_{S0}) =^{noted} d\Theta_t$ being independent of $\vec{y}_{S0}$. In particular $[\vec{B}_{it}]_{\vec{A}} = d\Theta_t.[\vec{B}_i]_{|\vec{B}}$. In other words, for all $\vec{y}_{S0}, \vec{y}_{S1} \in \mathcal{M}_{n1}(B)$ and all $t, u \in \mathbb{R}$,

$$\Theta_t((1-u)\vec{y}_{S0} + u\,\vec{y}_{S1}) = (1-u)\Theta_t(\vec{y}_{S0}) + u\,\Theta_t(\vec{y}_{S1}), \quad \text{and} \quad \Theta_t(\vec{y}_{S1}) = \Theta_t(\vec{y}_{S0}) + d\Theta_t.(\vec{y}_{S1}-\vec{y}_{S0}). \tag{10.27}$$

Proof. Consider the straight line (possible in classical mechanics in $\mathbb{R}^3$) $q_t : u \to q_t(u) = q_{t_0} + u\,\overrightarrow{q_{t_0} q_{t_1}} \in \mathbb{R}^n$ (fixed in $\mathcal{R}_B$), in particular, $q_t(0) = q_{t_0}$ and $q_t(1) = q_{t_1}$. Let $\vec{y}_S(u) = [\overrightarrow{O_B q_t(u)}]_{|\vec{B}}$ (positions stored by B), so $\vec{y}_S(u) = [\overrightarrow{O_B q_{t_0}} + u\,\overrightarrow{q_{t_0} q_{t_1}}]_{|\vec{B}} = [(1-u)\overrightarrow{O_B q_{t_0}} + u\,\overrightarrow{O_B q_{t_1}}]_{|\vec{B}} = (1-u)[\overrightarrow{O_B q_{t_0}}]_{|\vec{B}} + u\,[\overrightarrow{O_B q_{t_1}}]_{|\vec{B}} = (1-u)\vec{y}_{S0} + u\,\vec{y}_{S1}$, where $\vec{y}_{S0} = [\overrightarrow{O_B q_{t_0}}]_{|\vec{B}} = \vec{y}_S(0)$ and $\vec{y}_{S1} = [\overrightarrow{O_B q_{t_1}}]_{|\vec{B}} = \vec{y}_S(1)$. Idem for A: $\vec{y}_{Dt}(u) = [\overrightarrow{O_A q_t(u)}]_{|\vec{A}} = (1-u)\vec{y}_{Dt_0} + u\vec{y}_{Dt1}$ (positions stored by A). Thus

$$(1-u)\Theta_t(\vec{y}_{S0}) + u\Theta_t(\vec{y}_{S1}) \overset{(10.19)}{=} (1-u)\vec{y}_{Dt_0} + u\vec{y}_{Dt1} = \vec{y}_{Dt}(u) \overset{(10.19)}{=} \Theta_t(\vec{y}_S(u)) = \Theta_t((1-u)\vec{y}_{S0} + u\vec{y}_{S1}),$$

thus $(10.27)_1$, thus $(10.26)_1$. Hence (derivation in u): $-\Theta_t(\vec{y}_{S0}) + \Theta_t(\vec{y}_{S1}) = d\Theta_t(\vec{y}_S(u)).(-\vec{y}_{S0} + \vec{y}_{S1})$, true for all u, thus $d\Theta_t(\vec{y}_S(u))$ is independent of u, $d\Theta_t(\vec{y}_S(u)) = d\Theta_t(\vec{y}_{S0})$, true for all $\vec{y}_{S0}$, so $d\Theta_t(\vec{y}_{S0}) =^{\text{noted}} d\Theta_t$, thus $(10.27)_2$, thus $(10.26)_2$. Thus $[\overrightarrow{O_{Bt} P_{Bti}}]_{|\vec{A}} = d\Theta_t.[\overrightarrow{O_{Bt} P_{Bti}}]_{|\vec{B}}$ where P_{Bti} is s.t. $\vec{B}_{it} = \overrightarrow{O_{Bt} P_{Bti}}$, thus $[\vec{B}_{it}]_{\vec{A}} = d\Theta_t.[\vec{B}_i]_{|\vec{B}}$. ∎

Exercice 10.9 Call $Q_t = [Q_{t,ij}]$ the transition matrix from $(\vec{A}_{it})$ to $(\vec{B}_{it})$ in $\vec{\mathbb{R}}^n$, and $(\vec{E}_i)$ the canonical basis in $\mathcal{M}_{n1}$. Prove

$$[d\Theta_t]_{|\vec{E}} = Q_t, \quad \text{i.e.} \quad d\Theta_t.\vec{E}_j = \sum_{i=1}^{n} Q_{t,ij}\vec{E}_i, \quad \forall j. \tag{10.28}$$

Answer. $\vec{B}_{jt} = \sum_{i=1}^n Q_{t,ij}\vec{A}_{it}$ gives $[\vec{B}_{jt}]_{|\vec{A}} = \sum_{i=1}^n Q_{t,ij}\vec{E}_i$, and $[\vec{B}_{jt}]_{|\vec{A}} =^{(10.26)} d\Theta_t.[\vec{B}_{jt}]_{|\vec{B}} = d\Theta_t.\vec{E}_j$. ∎

10.4 Translated velocities

t is fixed, $\vec{v}_t(p_t) = \frac{\partial\widetilde{\Phi}}{\partial t}(t, P_{\mathit{Obj}})$ is the velocity of a particle $P_{\mathit{Obj}} \in \mathit{Obj}$ at t at p_t, $\vec{x}_{At} := [\overrightarrow{O_{At} p_t}]_{|\vec{A}}$, $\vec{x}_{Bt} := [\overrightarrow{O_{Bt} p_t}]_{|\vec{B}}$, and Θ_t affine.

Definition 10.10 The translated relative velocity and acceleration from B to A at t at p_t are the matrices

$$\boxed{\vec{v}_{Bt*}(\vec{x}_{At}) := d\Theta_t.\vec{v}_{Bt}(\vec{x}_{Bt})} \quad \text{and} \quad \boxed{\vec{\gamma}_{Bt*}(\vec{x}_{At}) = d\Theta_t.\vec{\gamma}_{Bt}(\vec{x}_{Bt})}. \tag{10.29}$$

I.e., $\vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t.[\vec{v}_t(p_t)]_{\vec{B}}$ and $\vec{\gamma}_{Bt*}(\vec{x}_{At}) = d\Theta_t.[\vec{\gamma}_t(p_t)]_{\vec{B}}$.

Interpretation: Let q_{t_0} and q_{t_1} be particles in $\mathit{Obj}\mathcal{R}_B$ s.t. $\vec{v}_{Bt}(\vec{x}_{Bt}) = [\overrightarrow{q_{t_0} q_{t_1}}]_{|\vec{B}}$ where $\vec{x}_{Bt} = [\overrightarrow{O_B q_{t_0}}]_{|\vec{B}}$ (here $\overrightarrow{q_{t_0} q_{t_1}}$ is a tangent vector at q_{t_0} to the curve $q_t : u \to q_t(u) = q_{t_0} + u\,\overrightarrow{q_{t_0} q_{t_1}}$ in the proof of prop. 10.8); Then $[\overrightarrow{q_{t_0} q_{t_1}}]_{|\vec{A}} =^{(10.26)} d\Theta_t.[\overrightarrow{q_{t_0} q_{t_1}}]_{|\vec{B}}$ gives $[\overrightarrow{q_{t_0} q_{t_1}}]_{|\vec{A}} = d\Theta_t.\vec{v}_{Bt}(\vec{x}_{Bt}) = \vec{v}_{Bt*}(\vec{x}_{At})$. Similarly for $\vec{\gamma}_{Bt*}(\vec{x}_{At})$.

Exercice 10.11 $(\vec{A}_i)$ and $(\vec{B}_i)$ are Euclidean basis (e.g. in foot and metre), $(\cdot,\cdot)_A$ and $(\cdot,\cdot)_B$ are the associated Euclidean dot products, $\lambda = ||\vec{B}_i||_A$ (e.g. $\simeq 3.28$), $(\vec{E}_i)$ is the canonical basis in $\mathcal{M}_{n1}$, and $(\cdot,\cdot)_{\mathcal{M}}$ is the canonical inner dot product in $\mathcal{M}_{n1}$. Call $\vec{E}_{it*} := d\Theta_t.\vec{E}_i$ and prove:

$$\forall i, j, \ \ (\vec{E}_{it*}, \vec{E}_{jt*})_{\mathcal{M}} = \lambda^2\delta_{ij}, \quad \text{and} \quad d\Theta_t{}^T.d\Theta_t = \lambda^2 I. \tag{10.30}$$

Answer. $(\vec{B}_{it})$ is a Euclidean basis for B, thus is a Euclidean orthogonal basis for all observers, in particular for A, with $||\vec{B}_{it}||_A = \lambda$ for all i. And $\vec{E}_{it*} = d\Theta_t.[\vec{B}_{jt}]_{|\vec{B}} =^{(10.26)} [\vec{B}_{it}]_{|\vec{A}}$. Thus $(\vec{E}_{it*}, \vec{E}_{jt*})_{\mathcal{M}} = [\vec{E}_{it*}]^T.[\vec{E}_{jt*}] = [\vec{B}_{it}]_{|\vec{A}}^T.[\vec{B}_{jt}]_{|\vec{A}} = (\vec{B}_{it}, \vec{B}_{jt})_A = \lambda^2(\vec{B}_{it}, \vec{B}_{jt})_B = \lambda^2\delta_{ij}$, thus $(10.30)_1$; Then $\lambda^2\delta_{ij} = (d\Theta_t.\vec{E}_i, d\Theta_t.\vec{E}_j)_{\mathcal{M}} = (d\Theta_t{}^T.d\Theta_t.\vec{E}_i, \vec{E}_j)_{\mathcal{M}}$, true for all i, j, thus $d\Theta_t{}^T.d\Theta_t = \lambda^2 I$, thus $(10.30)_2$. ∎

10.5 Definition of Θ

Definition 10.12 The translator from B to A is the function Θ defined with (10.20) by

$$\Theta : \begin{cases} \displaystyle\bigcup_{t\in[t_1,t_2]} (\{t\}\times \mathcal{M}_{n1}(\mathrm{B})) \to \mathcal{M}_{n1}(\mathrm{A}) \\ \qquad\qquad (t,\vec{y}_S) \to \boxed{\Theta(t,\vec{y}_S) := \Theta_t(\vec{y}_S)}, \end{cases} \tag{10.31}$$

i.e., for all $Q_{\mathcal{R}_B} \in Obj\mathcal{R}_B$ and all t,

$$\Theta(t, [\overrightarrow{O_B\widetilde{\Phi}_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec{B}}) = [\overrightarrow{O_A\widetilde{\Phi}_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec{A}}, \quad \text{i.e.} \quad \Theta(t,\vec{\varphi}_S(Q_{\mathcal{R}_B})) = \vec{\varphi}_D(t,Q_{\mathcal{R}_B}). \tag{10.32}$$

E.g., (10.22) gives $\Theta(t,\vec{0}) = [\overrightarrow{O_AO_B(t)}]_{|\vec{A}}$.

Remark 10.13 The translator Θ looks like a motion, but is not: A "usual" motion is defined by one observer and connects one particle to its position; While Θ connects two "matrix positions" of one particle relative to two referentials: Θ is an "inter-referential" function. ⁂

10.6 The "Θ-velocity" is the drive velocity

Definition 10.14 The "Θ-velocity" and "Θ-acceleration" are defined by (Eulerian type definition)

$$\text{with} \quad \vec{y}_{Dt} = \Theta(t,\vec{y}_S), \quad \begin{cases} \vec{v}_\Theta(t,\vec{y}_{Dt}) := \dfrac{\partial\Theta}{\partial t}(t,\vec{y}_S) \ \ (\in \mathcal{M}_{n1}), \\ \vec{\gamma}_\Theta(t,\vec{y}_{Dt}) = \dfrac{\partial^2\Theta}{\partial t^2}(t,\vec{y}_S) \ \ (\in \mathcal{M}_{n1}). \end{cases} \tag{10.33}$$

Proposition 10.15

$$\boxed{\vec{v}_\Theta = \vec{v}_D} \quad \textit{and} \quad \boxed{\vec{\gamma}_\Theta = \vec{\gamma}_D}, \tag{10.34}$$

i.e. $\vec{v}_\Theta(t,\vec{y}) = \vec{v}_D(t,\vec{y})$ *and* $\vec{\gamma}_\Theta(t,\vec{y}) = \vec{\gamma}_D(t,\vec{y})$ *in* $\mathcal{M}_{n1}$*, for all* t *and all* $\vec{y} \in \mathcal{M}_{n1}(\mathrm{A})$*.*

Proof. $\Theta(t,\vec{\varphi}_S(Q_{\mathcal{R}_B})) = \vec{\varphi}_D(t,Q_{\mathcal{R}_B})$ gives

$$\frac{\partial\Theta}{\partial t}(t,\vec{\varphi}_S(Q_{\mathcal{R}_B})) = \frac{\partial\vec{\varphi}_D}{\partial t}(t,Q_{\mathcal{R}_B}), \quad \text{i.e.} \quad \vec{v}_\Theta(t,\Theta(t,\vec{\varphi}_S(Q_{\mathcal{R}_B}))) = \vec{v}_D(t,\vec{\varphi}_D(t,Q_{\mathcal{R}_B})), \tag{10.35}$$

thus $\vec{v}_\Theta(t,\vec{\varphi}_D(t,Q_{\mathcal{R}_B})) = \vec{v}_D(t,\vec{\varphi}_D(t,Q_{\mathcal{R}_B}))$, thus $\vec{v}_\Theta(t,\vec{y}) = \vec{v}_D(t,\vec{y})$ for all $\vec{y} \in \mathcal{M}_{n1}(\mathrm{A})$. Idem with $\frac{\partial^2}{\partial t^2}$. ⁂

10.7 The velocity-addition formula

(10.24) gives

$$\vec{\varphi}_A(t,P_{Obj}) = \Theta(t,\vec{\varphi}_B(t,P_{Obj})), \tag{10.36}$$

thus

$$\frac{\partial\vec{\varphi}_A}{\partial t}(t,P_{Obj}) = \frac{\partial\Theta}{\partial t}(t,\vec{\varphi}_B(t,P_{Obj})) + d\Theta(t,\vec{\varphi}_B(t,P_{Obj})).\frac{\partial\vec{\varphi}_B}{\partial t}(t,P_{Obj}). \tag{10.37}$$

Thus

$$\vec{v}_A(t,\vec{x}_{At}) = \vec{v}_\Theta(t,\vec{x}_{At}) + d\Theta(t,\vec{x}_{Bt}).\vec{v}_B(t,\vec{x}_{Bt}), \tag{10.38}$$

where $\vec{x}_{Bt} = \vec{\varphi}_B(t,P_{Obj})$ and $\vec{x}_{At} = \vec{\varphi}_A(t,P_{Obj}) = \Theta_t(\vec{x}_{Bt})$. Thus, with $\vec{v}_\Theta =^{(10.34)} \vec{v}_D$,

$$\boxed{\vec{v}_{At} = \vec{v}_{Bt*} + \vec{v}_{Dt}} \quad \text{where} \quad \vec{v}_{Bt*}(\vec{x}_{At}) := d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}), \tag{10.39}$$

which is the velocity-addition formula in $\mathcal{R}_A$:

$$\begin{aligned} \vec{v}_{At} \text{ the absolute velocity} = \vec{v}_{Bt*} &\text{ the translated relative velocity from B to A} \\ + \vec{v}_{Dt} &\text{ the drive velocity.} \end{aligned} \tag{10.40}$$

In other words (relation between the numerical values of the velocities stored by A and B),

$$[\vec{v}_t(p_t)]_{|\vec{A}} = d\Theta_t.[\vec{v}_t(p_t)]_{\vec{B}} + [\vec{v}_{\mathcal{R}_{Bt}}(p_t)]_{|\vec{A}}. \tag{10.41}$$

10.8 Coriolis acceleration, and the acceleration-addition formula

(10.37) gives

$$\begin{aligned}\frac{\partial^2\vec{\varphi}_A}{\partial t^2}\,(t,P_{Obj}) &= \frac{\partial^2\Theta}{\partial t^2}(t,\vec{x}_{Bt}) + d\frac{\partial\Theta}{\partial t}(t,\vec{x}_{Bt}).\frac{\partial\vec{\varphi}_B}{\partial t}(t,P_{Obj})\\ &\quad + \big(\frac{\partial(d\Theta)}{\partial t}(t,\vec{x}_{Bt}) + d^2\Theta(t,\vec{x}_{Bt}).\frac{\partial\vec{\varphi}_B}{\partial t}(t,P_{Obj})\big).\frac{\partial\vec{\varphi}_B}{\partial t}(t,P_{Obj}) + d\Theta(t,\vec{x}_{Bt}).\frac{\partial^2\vec{\varphi}_B}{\partial t^2}(t,P_{Obj}).\end{aligned} \tag{10.42}$$

And $d^2\Theta_t = 0$ in our classical framework (Θ_t is affine); And $\frac{\partial\Theta}{\partial t}(t,\vec{y}_S) = \vec{v}_{\Theta t}(\Theta_t(\vec{y}_S))$ gives $\frac{\partial(d\Theta)}{\partial t}(t,\vec{y}_S) = d(\frac{\partial\Theta}{\partial t})(t,\vec{y}_S) = d\vec{v}_{\Theta t}(\Theta_t(\vec{y}_S)).d\Theta_t(\vec{y}_S)$; And $\frac{\partial\vec{\varphi}_B}{\partial t}(t,P_{Obj}) = \vec{v}_{Bt}(\vec{x}_{Bt})$ where $\vec{x}_{Bt} = \vec{\varphi}_B(t,P_{Obj})$; Thus

$$\begin{aligned}\vec{\gamma}_{At}(\vec{x}_{At}) &= \vec{\gamma}_{\Theta t}(\vec{x}_{At}) + 2d\vec{v}_{\Theta t}(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) + d\Theta_t(\vec{x}_{Bt}).\vec{\gamma}_{Bt}(\vec{x}_{Bt})\\ &= \vec{\gamma}_{Dt}(\vec{x}_{At}) + 2d\vec{v}_{Dt}(\vec{x}_{At}).\vec{v}_{Bt*}(\vec{x}_{At}) + \vec{\gamma}_{Bt*}(\vec{x}_{At}).\end{aligned} \tag{10.43}$$

Definition 10.16 At t, the Coriolis acceleration $\vec{\gamma}_{Ct}$ at $\vec{x}_{At}$ is

$$\vec{\gamma}_{Ct}(\vec{x}_{At}) = 2d\vec{v}_{Dt}(\vec{x}_{At}).\vec{v}_{Bt*}(\vec{x}_{At}), \quad \text{i.e.} \quad \boxed{\vec{\gamma}_{Ct} = 2d\vec{v}_{Dt}.\vec{v}_{Bt*}}. \tag{10.44}$$

And the Coriolis acceleration $\vec{\gamma}_C$ at t at $\vec{x}_{At}$ is $\vec{\gamma}_C(t,\vec{x}_{At}) := \vec{\gamma}_{Ct}(\vec{x}_{At})$.

Thus (10.43) gives the acceleration-addition formula in $\mathcal{R}_A$:

$$\boxed{\vec{\gamma}_{At} = \vec{\gamma}_{Bt*} + \vec{\gamma}_{Dt} + \vec{\gamma}_{Ct}}, \quad \text{i.e.} \tag{10.45}$$

$$\begin{aligned}\vec{\gamma}_{At} \text{ the absolute acceleration} &= \vec{\gamma}_{Bt*} \text{ the translated relative acceleration from B to A}\\ &\quad + \vec{\gamma}_{Dt} \text{ the drive acceleration} + \vec{\gamma}_{Ct} \text{ the Coriolis acceleration.}\end{aligned} \tag{10.46}$$

In other words (relation between the numerical values of the acceletations stored by A and B),

$$[\vec{\gamma}_t(p_t)]_{|\vec{A}} = d\Theta_t.[\vec{\gamma}_t(p_t)]_{\vec{B}} + [\vec{\gamma}_{\mathcal{R}_{Bt}}(p_t)]_{|\vec{A}} + 2[d\vec{v}_{\mathcal{R}_{Bt}}]_{|\vec{A}}.d\Theta_t.[\vec{v}_t(p_t)]_{|\vec{B}}. \tag{10.47}$$

10.9 With an initial time

Let $t_0, t \in \mathbb{R}$. Consider the Lagrangian associated function $\Phi_t^{t_0}$ with the motion $\widetilde{\Phi}$ of *Obj*:

$$\Phi_t^{t_0} : \begin{cases} \Omega_{t_0} & \to \Omega_t \\ p_{t_0}{=}\widetilde{\Phi}(t_0,P_{Obj}) & \to p_t = \Phi_t^{t_0}(p_{t_0}) := \widetilde{\Phi}(t,P_{Obj}). \end{cases} \tag{10.48}$$

And, with $\vec{x}_{At} = \vec{\varphi}_A(t,P_{Obj}) = [\overrightarrow{O_Ap_t}]_{|\vec{A}}$ and $\vec{x}_{Bt} = \vec{\varphi}_B(t,P_{Obj}) = [\overrightarrow{O_Bp_t}]_{|\vec{B}}$, define the "matrix motions" $\vec{\varphi}_{At}^{t_0} : \mathcal{M}_{n1}(\text{A}) \to \mathcal{M}_{n1}(\text{A})$ and $\vec{\varphi}_{Bt}^{t_0} : \mathcal{M}_{n1}(\text{B}) \to \mathcal{M}_{n1}(\text{B})$ by

$$\begin{cases} \vec{\varphi}_{At}^{t_0}(\vec{x}_{At_0}) := \vec{x}_{At} & (= [\overrightarrow{O_A\widetilde{\Phi}(t,P_{Obj})}]_{|\vec{A}} = [\overrightarrow{O_A\Phi_t^{t_0}(p_{t_0})}]_{|\vec{A}} = \vec{\varphi}_{At}(P_{Obj})), \\ \vec{\varphi}_{Bt}^{t_0}(\vec{x}_{Bt_0}) := \vec{x}_{Bt} & (= [\overrightarrow{O_B\widetilde{\Phi}(t,P_{Obj})}]_{|\vec{B}} = [\overrightarrow{O_B\Phi_t^{t_0}(p_{t_0})}]_{|\vec{B}} = \vec{\varphi}_{Bt}(P_{Obj})). \end{cases} \tag{10.49}$$

And $\Theta_t(\vec{x}_{Bt}) = \vec{x}_{At}$, i.e. $\Theta_t(\vec{\varphi}_{Bt}^{t_0}(\vec{x}_{Bt_0})) = \vec{\varphi}_{At}^{t_0}(\vec{x}_{At_0})$ with $\vec{x}_{At_0} = \Theta_{t_0}(\vec{x}_{Bt_0})$, thus

$$\boxed{\Theta_t \circ \vec{\varphi}_{Bt}^{t_0} = \vec{\varphi}_{At}^{t_0} \circ \Theta_{t_0}} : \mathcal{M}_{n1}(\text{B}) \to \mathcal{M}_{n1}(\text{A}). \tag{10.50}$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
& \vec{x}_{Bt_0} = \vec{\varphi}_B(t_0,P_{Obj}) \arrow[rr, "\vec{\varphi}_{Bt}^{t_0}"'] \arrow[dd, "\Theta_{t_0}"'] & & \vec{x}_{Bt} = \vec{\varphi}_{Bt}^{t_0}(\vec{x}_{Bt_0}) \arrow[dd, "\Theta_t"] \\
P_{Obj} \in Obj \arrow[ru, "\vec{\varphi}_{Bt_0}"] \arrow[rd, "\vec{\varphi}_{At_0}"] & & & \\
& \vec{x}_{At_0} = \vec{\varphi}_A(t_0,P_{Obj}) = \Theta_{t_0}(\vec{x}_{Bt_0}) \arrow[rr, "\vec{\varphi}_{At}^{t_0}"] & & \vec{x}_{At} = \vec{\varphi}_{At}^{t_0}(\vec{x}_{At_0}) = \Theta_t(\vec{x}_{Bt}).
\end{tikzcd} \tag{10.51}$$

Thus, for any vector field $\vec{u}_{Bt_0}$ in $\mathcal{R}_B$,

$$\underbrace{d\Theta_t(\vec{x}_{Bt})}_{\text{(translation at t)}} . \underbrace{d\vec{\varphi}_{Bt}^{t_0}(\vec{x}_{Bt_0}).\vec{u}_{Bt_0}(\vec{x}_{Bt_0})}_{\text{(deformation from } t_0 \text{ to t)}} = \underbrace{d\vec{\varphi}_{At}^{t_0}(\vec{x}_{At_0})}_{\text{(deformation from } t_0 \text{ to t)}} . \underbrace{d\Theta_{t_0}(\vec{x}_{Bt_0}).\vec{u}_{Bt_0}(\vec{x}_{Bt_0})}_{\text{(translation at } t_0\text{)}} . \tag{10.52}$$

Exercice 10.17 Redo the above steps with $\mathit{Obj}\mathcal{R}_B$ instead of Obj.

Answer. Consider the Lagrangian associated function $\Phi_{RBt}^{t_0}$ with the motion $\widetilde{\Phi}_{\mathcal{R}_B}$ of $\mathit{Obj}\mathcal{R}_B$:

$$\Phi_{RBt}^{t_0} : \left\{ \begin{array}{rl} \Omega_{RBt_0} = \mathbb{R}^n & \to \Omega_{RBt} = \mathbb{R}^n \\ q_{t_0} = \widetilde{\Phi}_{\mathcal{R}_B}(t_0, Q_{\mathcal{R}_B}) & \to q_t = \Phi_{RBt}^{t_0}(q_{t_0}) := \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B}), \end{array} \right\} \tag{10.53}$$

then define the "matrix motions" $\vec{\varphi}_{Dt}^{t_0} : \mathcal{M}_{n1}(\mathrm{A}) \to \mathcal{M}_{n1}(\mathrm{A})$ and $\vec{\varphi}_{St}^{t_0} : \mathcal{M}_{n1}(\mathrm{B}) \to \mathcal{M}_{n1}(\mathrm{B})$ by

$$\left\{ \begin{array}{ll} \vec{\varphi}_{Dt}^{t_0}(\vec{y}_{Dt_0}) := \vec{y}_{Dt} & (= [\overrightarrow{O_A\widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})}]_{|\vec{A}} = [\overrightarrow{O_A\Phi_{RBt}^{t_0}(p_{t_0})}]_{|\vec{A}} = \vec{\varphi}_{Dt}(Q_{\mathcal{R}_B})), \\ \vec{\varphi}_{St}^{t_0}(\vec{y}_S) := \vec{y}_S & (= [\overrightarrow{O_B\widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})}]_{|\vec{B}} = [\overrightarrow{O_B\Phi_{RBt}^{t_0}(q_{t_0})}]_{|\vec{B}} = \vec{\varphi}_S(Q_{\mathcal{R}_B})), \end{array} \right. \tag{10.54}$$

Thus $\vec{\varphi}_S$ is a time-shift, which is also abusively noted $\vec{\varphi}_{St}^{t_0} = I$ (algebraic identity). So with $\Theta_t(\vec{y}_S) = \vec{y}_{Dt}$ we get $\Theta_t(\vec{\varphi}_{Dt}^{t_0}(\vec{y}_S)) = \vec{\varphi}_{Dt}^{t_0}(\vec{y}_{Dt_0})$, with $\vec{y}_{Dt_0} = \Theta_{t_0}(\vec{y}_S)$, thus

$$\boxed{\Theta_t \circ \vec{\varphi}_{St}^{t_0} = \vec{\varphi}_{Dt}^{t_0} \circ \Theta_{t_0}} : \mathcal{M}_{n1}(\mathrm{B}) \to \mathcal{M}_{n1}(\mathrm{A}) \tag{10.55}$$

(also abusively written $\Theta_t = \vec{\varphi}_{Dt}^{t_0} \circ \Theta_{t_0}$). In other words, the following diagram commutes:

$$\begin{tikzcd}
 & \vec{y}_S = \vec{\varphi}_S(Q_{\mathcal{R}_B}) \arrow[r, "\vec{\varphi}_{St}^{t_0} = \text{time shift}"'] \arrow[dd, "\Theta_{t_0}"] & \vec{y}_S = \vec{\varphi}_S(Q_{\mathcal{R}_B}) \arrow[dd, "\Theta_t"'] \\
Q_{\mathcal{R}_B} \in \mathit{Obj}\mathcal{R}_B \arrow[ur, "\vec{\varphi}_S"] \arrow[dr, "\vec{\varphi}_D^{t_0}"'] & & \\
 & \vec{y}_{Dt_0} = \vec{\varphi}_{Dt_0}(Q_{\mathcal{R}_B}) = \Theta_{t_0}(\vec{y}_S) \arrow[r, "\vec{\varphi}_{Dt}^{t_0}"] & \vec{y}_{Dt} = \vec{\varphi}_{Dt}(Q_{\mathcal{R}_B}) = \vec{\varphi}_{Dt}^{t_0}(\vec{y}_{Dt_0}) = \Theta_t(\vec{y}_S).
\end{tikzcd} \tag{10.56}$$

And (10.55) gives, for any $\vec{y}_S = \vec{\varphi}_S(Q_{\mathcal{R}_B})$ and all vector field $\vec{u}_S$ (static in $\mathcal{R}_B$), with $\vec{y}_{Dt_0} = \Theta_{t_0}(\vec{y}_S)$,

$$\underbrace{d\Theta_t(\vec{y}_S)}_{\text{(translation at t)}} \quad . \quad \underbrace{d\vec{\varphi}_{St}^{t_0}(\vec{y}_S).\vec{u}_S(\vec{y}_S)}_{\text{(time shift from } t_0 \text{ to t)}} \quad = \quad \underbrace{d\vec{\varphi}_{Dt}^{t_0}(\vec{y}_{Dt_0})}_{\text{(Drive motion from } t_0 \text{ to t)}} \quad . \underbrace{d\Theta_{t_0}(\vec{y}_S).\vec{u}_S(\vec{y}_S)}_{\text{(translation at } t_0\text{)}} . \tag{10.57}$$

∴

10.10 Drive and Coriolis forces

10.10.1 Fundamental principal: requires a Galilean referential

Second Newton's law of motion (fundamental principle of dynamics): In a Galilean referential, the sum of the external forces $\vec{f}$ on an object is equal to its mass multiplied by its acceleration:

$$\sum \text{external}\vec{f} = m\vec{\gamma} \quad \text{(in a Galilean referential)}. \tag{10.58}$$

Question: And in a Non Galilean referential?

Answer: Then you have to add "observer dependent forces", i.e. you have to add "apparent forces" due to the motion of the non Galilean observer. Indeed, the motion of an object in our Universe does not care about the observer motion (his accelerations and velocities).

See e.g. `https://www.youtube.com/watch?v=_36MiCUS1ro` for a carousel (a merry-go-round),

See e.g. `https://www.youtube.com/watch?v=aeY9tY9vKgs` for tornadoes.

10.10.2 Drive + Coriolis forces = the inertial force

Consider $\vec{f}(t, p_t)$ = the sum of the external forces acting on P_{Obj} at t at $p_t = \widetilde{\Phi}(t, P_{\mathit{Obj}})$.

In a Galilean referential $\mathcal{R}_A$, Newton laws (10.58) means

$$[\vec{f}_t(p_t)]_{|\vec{A}} = m\,[\vec{\gamma}_t(p_t)]_{|\vec{A}}, \quad \text{written} \quad \boxed{\vec{f}_{At}(\vec{x}_{At}) = m\,\vec{\gamma}_{At}(\vec{x}_{At})} \in \mathcal{M}_{n1}, \tag{10.59}$$

with $\vec{x}_{At} := [\overrightarrow{O_Ap_t}]_{|\vec{A}}$, $\vec{f}_{At}(\vec{x}_{At}) := [\vec{f}_t(p_t)]_{|\vec{A}}$ and $\vec{\gamma}_{At}(\vec{x}_{At}) = [\vec{\gamma}_t(p_t)]_{|\vec{A}}$. With $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$, the acceleration addition formula gives $\vec{f}_{At}(\vec{x}_{At}) = m(d\Theta_t.\vec{\gamma}_B(\vec{x}_{Bt}) + \vec{\gamma}_{Dt}(\vec{x}_{At}) + \vec{\gamma}_{Ct}(\vec{x}_{At})) \in \mathcal{R}_A$, thus, in $\mathcal{R}_B$,

$$\underbrace{d\Theta_t{}^{-1}.\vec{f}_{At}(\vec{x}_{At})}_{\vec{f}_{At}{}^*(\vec{x}_{Bt}) = \vec{f}_{Bt}(\vec{x}_{Bt})} = m\,\vec{\gamma}_B(\vec{x}_{Bt}) + \underbrace{m\,d\Theta_t{}^{-1}.\vec{\gamma}_{Dt}(\vec{x}_{At})}_{m\,\vec{\gamma}_{Dt}{}^*(\vec{x}_{Bt})} + \underbrace{m\,d\Theta_t{}^{-1}.\vec{\gamma}_{Ct}(\vec{x}_{At})}_{m\,\vec{\gamma}_{Ct}{}^*(\vec{x}_{Bt})}, \tag{10.60}$$

and $d\Theta_t{}^{-1}.[\vec{f}_t(p_t)]_{|\vec{A}} = d\Theta_t{}^{-1}.\vec{f}_{At}(\vec{x}_{At}) =^{(10.26)} [\vec{f}_t(p_t)]_{|\vec{B}} =^{\text{noted}} \vec{f}_{Bt}(\vec{x}_{Bt})$ is the external forces as quantified by B at t, cf. (10.26) (with Θ_t supposed to be affine). And with the pull-back notation, cf. (10.26):

Definition 10.18 At t on p_t, define

- The drive force $\vec{f}_{BDt}(\vec{x}_{Bt}) := -m\, d\Theta_t{}^{-1}.\vec{\gamma}_{Dt}(\vec{x}_{At}) \quad (= -m\, \vec{\gamma}_{Dt}{}^*(\vec{x}_{Bt}))$.
- The Coriolis force $\vec{f}_{BCt}(\vec{x}_{Bt}) := -m\, d\Theta_t{}^{-1}.\vec{\gamma}_{Ct}(\vec{x}_{At}) \quad (= -m\, \vec{\gamma}_{Ct}{}^*(\vec{x}_{Bt}))$. (10.61)
- (The inertial, or fictitious, force $:= \vec{f}_{BDt}(\vec{x}_{Bt}) + \vec{f}_{BCt}(\vec{x}_{Bt}) = -m\, d\Theta_t{}^{-1}.(\vec{\gamma}_{Dt} + \vec{\gamma}_{Ct})(\vec{x}_{At})$.)

Then (10.60) gives the fundamental principle quantified in $\mathcal{R}_B$ (non Galilean referential):

$$\boxed{\vec{f}_{Bt}(\vec{x}_{Bt}) + \vec{f}_{BDt}(\vec{x}_{Bt}) + \vec{f}_{BCt}(\vec{x}_{Bt}) = m\, \vec{\gamma}_B(\vec{x}_{Bt})}, \tag{10.62}$$

i.e., at t, in $\mathcal{R}_B$: The external force + the Drive and Coriolis forces = m times the acceleration.

10.11 Summary for "Sun and Earth"

Illustation with a simplified (circular) motion of the Earth around the Sun.

10.11.1 Coriolis forces on the Earth

1. **Referentials.**

1.1. Relative referential $\mathcal{R}_B = (O_B, (\vec{B}_1, \vec{B}_2, \vec{B}_3))$ chosen by the observer B fixed on the Earth, where $O_{Bt} = \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{O_B})$ is the position of the particle Q_{O_B} at the center of the Earth, written O_B by B (fixed for B), and $(\vec{B}_{1t}, \vec{B}_{2t}, \vec{B}_{3t})$ is a Euclidean basis (e.g. built with the metre) fixed in the Earth, written $(\vec{B}_1, \vec{B}_2, \vec{B}_3)$ by B (fixed for B), with $\vec{B}_3$ chosen to be along the rotation axis of the Earth and oriented from the south pole to the north pole; And $(\cdot,\cdot)_B$ is the associated Euclidean dot product. So, a fixed particle $Q_{\mathcal{R}_B}$ in the Earth at longitude $\theta_{Q_{\mathcal{R}_B}} \in]-\pi, \pi]$ and latitude $\varphi_{Q_{\mathcal{R}_B}} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is referenced by observer B as the matrix $\vec{y}_S = \vec{\varphi}_S(Q_{\mathcal{R}_B}) = [\overrightarrow{O_B \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})}]_{|\vec{B}} = R_B \begin{pmatrix} \cos(\theta_{Q_{\mathcal{R}_B}}) \cos(\varphi_{Q_{\mathcal{R}_B}}) \\ \sin(\theta_{Q_{\mathcal{R}_B}}) \cos(\varphi_{Q_{\mathcal{R}_B}}) \\ \sin(\varphi_{Q_{\mathcal{R}_B}}) \end{pmatrix}$ where $R_B = ||\overrightarrow{O_B \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})}||_B$ is the distance between Q_{O_B} and $Q_{\mathcal{R}_B}$ (e.g. if $Q_{\mathcal{R}_B}$ is on the surface of the Earth then $R_B \simeq 6371$ km).

1.2. Initial Galilean referential $\mathcal{R}_{A0} = (O_{A0}, (\vec{A}_1, \vec{A}_2, \vec{A}_3))$: O_{A0} is at the center of the Sun and $(\vec{A}_1, \vec{A}_2, \vec{A}_3)$ is a Euclidean basis (e.g. built with the foot) fixed relative to the stars, such that $\vec{A}_3 = \mu \vec{B}_3$ with $\mu > 0$ (e.g. $\mu = 0.3048$ and $\lambda = \frac{1}{\mu} \simeq 3.28$); And $(\cdot,\cdot)_A$ is the associated Euclidean dot product.

1.3. Deduced absolute Galilean referential $\mathcal{R}_A = (O_{At}, (\vec{A}_1, \vec{A}_2, \vec{A}_3))$ chosen by observer A fixed on Earth, where $O_{At} = O_{Bt}$, written O_A by A (fixed for A). Since it takes more that 365 days for Q_{O_B} to complete a rotation around the Sun, the motion of Q_{O_B} will be considered to be rectilinear at constant velocity "in a short interval of time" sufficient for the computation of the Coriolis acceleration with "sufficient accuracy" (simplifies the calculations).

(If A prefers to work with the initial Galilean referential $\mathcal{R}_{A0}$, then the absolute matrix motion $\vec{\varphi}_A(t, P_{Obj}) = [\overrightarrow{O_A \widetilde{\Phi}(t, P_{Obj})}]_{|\vec{A}}$ has to be replaced by $\vec{\varphi}_A(t, P_{Obj}) = [\overrightarrow{O_{A0} O_B(t)}]_{|\vec{A}} + [\overrightarrow{O_B(t) \widetilde{\Phi}(t, P_{Obj})}]_{|\vec{A}}$, idem for the drive motion $\vec{\varphi}_D$.)

2. **Drive motion.**

2.1. The motion $t \to q_t = \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})$ of a particle $Q_{\mathcal{R}_B}$ in the Earth is stored by A as the drive motion $\vec{\varphi}_D$ given by (matrix valued), with ω the angular velocity of the Earth in $\mathcal{R}_A$,

$$\vec{y}_D(t) = \vec{\varphi}_D(t, Q_{\mathcal{R}_B}) = R_A(Q_{\mathcal{R}_B}) \begin{pmatrix} \cos(\omega t) \cos \varphi_{Q_{\mathcal{R}_B}} \\ \sin(\omega t) \cos \varphi_{Q_{\mathcal{R}_B}} \\ \sin \varphi_{Q_{\mathcal{R}_B}} \end{pmatrix} = [\overrightarrow{O_A q(t)}]_{|\vec{A}} = \begin{pmatrix} y_{D1}(t) \\ y_{D2}(t) \\ y_{D3} \end{pmatrix}, \tag{10.63}$$

where $R_A(Q_{\mathcal{R}_B}) = ||\overrightarrow{Q_{O_B} Q_{\mathcal{R}_B}}||_{|\vec{A}}$ is the distance between Q_{O_B} and $Q_{\mathcal{R}_B}$ for A (e.g. $R_A \simeq 20902231$ foot if $Q_{\mathcal{R}_B}$ is on the surface of the Earth). (And (ωt) by replaced by $(\alpha_0 + \omega(t - t_0))$ to be more general.)

2.2. Drive velocity: With $\vec{\omega}_D := \omega \vec{A}_3$,

$$\vec{v}_D(t, \vec{y}_D(t)) = \vec{y}_D{}'(t) = \omega R_A \begin{pmatrix} -\sin(\omega t) \cos \varphi_{Q_{\mathcal{R}_B}} \\ \cos(\omega t) \cos \varphi_{Q_{\mathcal{R}_B}} \\ 0 \end{pmatrix} = \omega \begin{pmatrix} -y_2(t) \\ y_1(t) \\ 0 \end{pmatrix} = \omega \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .\vec{y}_D(t) = \vec{\omega}_D \wedge \vec{y}_D(t). \tag{10.64}$$

2.3. Drive acceleration:

$$\vec{\gamma}_D(t,\vec{y}_{Dt}) = \vec{y}_D''(t) = \vec{\omega}_D \wedge \vec{y}_D'(t) = \vec{\omega}_D \wedge \vec{v}_D(t,\vec{y}_{Dt}) = \vec{\omega}_D \wedge (\vec{\omega}_D \wedge \vec{y}_D(t)) = -\omega^2 \begin{pmatrix} y_{D1}(t) \\ y_{D2}(t) \\ 0 \end{pmatrix} \tag{10.65}$$

= the usual centrifugal acceleration (in a plane parallel to the equatorial plane, drawing).

2.4. Differential of the drive velocity (time and space independent here): (10.64) gives

$$d\vec{v}_D(t,\vec{y}_{Dt}) = d\vec{v}_D = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \vec{\omega}_D \wedge . \tag{10.66}$$

3. **Translator.**

3.1. Here $O_{At} = O_{Bt}$, thus $\Theta_t(\vec{0}) = \vec{0}$ (with $[\vec{0}] =^{\text{noted}} \vec{0} =$ the null matrix), cf. (10.22).

3.2. Calculation of $d\Theta_t$. With Θ_t affine, $d\Theta_t.[\vec{B}_{it}]_{|\vec{B}} = [\vec{B}_{it}]_{|\vec{A}}$. Thus $\vec{B}_3 = \lambda\vec{A}_3$ (hypothesis) and $d\Theta_t.[\vec{B}_3]_{|\vec{B}} = [\vec{B}_{3t}]_{|\vec{A}}$ give $d\Theta_t.\vec{E}_3 = \lambda\vec{E}_3$ where $(\vec{E}_i)$ is the canonical basis in $\mathcal{M}_{n1}$. Then let $Q_{Bi} \in Obj\mathcal{R}_B$ be the Earth particle which position $q_{ti} = \widetilde{\Phi}_{\mathcal{R}_B}(t,Q_{Bi})$ makes $\vec{B}_{it} := \overrightarrow{O_{Bt}q_{ti}}$. So, $\vec{B}_1$ and $\vec{B}_2$ being in the equatorial plane, (10.63) gives $d\Theta_t.\vec{E}_1 = d\Theta_t.[\vec{B}_1]_{|\vec{B}} = [\vec{B}_1]_{|\vec{A}} = [\overrightarrow{O_Aq_{t1}}]_{|\vec{A}} = \lambda \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{pmatrix}$, and $d\Theta_t.\vec{E}_2 = d\Theta_t.[\vec{B}_2]_{|\vec{B}} = [\vec{B}_2]_{|\vec{A}} = [\overrightarrow{O_Aq_{t2}}]_{|\vec{A}} = \lambda \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \\ 0 \end{pmatrix}$. Thus $[d\Theta_t]_{|\vec{E}} = \lambda \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ = the expected rotation matrix expanded by λ (change of unit of measurement).

3.3. Calculation of Θ_t (affine): $\Theta_t(\vec{y}_S) = \Theta_t(\vec{0}) + d\Theta_t.\vec{y}_S$, so, with $O_{At} = O_{Bt}$ here,

$$\vec{y}_{Dt} := \Theta_t(\vec{y}_S) = d\Theta_t.\vec{y}_S \tag{10.67}$$

4. **Motions of *Obj*.**

4.1. B quantifies the motion $\widetilde{\Phi}$ of *Obj*, i.e. he stores the relative motion $\vec{\varphi}_B$ of *Obj*, and the relative velocities and accelerations $\vec{v}_{Bt}$ and $\vec{\gamma}_B$ (matrices), cf. (10.10)-(10.12).

4.2. Translations for A: With $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$,

$$\vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) \quad \text{and} \quad \vec{\gamma}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{\gamma}_{Bt}(\vec{x}_{Bt}). \tag{10.68}$$

5. **Drive force** (apparent force in $\mathcal{R}_B$ due to the motion of B):

$$\vec{f}_{BDt}(\vec{x}_{Bt}) = -m\, d\Theta_t{}^{-1}.\vec{\gamma}_{Dt}(\vec{x}_{At}) \overset{(10.65)}{=} \lambda m\omega^2 d\Theta_t{}^{-1}. \begin{pmatrix} x_{A1}(t) \\ x_{A2}(t) \\ 0 \end{pmatrix} \overset{(10.67)}{=} \lambda m\omega^2 \begin{pmatrix} x_{B1}(t) \\ x_{B2}(t) \\ 0 \end{pmatrix}, \tag{10.69}$$

centrifugal force (in a "parallel plane" at latitude of P_{Obj}).

6. **Coriolis acceleration** (apparent acceleration due to the motion of B):

$$\vec{\gamma}_{Ct}(\vec{x}_{At}) = 2\, d\vec{v}_{Dt}.(d\Theta_t.\vec{v}_{Bt}(\vec{x}_{Bt})) = 2\, d\Theta_t.d\vec{v}_{Dt}.\vec{v}_{Bt}(\vec{x}_{Bt}) \tag{10.70}$$

because $d\Theta_t$ commutes with $d\vec{v}_{Dt}$ (composition of "rotations along the same south-north axis" which reads as $e^{i\omega t}.e^{i\frac{\pi}{2}} = e^{i\frac{\pi}{2}}e^{i\omega t} = e^{i(\frac{\pi}{2}+\omega t)}$ in the equatorial plane).

7. **Coriolis force** (apparent force due to the motion of B):

$$\vec{f}_{BCt}(\vec{x}_{Bt}) = -m\, d\Theta_t{}^{-1}.\vec{\gamma}_{Ct}(\vec{x}_{At}) = -2m\, d\vec{v}_{Dt}.\vec{v}_{Bt}(\vec{x}_{Bt}) = -2m\vec{\omega} \wedge \vec{v}_{Bt}(\vec{x}_{Bt}). \tag{10.71}$$

11 Objectivities

Goal: To give an objective expression of the laws of mechanics; As Maxwell [13] said: “The formula at which we arrive must be such that a person of any nation, by substituting for the different symbols the numerical value of the quantities as measured by his own national units, would arrive at a true result”.

Generic notation: if a function z is given as $z(t,x)$, then $z_t(x) := z(t,x)$, and conversely.

11.1 “Isometric objectivity” and “Frame Invariance Principle”

This manuscript is not intended to describe “isometric objectivity”:

“Isometric objectivity” is the framework in which the “principle of material frame-indifference” (“frame invariance principle”) is settled, principle which states that “Rigid body motions should not affect the stress constitutive law of a material”. E.g., Truesdell–Noll [19] p. 41:

> « Constitutive equations must be invariant under changes of frame of reference. »

Or Germain [9] :

> « Axiom of power of internal forces. The virtual power of the "internal forces" acting on a system S for a given virtual motion is an objective quantity; i.e., it has the same value whatever be the frame in which the motion is observed. »

NB: Both of these affirmations are limited to “isometric changes of frame” (the same metric for all), as Truesdell–Noll [19] page 42-43 explain: The “isometric objectivity” concern one observer who defines his Euclidean dot product and consider only orthonormal change of bases to validate a constitutive law.

If you want to interpret “isometric objectivity” in the “covariant objectivity” framework, then “isometric objectivity” corresponds to a dictatorial management: One observer with his Euclidean referential (e.g. based on the English foot), imposes his unit of length to all other users (isometry hypothesis). (Note: The metre was not adopted by the scientific community until after 1875.)

Moreover, isometric objectivity leads to despise the difference between covariance and contravariance, due to the uncontrolled use of the Riesz representation theorem.

Remark 11.1 Marsden and Hughes [12] p. 8 use this isometric framework to begin with. But, pages 22 and 163, they write that a “good modelization” has to be “covariant objective” (observer independent) to begin with; And they propose a covariant modelization for elasticity at § 3.3. ⁂

11.2 Definition and characterization of the covariant objectivity

11.2.1 Framework of classical mechanics

Framework of classical mechanics to simplify. Consider two observers A and B and their referentials $\mathcal{R}_A = (O_A, (\vec{A}_i))$ and $\mathcal{R}_B = (O_B, (\vec{B}_i))$. E.g., $(\vec{A}_i)$ and $(\vec{B}_i)$ are Euclidean bases in foot and metre, $(\cdot,\cdot)_A$ and $(\cdot,\cdot)_B$ is their associated Euclidean dot products. And Θ is the translator, cf. (10.19).

Consider a regular motion $\widetilde{\Phi}$ of an object Obj, $p_t = \widetilde{\Phi}(t, P_{\mathit{Obj}}) \in \mathbb{R}^n$ the position at t of a particle in our Universe, $\Omega_t = \widetilde{\Phi}(t, \mathit{Obj})$ the configuration at t, and $\mathcal{C} = \bigcup_{t\in[a,b]}(\{t\}\times\Omega_t)$ the set of configurations. And $\vec{x}_{At} := [\overrightarrow{O_A p_t}]_{|\vec{A}} \in \mathcal{M}_{n1}(\mathrm{A})$ and $\vec{x}_{Bt} := [\overrightarrow{O_B p_t}]_{|\vec{B}} \in \mathcal{M}_{n1}(\mathrm{B})$ are the stored components of p_t relative to the chosen referentials, $\mathcal{M}_{n1}(\mathrm{A})$ and $\mathcal{M}_{n1}(\mathrm{B})$ being the spaces of $n*1$ matrices as referred to by A and B.

11.2.2 Covariant objectivity of a scalar function

Let $f : \left\{\begin{array}{ll} \mathcal{C} & \to \mathbb{R} \\ (t,p_t) & \to f(t,p_t) \end{array}\right\}$ be a Eulerian scalar function (e.g., a temperature field). f is quantified by A and B as the functions $f_A : \left\{\begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1}(\mathrm{A}) & \to \mathbb{R} \\ (t,\vec{x}_{At}) & \to f_A(t,\vec{x}_{At}) := f(t,p_t) \end{array}\right\}$ and $f_B : \left\{\begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1}(\mathrm{B}) & \to \mathbb{R} \\ (t,\vec{x}_{Bt}) & \to f_B(t,\vec{x}_{Bt}) := f(t,p_t) \end{array}\right\}$.

Definition 11.2 f is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all t,

$$f_{At}(\vec{x}_{At}) = f_{Bt}(\vec{x}_{Bt}) \quad \text{when} \quad \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}), \tag{11.1}$$

i.e. $f_{At} = f_{Bt*}$ is the push-forward of f_{Bt} by Θ_t cf. (6.8).

11.2.3 Covariant objectivity of a vector field

Let $\vec{w} : \left\{ \begin{array}{ll} \mathcal{C} & \to \vec{\mathbb{R}}^n \\ (t,p_t) & \to \vec{w}(t,p_t) \end{array} \right\}$ be a Eulerian vector field (e.g., a force field). $\vec{w}$ is quantified by A and B as the functions $\vec{w}_A : \left\{ \begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1}(\mathrm{A}) & \to \mathcal{M}_{n1}(\mathrm{A}) \\ (t,\vec{x}_{At}) & \to \vec{w}_A(t,\vec{x}_{At}) := [\vec{w}(t,p_t)]_{\vec{A}} \end{array} \right\}$ and $\vec{w}_B : \left\{ \begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1}(\mathrm{B}) & \to \mathcal{M}_{n1}(\mathrm{B}) \\ (t,\vec{x}_{Bt}) & \to \vec{w}_B(t,\vec{x}_{Bt}) := [\vec{w}(t,p_t)]_{\vec{B}} \end{array} \right\}$. So $\vec{w}_A(t,\vec{x}_{At})$ and $\vec{w}_B(t,\vec{x}_{Bt})$ are the column matrices of the components of $\vec{w}(t,p_t)$ in $\mathcal{R}_A$ and $\mathcal{R}_B$.

Definition 11.3 $\vec{w}$ is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all t,

$$\vec{w}_{At}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt}) \quad \text{when} \quad \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}), \tag{11.2}$$

i.e. $\vec{w}_{At} = \vec{w}_{Bt*}$ is the push-forward of $\vec{w}_{Bt}$ by Θ_t cf. (6.20).

Example 11.4 Fundamental counter-example: A Eulerian velocity field is not objective, cf. (10.39), because of the drive velocity $\vec{v}_D \neq \vec{0}$ in general. Neither is a Eulerian acceleration field, cf. (10.45). ∴

Example 11.5 The field of gravitational forces (external forces) is objective covariant. ∴

11.2.4 Covariant objectivity of differential forms

Let $\alpha : \left\{ \begin{array}{ll} \mathcal{C} & \to \mathbb{R}^{n*} \\ (t,p_t) & \to \alpha(t,p_t) \end{array} \right\}$ be a Eulerian differential form (e.g. a measuring device used to get the internal power). α is quantified by A and B as the functions $\alpha_A : \left\{ \begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1}(\mathrm{A}) & \to \mathcal{M}_{n1}(\mathrm{A}) \\ (t,\vec{x}_{At}) & \to \alpha_A(t,\vec{x}_{At}) := [\alpha(t,p_t)]_{\vec{A}} \end{array} \right\}$ and $\alpha_B : \left\{ \begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1}(\mathrm{B}) & \to \mathcal{M}_{n1}(\mathrm{B}) \\ (t,\vec{x}_{Bt}) & \to \alpha_B(t,\vec{x}_{Bt}) := [\alpha(t,p_t)]_{\vec{B}} \end{array} \right\}$. So $\alpha_A(t,\vec{x}_{At})$ and $\alpha_B(t,\vec{x}_{Bt})$ are the row matrices of the components of $\alpha(t,p_t)$ in $\mathcal{R}_A$ and $\mathcal{R}_B$.

Definition 11.6 α is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all t,

$$\alpha_{At}(\vec{x}_{At}) = \alpha_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1} \quad \text{when} \quad \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}). \tag{11.3}$$

i.e. $\alpha_{At} = \alpha_{Bt*}$ is the push-forward of α_{Bt} by Θ_t cf. (7.3).

NB: (11.3) and (11.2) are compatible: If $\vec{w}$ is an objective vector field and if α is an objective differential form, then the scalar function $\alpha.\vec{w}$ is objective:

$$\alpha_{At}(\vec{x}_{At}).\vec{w}_{At}(\vec{x}_{At}) = \alpha_{Bt}(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt}) \quad (= (\alpha(t,p_t).\vec{w}(t,p_t)), \tag{11.4}$$

since $\alpha_{At}(\vec{x}_{At}).\vec{w}_{At}(\vec{x}_{At}) = (\alpha_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1}).(d\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})) = \alpha_{Bt}(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})$.

11.2.5 Covariant objectivity of tensors

A tensor acts on both vector fields and differential forms, and its objectivity is deduced from the previous §.

So, let T be a (Eulerian) tensor corresponding to a "physical quantity". The observers A and B describe T as being the functions T_A and T_B.

Definition 11.7 T is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all t,

$$T_{At}(\vec{x}_{At}) = T_{Bt*}(\vec{x}_{At}) \tag{11.5}$$

i.e. T_{At} is the push-forward of T_{Bt} by Θ_t.

(Recall: $T_{Bt*}(\vec{x}_{At})(\alpha_1(\vec{x}_{At}),...,\vec{w}_1(\vec{x}_{At})) := T_{Bt}(\vec{x}_{Bt})(\alpha_1{}^*(\vec{x}_{Bt}),...,\vec{w}_1{}^*(\vec{x}_{Bt}))$.)

Example 11.8 (Non covariant objectivity of a differential $d\vec{w}$) Let $\vec{w}$ be an objective vector field, seen as $\vec{w}_A$ by A and $\vec{w}_B$ by B; So $\vec{w}_{At}(\vec{x}_{At}) =^{(11.2)} d\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})$ when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$, thus

$$d\vec{w}_{At}(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt}) = d\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}) + (d^2\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})), \tag{11.6}$$

hence

$$\begin{aligned} d\vec{w}_{At}(\vec{x}_{At}) &= d\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1} + (d^2\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})).d\Theta_t(\vec{x}_{Bt})^{-1} \\ &\neq d\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1} \quad \text{when} \quad d^2\Theta_t \neq 0. \end{aligned} \tag{11.7}$$

Thus $d\vec{w}$ is not covariant objective in general. However in classical mechanics for "change of Cartesian referentials" Θ_t is affine, so $d^2\Theta_t = 0$, and in particular $d\vec{w}$ is objective when $\vec{w}$ is. And

$$\begin{aligned} &(d^2\vec{w}_{At}(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt})).d\Theta_t(\vec{x}_{Bt}) + d\vec{w}_{At}(\vec{x}_{At}).d^2\Theta_t(\vec{x}_{Bt}) \\ &= d\Theta_t(\vec{x}_{Bt}).d^2\vec{w}_{Bt}(\vec{x}_{Bt}) + 2\, d^2\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}) + d^3\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt}). \end{aligned} \tag{11.8}$$

Thus $d^2\vec{w}$ is not covariant objective in general (but if Θ_t is affine then $d^2\vec{w}$ is objective if $\vec{w}$ is). ∴

11.3 Non objectivity of the velocities

11.3.1 Eulerian velocity $\vec{v}$: not covariant (and not isometric) objective

Velocity addition formala: With $\vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{w}(\vec{x}_{Bt})$ when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$, cf. (10.39),

$$\begin{aligned} \vec{v}_{At}(\vec{x}_{At}) &= \vec{v}_{Bt*}(\vec{x}_{At}) + \vec{v}_{Dt}(\vec{x}_{At}) \\ &\neq \vec{v}_{Bt*}(\vec{x}_{At}) \quad \text{when} \quad \vec{v}_{Dt}(\vec{x}_{At}) \neq \vec{0}, \end{aligned} \tag{11.9}$$

thus a Eulerian velocity field is not covariant objective (and not isometric objective).

11.3.2 $d\vec{v}$ is not objective

The velocity addition formula, $(\vec{v}_{At} - \vec{v}_{Dt})(\vec{x}_{At}) = \vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})$ when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$, gives

$$d(\vec{v}_{At} - \vec{v}_{Dt})(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt}) = d\Theta_t(\vec{x}_{Bt}).d\vec{v}_{Bt}(\vec{x}_{Bt}) + d^2\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}), \tag{11.10}$$

thus $d\vec{v}$ is neither covariant objective nor isometric objective because of $d\vec{v}_D$:

$$d\vec{v}_{At}(\vec{x}_{At}) = d\vec{v}_{Bt*}(\vec{x}_{At}) + d\vec{v}_{Dt}(\vec{x}_{At}) + d^2\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1} \neq d\vec{v}_{Bt*}(\vec{x}_{At}) \quad \text{in general.} \tag{11.11}$$

Remark 11.9 Recall: "Isometric objective" implies

- The use of the same Euclidean metric in $\mathcal{R}_B$ and $\mathcal{R}_A$, i.e. $(\cdot,\cdot)_A = (\cdot,\cdot)_B$,
- $\widetilde{\Phi}_{\mathcal{R}_B}$ (motion of $\mathcal{R}_B$) is a solid body motion, and
- Θ_t is affine (so $d^2\Theta_t = 0$ for all t). ∴

Exercice 11.10 Prove, with Q_t the (orthonormal) transition matrix from $(\vec{A}_i)$ to $(\vec{B}_i)$:

$$[d\vec{v}_t]_{|\vec{B}} = Q_t.[d\vec{v}_t]_{|\vec{A}}.Q_t^{-1} + Q'(t).Q_t^{-1}, \quad \text{written} \quad [L]_{|\vec{B}} = Q.[L]_{|\vec{A}}.Q^T + \dot{Q}.Q^T. \tag{11.12}$$

(Used in classical mechanics courses, to prove that $d\vec{v}$ isn't "isometric objective" because of $\dot{Q}.Q^T$.)

Answer. $t_0, t \in \mathbb{R}$, $p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj})$, $p_t = \widetilde{\Phi}(t, P_{Obj}) = \Phi_t^{t_0}(p_{t_0})$, $\vec{v}(t,p_t) = \frac{\partial\widetilde{\Phi}}{\partial t}(t, P_{Obj})$, and $F_t^{t_0}(p_{t_0}) = d\Phi_t^{t_0}(p_{t_0})$. So $\vec{v}(t, \Phi_t^{t_0}(p_{t_0})) = \frac{\partial\Phi_{p_{t_0}}^{t_0}}{\partial t}(t, p_{t_0})$, thus $d\vec{v}(t,p_t).F_{p_{t_0}}^{t_0}(t) = \frac{\partial F_{p_{t_0}}^{t_0}}{\partial t}(t)$. And (4.30), with $F_{p_{t_0}}^{t_0} =^{\text{noted}} F$, gives $[F(t)]_{|\vec{a}_{t_0},\vec{B}} = Q(t).[F(t)]_{|\vec{a}_{t_0},\vec{A}}$, thus $[F'(t)]_{|\vec{a}_{t_0},\vec{B}} = Q'(t).[F(t)]_{|\vec{a}_{t_0},\vec{A}} + Q(t).[F'(t)]_{|\vec{a}_{t_0},\vec{A}}$. Thus $[d\vec{v}(t,p_t)]_{|\vec{B}} = [F_{p_{t_0}}^{t_0}{}'(t).F_{p_{t_0}}^{t_0}(t)]_{|\vec{B}} = [F_{p_{t_0}}^{t_0}{}'(t)]_{|\vec{B}}.[F_{p_{t_0}}^{t_0}(t)]_{|\vec{B}} = (Q'(t).[F(t)]_{|\vec{a}_{t_0},\vec{A}} + Q(t).[F'(t)]_{|\vec{a}_{t_0},\vec{A}}).[F(t)]_{|\vec{a}_{t_0},\vec{A}}^{-1}.Q(t)^{-1} = Q'(t).Q(t)^{-1} + Q(t).[F'(t)]_{|\vec{a}_{t_0},\vec{A}}.[F(t)]_{|\vec{a}_{t_0},\vec{A}}^{-1}.Q(t)^{-1} = Q'(t).Q(t)^{-1} + Q(t).[d\vec{v}(t,p_t)]_{|\vec{A}}.Q(t)^{-1}$. And cf. (3.34)-(3.35). ∴

Exercice 11.11 Prove that $d^2\vec{v}$ is "isometric objective" when $\widetilde{\Phi}_{\mathcal{R}_B}$ is a rigid body motion.

Answer. (11.8) with $\vec{v}_A - \vec{v}_D$ instead of $\vec{w}_A$, and $\vec{v}_B$ instead of $\vec{w}_B$ give, in an "isometric objective" framework,

$$d^2(\vec{v}_{At} - \vec{v}_{Dt})(\vec{x}_{At}).(\vec{u}_{Bt*}, \vec{w}_{Bt*}) = d\Theta_t(\vec{x}_{Bt}).d^2\vec{v}_{Bt}(\vec{x}_{Bt})(\vec{u}_B, \vec{w}_B). \tag{11.13}$$

Here $d^2\vec{v}_{Dt} = 0$ (rigid body motion), thus $d^2\vec{v}$ is "isometric objective". ∴

11.3.3 $d\vec{v}+d\vec{v}^T$ is "isometric objective"

Proposition 11.12 *If $\widetilde{\Phi}_{\mathcal{R}_B}$ is a rigid body motion then $d\vec{v}_t+d\vec{v}_t^T$ is "isometric objective"*

$$d\vec{v}_{At}+d\vec{v}_{At}^T=(d\vec{v}_{Bt}+d\vec{v}_{Bt}^T)_*. \tag{11.14}$$

(Isometric framework: The rate of deformation tensor is independent of an added added rigid motion.)

Proof. $Q.Q^T=I$ gives $\overset{\bullet}{Q}.Q^T+(\overset{\bullet}{Q}.Q^T)^T=0$, then apply (11.12). ∎

Exercice 11.13 Prove that $\Omega=\frac{d\vec{v}-d\vec{v}^T}{2}$ is not isometric objective.

Answer. (11.11) gives $d\vec{v}_{At}^T=d\vec{v}_{Bt*}^T+d\vec{v}_{Dt}^T$, thus $\frac{d\vec{v}_{At}-d\vec{v}_{At}^T}{2}=\frac{d\vec{v}_{Bt*}-d\vec{v}_{Bt*}^T}{2}+\frac{d\vec{v}_{Dt}-d\vec{v}_{Dt}^T}{2}\neq\frac{d\vec{v}_{Bt*}-d\vec{v}_{Bt*}^T}{2}$, even if $\widetilde{\Phi}_{\mathcal{R}_B}$ is a solid body motion (then $\frac{d\vec{v}_{Dt}-d\vec{v}_{Dt}^T}{2}=\vec{\omega}\wedge$ is a rotation time a dilation). ∎

11.3.4 Lagrangian velocities

The Lagrangian velocities do not define a vector field, cf. § 3.2.2. Thus asking about the objectivity of Lagrangian velocities is meaningless.

11.4 The Lie derivatives are covariant objective

Framework of § 10. In particular we have the velocity-addition formula $\vec{v}_{At}=\vec{v}_{Bt*}+\vec{v}_{Dt}$ in $\mathcal{R}_A$ where $\vec{v}_{Bt*}(\vec{x}_{At})=d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})$ and $\vec{x}_{Bt}=\Theta_t(\vec{x}_{At})$, cf. (10.39).

The objectivity under concern is the covariant objectivity (no inner dot product or basis required). The Lie derivatives are also called "objective rates" because they are covariant objectives. Easy proofs.

11.4.1 Scalar functions

Proposition 11.14 *If f be a covariant objective function, cf. (11.1), then its Lie derivative $\mathcal{L}_{\vec{v}}f$ is covariant objective:*

$$\mathcal{L}_{\vec{v}_A}f_A=\Theta_*(\mathcal{L}_{\vec{v}_B}f_B),\quad \text{i.e.}\quad \mathcal{L}_{\vec{v}_A}f_A(t,\vec{x}_{At})=\mathcal{L}_{\vec{v}_B}f_B(t,\vec{x}_{Bt})\quad \text{when}\quad \vec{x}_{At}=\Theta_t(\vec{x}_{Bt}), \tag{11.15}$$

i.e., $\frac{Df_A}{Dt}(t,\vec{x}_{At})=\frac{Df_B}{Dt}(t,\vec{x}_{Bt})$, *i.e.* $(\frac{\partial f_A}{\partial t}+df_A.\vec{v}_A)(t,\vec{x}_{At})=(\frac{\partial f_B}{\partial t}+df_B.\vec{v}_B)(t,\vec{x}_{Bt})$.

Proof. Consider the motion $t\to p(t)=\widetilde{\Phi}(tP_{Obj})$ of a particle P_{Obj}, and $\vec{x}_A(t)=[\overrightarrow{O_Ap(t)}]_{|\vec{A}}$ and $\vec{x}_B(t)=[\overrightarrow{O_Bp(t)}]_{|\vec{B}}$. With f objective, (11.1) gives $f_B(t,\vec{x}_B(t))=f_A(t,\Theta(t,\vec{x}_B(t)))$ $(=f_A(t,\vec{x}_A(t)))$, thus

$$\begin{aligned}\frac{Df_B}{Dt}(t,\vec{x}_B(t))&=\frac{\partial f_A}{\partial t}(t,\vec{x}_A(t))+df_{At}(\vec{x}_A(t)).(\underbrace{\frac{\partial\Theta}{\partial t}(t,\vec{x}_B(t))}_{\vec{v}_{Dt}(\vec{x}_{At})}+\underbrace{d\Theta_t(\vec{x}_B(t)).\vec{v}_{Bt}(\vec{x}_B(t)))}_{\vec{v}_{Bt*}(\vec{x}_{At})}\\&=\frac{\partial f_A}{\partial t}(t,\vec{x}_{At})+df_{At}(\vec{x}_{At}).\vec{v}_{At}(\vec{x}_{At})=\frac{Df_A}{Dt}(t,\vec{x}_{At}),\end{aligned} \tag{11.16}$$

thanks to velocity addiction formula $\vec{v}_{At}=\vec{v}_{Bt*}+\vec{v}_{Dt}$. ∎

11.4.2 Vector fields

Proposition 11.15 *Let $\vec{w}$ be a covariant objective vector field, cf. (11.2). Then its Lie derivative $\mathcal{L}_{\vec{v}}\vec{w}$ is covariant objective:*

$$\mathcal{L}_{\vec{v}_A}\vec{w}_A=\Theta_*(\mathcal{L}_{\vec{v}_B}\vec{w}_B), \tag{11.17}$$

i.e., when $\vec{x}_{At}=\Theta_t(\vec{x}_{Bt})$,

$$\mathcal{L}_{\vec{v}_A}\vec{w}_A(t,\vec{x}_{At})=d\Theta_t(\vec{x}_{Bt}).\mathcal{L}_{\vec{v}_B}\vec{w}_B(t,\vec{x}_{Bt}), \tag{11.18}$$

i.e.,

$$(\frac{D\vec{w}_A}{Dt}-d\vec{v}_A.\vec{w}_A)(t,\vec{x}_{At})=d\Theta(t,\vec{x}_{Bt}).(\frac{D\vec{w}_B}{Dt}-d\vec{v}_B.\vec{w}_B)(t,\vec{x}_{Bt}), \tag{11.19}$$

i.e.,

$$(\frac{\partial\vec{w}_A}{\partial t}+d\vec{w}_A.\vec{v}_A-d\vec{v}_A.\vec{w}_A)(t,\vec{x}_{At})=d\Theta(t,\vec{x}_{Bt}).(\frac{\partial\vec{w}_B}{\partial t}+d\vec{w}_B.\vec{v}_B-d\vec{v}_B.\vec{w}_B)(t,\vec{x}_{Bt}). \tag{11.20}$$

But the partial, convected, material, and Lie autonomous derivatives are not covariant objective (not *even isometric objective because of the drive velocity* $\vec v_D$*): We have*

$$(d\vec w_{At}.(\vec v_{At}-\vec v_{Dt}))(\vec x_{At}) = (d\Theta_t.(d\vec w_{Bt}.\vec v_{Bt}) + (d^2\Theta_t.\vec w_{Bt}).\vec v_{Bt})(\vec x_{Bt}), \tag{11.21}$$

$$(d(\vec v_{At}-\vec v_{Dt}).\vec w_{At})(\vec x_{At}) = (d\Theta_t.(d\vec v_{Bt}.\vec w_{Bt}) + (d^2\Theta_t.\vec v_{Bt}).\vec w_{Bt})(\vec x_{Bt}), \tag{11.22}$$

$$(d(\vec v_{At}-\vec v_{Dt}).(\vec v_{At}-\vec v_{Dt}))(\vec x_{At}) = (d\Theta_t.(d\vec v_{Bt}.\vec v_{Bt}) + d^2\Theta_t(\vec v_{Bt},\vec v_{Bt}))(\vec x_{Bt}), \tag{11.23}$$

$$\mathcal{L}^0_{(\vec v_{At}-\vec v_{Dt})}\vec w_{At}(\vec x_{At}) = d\Theta_t(\vec x_{Bt}).\mathcal{L}^0_{\vec v_{Bt}}\vec w_{Bt}(\vec x_{Bt}), \tag{11.24}$$

$$\frac{\partial \vec w_A}{\partial t}(t,\vec x_{At}) + \mathcal{L}^0_{\vec v_D}\vec w_{At}(\vec x_{At}) = d\Theta_t(\vec x_{Bt}).\frac{\partial \vec w_B}{\partial t}(t,\vec x_{Bt}), \tag{11.25}$$

$$\frac{D\vec w_A}{Dt}(t,\vec x_{At}) - d\vec v_{Dt}.\vec w_{At}(\vec x_{At}) = d\Theta_t.(\vec x_{Bt}).\frac{D\vec w_B}{Dt}(t,\vec x_{Bt}) + d^2\Theta_t(\vec v_{Bt},\vec w_{Bt})(\vec x_{Bt}), \tag{11.26}$$

$$\frac{\partial(\vec v_A-\vec v_D)}{\partial t}(t,\vec x_{At}) + \mathcal{L}^0_{\vec v_D}(\vec v_A-\vec v_D)(t,\vec x_{At}) = d\Theta_t(\vec x_{Bt}).\frac{\partial \vec v_B}{\partial t}(t,\vec x_{Bt}). \tag{11.27}$$

Proof. • $\vec w_{At}(\Theta_t(\vec x_{Bt})) = d\Theta_t(\vec x_{Bt}).\vec w_{Bt}(\vec x_{Bt})$ gives

$$d\vec w_{At}(\vec x_{At}).d\Theta_t(\vec x_{Bt}) = d^2\Theta_t(\vec x_{Bt}).\vec w_{Bt}(\vec x_{Bt}) + d\Theta_t(\vec x_{Bt}).d\vec w_B(\vec x_{Bt}), \tag{11.28}$$

thus, with $d\Theta_t(\vec x_{Bt}).\vec v_{Bt}(\vec x_{Bt}) = (\vec v_{At}-\vec v_{Dt})(\vec x_{At}) = \vec v_{Bt*}(\vec x_{At})$ (velocity-addition formula),

$$d\vec w_{At}(\vec x_{At}).(\vec v_{At}-\vec v_{Dt})(\vec x_{At}) = (d^2\Theta_t(\vec x_{Bt}).\vec v_{Bt}(\vec x_{Bt})).\vec w_{Bt}(\vec x_{Bt}) + d\Theta_t(\vec x_{Bt}).d\vec w_{Bt}(\vec x_{Bt}).\vec v_{Bt}(\vec x_{Bt}),$$

hence (11.21). In particular $d\vec w_{At}(\vec x_{At}).\vec v_{At}(\vec x_{At}) \neq d\Theta_t(\vec x_{Bt}).(d\vec w_{Bt}(\vec x_{Bt}).\vec v_{Bt}(\vec x_{Bt}))$ (the vector field $d\vec w.\vec v$ is not objective).

• $(\vec v_{At}-\vec v_{Dt})(\Theta_t(\vec x_{Bt})) = d\Theta_t(\vec x_{Bt}).\vec v_{Bt}(\vec x_{Bt})$ gives

$$d(\vec v_{At}-\vec v_{Dt})(\vec x_{At}).d\Theta_t(\vec x_{Bt}) = d^2\Theta_t(\vec x_{Bt}).\vec v_{Bt}(\vec x_{Bt}) + d\Theta_t(\vec x_{Bt}).d\vec v_{Bt}(\vec x_{Bt}),$$

so, applied to $\vec w_{Bt}$ (resp. $\vec v_{Bt}$), we get (11.22) (resp. (11.23)). Hence (11.24).

• If $\vec x_{At} = \Theta_t(\vec x_B)$, then $\vec w_A(t,\Theta(t,\vec x_B)) = d\Theta(t,\vec x_B).\vec w_B(t,\vec x_B)$, so, with $\frac{\partial\Theta}{\partial t}(t,\vec x_B) = \vec v_{\Theta t}(\vec x_{At})$, we get

$$\begin{aligned}\frac{\partial \vec w_A}{\partial t}(t,\vec x_{At}) + d\vec w_{At}(\vec x_{At}).\vec v_{\Theta t}(\vec x_{At}) &= d\frac{\partial\Theta}{\partial t}(t,\vec x_B).\vec w_{Bt}(\vec x_B) + d\Theta_t(\vec x_B).\frac{\partial \vec w_B}{\partial t}(t,\vec x_B)\\ &= (d\vec v_{\Theta t}(\vec x_{At}).d\Theta_t(\vec x_B)).\vec w_{Bt}(\vec x_B) + d\Theta_t(\vec x_B).\frac{\partial \vec w_B}{\partial t}(t,\vec x_B),\end{aligned}$$

Thus (11.25) since $\vec v_\Theta = \vec v_D$; Then (11.21) gives (11.26).

• $\vec v_{B*}(t,\Theta(t,\vec x_B)) = d\Theta(t,\vec x_B).\vec v_B(t,\vec x_B)$ gives

$$\frac{\partial \vec v_{B*}}{\partial t}(t,\vec x_{At}) + d\vec v_{B*}(\vec x_{At}).\vec v_\Theta(t,\vec x_{At}) = \underbrace{\frac{\partial d\Theta}{\partial t}(t,\vec x_B)}_{d\vec v_{\Theta t}(\vec x_{At}).d\Theta_t(\vec x_B)}.\vec v_{Bt}(\vec x_B) + d\Theta(t,\vec x_B).\frac{\partial \vec v_B}{\partial t}(t,\vec x_B,)$$

since $\frac{\partial d\Theta}{\partial t}(t,\vec x_B) = d(\frac{\partial\Theta}{\partial t})(t,\vec x_B)$ and $\frac{\partial\Theta}{\partial t}(t,\vec x_B) = \vec v_\Theta(t,\vec x_{At}) = \vec v_{\Theta_t}(\Theta_t(\vec x_B))$; hence (11.27). ∎

11.4.3 Tensors

Proposition 11.16 *It T is a covariant objective tensor, then its Lie derivatives are covariant objectives:*

$$\mathcal{L}_{\vec v_A}T_A = \Theta_*(\mathcal{L}_{\vec v_B}T_B). \tag{11.29}$$

Proof. Corollary of (11.15) and (11.18) to get $\mathcal{L}_{\vec v}(\alpha.\vec w) = (\mathcal{L}_{\vec v}\alpha).\vec w + \alpha.(\mathcal{L}_{\vec v}\vec w)$; Then use $\mathcal{L}_{\vec v}(t_1\otimes t_2) = (\mathcal{L}_{\vec v}t_1)\otimes t_2 + t_1\otimes(\mathcal{L}_{\vec v}t_2)$. ∎

11.5 Taylor expansions and ubiquity gift

11.5.1 In $\mathbb{R}^n$ with ubiquity

Generic formula:

$$f(t) = f(t_0) + (t{-}t_0)\, f'(t_0) + \frac{(t{-}t_0)^2}{2}\, f'(t_0)^2 + o((t{-}t_0)^2). \tag{11.30}$$

In particular $f(t) = \vec{w}(t, p(t))$ gives

$$\vec{w}(t, p(t)) = \vec{w}(t_0, p(t_0)) + (t{-}t_0)\, \frac{D\vec{w}}{Dt}(t_0, p(t_0)) + \frac{(t{-}t_0)^2}{2}\, \frac{D\vec{w}}{Dt}(t_0, p(t_0))^2 + o((t{-}t_0)^2). \tag{11.31}$$

Problem : $\vec{w}(t, p(t))$ is a vecteur at t at $p(t)$ while $\vec{w}(t_0, p(t_0))$ is a vecteur at t_0 at $p(t_0)$, so (11.31) cannot be written

$$\vec{w}(t, p(t)) - \big(\vec{w}(t_0, p(t_0)) + (t{-}t_0)\, \frac{D\vec{w}}{Dt}(t_0, p(t_0)) + \frac{(t{-}t_0)^2}{2}\, \frac{D\vec{w}}{Dt}(t_0, p(t_0))^2\big) = o((t{-}t_0)^2), \tag{11.32}$$

since the left-hand side supposes the ubiquity gift.

E.g. in a non-planar manifold (e.g. a surface in $\mathbb{R}^3$ considered on its own), $\vec{w}(t, p_t) \in T_{p_t}(\Omega_t)$ = the linear tangent space at $p(t) = p_t$, whereas $\vec{w}(t_0, p_{t_0}) \in T_{p_{t_0}}(\Omega_{t_0})$ = the linear tangent space at $p(t_0) = p_{t_0}$, and the tangent spaces $T_{p_t}(\Omega_t)$ and $T_{p_{t_0}}(\Omega_{t_0})$ are distinct at two distinct points in general; Thus the left-hand side of (11.32) is meaningless.

In $\mathbb{R}^3$ our affine space (our Universe), $T_{p_t}(\Omega_t)$ and $T_{p_{t_0}}(\Omega_{t_0})$ are identified with $\vec{\mathbb{R}}^3$, and (11.31) is well defined, and very useful!

11.5.2 General case

By definition, cf. (9.10), with $p(t) = \Phi^{t_0}(t, p_{t_0}) = \Phi^{t_0}_t(p_{t_0})$,

$$\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}) = \frac{d\Phi^{t_0}_t(p_{t_0})^{-1}.\vec{w}(t, p(t)) - \vec{w}(t_0, p_{t_0})}{t - t_0} + o(1). \tag{11.33}$$

Thus,

$$d\Phi^{t_0}_t(p_{t_0})^{-1}.\vec{w}(t, p(t)) = \vec{w}(t_0, p_{t_0}) + (t{-}t_0)\, \mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}) + o(t{-}t_0). \tag{11.34}$$

Hence we get the first order Taylor expansion without ubiquity gift:

$$\vec{w}(t, p(t)) = d\Phi^{t_0}_t(p_{t_0}).\Big(\vec{w} + (t{-}t_0)\, \mathcal{L}_{\vec{v}}\vec{w}\Big)(t_0, p_{t_0}) + o(t{-}t_0), \tag{11.35}$$

both side of the equality being in $T_{p_t}(\Omega_t)$ (meaningful in any manifold).

Proposition 11.17 *In $\mathbb{R}^n$, with the gift of ubiquity, (11.35) gives (11.31).*

Proof. $d\Phi^{t_0}(t_0{+}h, p_{t_0})).\vec{w}(t_0, p_{t_0}) =^{(4.35)} \vec{w}(t_0, p_{t_0}) + h\, d\vec{v}(t_0, p_{t_0}).\vec{w}(t_0, p_{t_0}) + o(h)$, thus

$$\begin{aligned}\vec{w}(t, p_t) &\overset{(11.35)}{=} (I + h\, d\vec{v}(t_0, p_{t_0}) + o(h)).\Big((\vec{w} + h\, \frac{D\vec{w}}{Dt} - h\, d\vec{v}.\vec{w})(t_0, p_{t_0}) + o(h)\Big)\\ &= (\vec{w} + h\, d\vec{v}.\vec{w} + h\, \frac{D\vec{w}}{Dt} - h\, d\vec{v}.\vec{w})(t_0, p_{t_0}) + o(h) = (\vec{w} + h\, \frac{D\vec{w}}{Dt})(t_0, p_{t_0}) + o(h),\end{aligned}$$

which is (11.31). ∴

Proposition 11.18 *In $\mathbb{R}^n$, at second order,*

$$\vec{w}(t, p(t)) = d\Phi^{t_0}_t(p_{t_0}).\big((\vec{w} + h\mathcal{L}_{\vec{v}}\vec{w} + \frac{h^2}{2}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w}))(t_0, p_{t_0}) + o(h^2)\big). \tag{11.36}$$

So if the values $\vec{w}(t_0, p_{t_0})$, $\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0})$ and $\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w})(t_0, p_{t_0})$ are known, then $\vec{w}(t, p(t))$ is estimated at second order thanks to the push-forward of $(\vec{w} + h\mathcal{L}_{\vec{v}}\vec{w} + \frac{h^2}{2}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w}))(t_0, p_{t_0})$ by $\Phi^{t_0}_t$.

Proof. Let $d\Phi^{t_0}(t,p_{t_0}) = F^{t_0}_{p_{t_0}}(t)$. Let $\vec{g}(t) = d\Phi^{t_0}(t,p_{t_0})^{-1}.\vec{w}(t,p(t))$ when $p(t) = \Phi^{t_0}(t,p_{t_0})$. So $\mathcal{L}_{\vec{v}}\vec{w}(t_0,p_{t_0}) = \vec{g}\,'(t_0)$, cf. (9.11). And $\frac{D\vec{w}}{Du}(u,p(u)) = d\vec{v}(u,p(u)).\vec{w}(u,p(u)) + F^t_u(p_t).\vec{g}\,'(u)$, cf. (9.16). Thus $\frac{D^2\vec{w}}{Du^2}(u,p(u)) = \frac{D(d\vec{v})}{Du}(u,p(u)).\vec{w}(u,p(u)) + d\vec{v}(u,p(u)).\frac{D\vec{w}}{Du}(u,p(u)) + d\vec{v}(u,p(u).F^t_u(p_t).\vec{g}\,'(u) + F^t_u(p_t).\vec{g}\,''(u)$. Thus $\frac{D^2\vec{w}}{Dt^2}(t,p(t)) = (\frac{D(d\vec{v})}{Dt}.\vec{w} + d\vec{v}.\frac{D\vec{w}}{Dt} + d\vec{v}.\mathcal{L}_{\vec{v}}\vec{w})(t,p(t)) + \vec{g}\,''(t)$. With (9.39) we get $\vec{g}\,''(t) = \mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w})(t,p(t))$, cf. (9.39).

Alternate proof (calculation): (4.34) gives $F^{t_0}_{p_{t_0}}(t) = I_{t_0} + h\,d\vec{v}(t_0,p_{t_0}) + \frac{h^2}{2}\,d\vec{\gamma}(t_0,p_{t_0}) + o(h^2)$. Thus, omitting the reference to (t_0,p_{t_0}) to lighten the writing,

$$\begin{aligned}&d\Phi^{t_0}_t(p_{t_0}).(\vec{w} + h\mathcal{L}_{\vec{v}}\vec{w} + \frac{h^2}{2}\mathcal{L}_{\vec{v}}\mathcal{L}_{\vec{v}}\vec{w} + o(h^2))\\ &= \Big(I + h\,d\vec{v} + \frac{h^2}{2}d(\frac{D\vec{v}}{Dt}) + o(h^2)\Big).\Big(\vec{w} + h\mathcal{L}_{\vec{v}}\vec{w} + \frac{h^2}{2}\mathcal{L}_{\vec{v}}\mathcal{L}_{\vec{v}}\vec{w} + o(h^2)\Big)\end{aligned} \tag{11.37}$$

The h^0 term is $I.\vec{w} = \vec{w}$. The h term is $\mathcal{L}_{\vec{v}}\vec{w} + d\vec{v}.\vec{w} = \frac{D\vec{w}}{Dt}$. The h^2 term is the sum of

- $\frac{1}{2}\mathcal{L}_{\vec{v}}\mathcal{L}_{\vec{v}}\vec{w} = \frac{1}{2}(\frac{D^2\vec{w}}{Dt^2} - 2\,d\vec{v}.\frac{D\vec{w}}{Dt} - \frac{D(d\vec{v})}{Dt}.\vec{w} + d\vec{v}.d\vec{v}.\vec{w})$, cf.(9.39),
- $d\vec{v}.\mathcal{L}_{\vec{v}}\vec{w} = d\vec{v}.\frac{D\vec{w}}{Dt} - d\vec{v}.d\vec{v}.\vec{w} = \frac{1}{2}(2d\vec{v}.\frac{D\vec{w}}{Dt} - 2d\vec{v}.d\vec{v}.\vec{w})$,
- $\frac{1}{2}d(\frac{D\vec{v}}{Dt}).\vec{w} = \frac{1}{2}(\frac{D(d\vec{v})}{Dt}.\vec{w} + d\vec{v}.d\vec{v}.\vec{w})$, cf.(2.36).

And the sum gives $\frac{D^2\vec{w}}{Dt^2}$. ∴

Part V
Appendix

In this appendix, we tried to give standard results useful in mechanics, results that are scattered in the existing literature, and sometimes difficult to find except in math books (differential geometry).

The definitions, notations and results are detailed, so that no ambiguity is possible (some notations can be nightmarish when not understood, or misused, or come like a bull in a china-shop). All the results presented apply to solids, fluids, thermodynamics, general relativity, electromagnetism, quantum mechanics, chemistry... (the same math applies to all... even applies to mechanical engineers...).

A Classical and duality notations

A.1 Contravariant vector and basis

A.1.1 Contravariant vector

Let $(E,+,.) =^{\text{noted}} E$ be a real vector space (= a linear space on the field $\mathbb{R}$).

Definition A.1 An element $\vec{x} \in E$ is called a vector, or a "contravariant vector".

A vector is a vector... So why this name contravariant? Historical answer: Because of the change of basis formula $[\vec{x}]_{|new} = P^{-1}.[\vec{x}]_{|old}$, see (A.28), which uses P^{-1}.

So, what is a covariant vector? Answer: From the vector space E, you can build the vector space (an overlay) $\mathcal{L}(E;\mathbb{R}) =^{\text{noted}} E^*$ = the space of linear forms on E (a linear form is a measuring instrument that gives values to vectors). Then an element $\ell \in E^*$ will be called a covariant vector, because of the change of basis formula $[\ell]_{|new} = [\ell]_{|old}.P$. See § A.5 for details.

A.1.2 Basis

Definitions: • n vectors $\vec{e}_1,...,\vec{e}_n \in E$ are linearly independent iff, for all $\lambda,...,\lambda_n \in \mathbb{R}$, the equality $\sum_{i=1}^n \lambda_i \vec{e}_i = \vec{0}$ implies $\lambda_i = 0$ for all $i = 1,...,n$.

• n vectors $\vec{e}_1,...,\vec{e}_n \in E$ span E iff, for all $\vec{x} \in E$, $\exists \lambda_1,...,\lambda_n \in \mathbb{R}$ such that $\vec{x} = \sum_{i=1}^n \lambda_i \vec{e}_i$.

• A basis in E is a set $\{\vec{e}_1,...,\vec{e}_n\} \subset E$ made of n linearly independent vectors which span E, in which case the dimension of E is n.

A.1.3 Canonical basis

Consider the field $\mathbb{R}$ of reals and the Cartesian product $\vec{\mathbb{R}}^n = \mathbb{R} \times ... \times \mathbb{R}$, n times. The canonical basis is

$$\vec{e}_1 = (1,0,...,0), \ ..., \ \vec{e}_n = (0,...,0,1), \tag{A.1}$$

with 0 = the addition identity element used $n-1$ times, and 1 = the multiplication identity element used once.

Remark A.2 The 3-D geometric space we live in has no canonical basis: What would the identity element 1 mean? 1 metre? 1 foot? And there is no "intrinsic" preferred direction to define $\vec{e}_1$. So the Cartesian product $\vec{\mathbb{R}}^n = \mathbb{R} \times ... \times \mathbb{R}$ and its canonical basis form an abstract mathematical model. ∴

A.1.4 Cartesian basis

(René Descartes 1596-1650.) Let $n = 1,2,3$, let $\mathbb{R}^n$ be the usual affine space (space of points), and let $\vec{\mathbb{R}}^n = (\vec{\mathbb{R}}^n,+,.)$ be the usual real vector space of bipoint vectors with its usual algebraic operations.

Let $p \in \mathbb{R}^n$, and let $(\vec{e}_i(p))$ be a basis at p.

A Cartesian basis in $\vec{\mathbb{R}}^n$ is a basis independent of p (the same at all p), and then $(\vec{e}_i(p)) =^{\text{noted}} (\vec{e}_i)$.

Example of a non Cartesian basis: The polar basis, see example 6.11 (polar coordinate system).

And a Euclidean basis is a particular Cartesian basis described in § B.1.

More generally, a Cartesian basis refers to $E^n = E \times ... \times E$ (n-times) where E is a dimension 1 vector space.

A.2 Representation of a vector relative to a basis

We give:

- the classical notation (non ambiguous), e.g. used by Arnold [3] and Germain [8], and
- the duality notation (can be ambiguous because of misuses), e.g. used by Marsden and Hughes [12].

Both classical and duality notation are equally good, but if you have any doubt, use the classical notations.

Definition A.3 Let $\vec{x} \in E$. Let $(\vec{e}_i)$ be a basis in E. The components of $\vec{x}$ relative to the basis $(\vec{e}_i)$ are the n real numbers $x_1, ..., x_n$ (classical notation) also named $x^1, ..., x^n$ (duality notation) such that

$$\vec{x} = \underbrace{x_1\vec{e}_1 + ... + x_n\vec{e}_n}_{\text{clas.}} = \underbrace{x^1\vec{e}_1 + ... + x^n\vec{e}_n}_{\text{dual}}, \quad \text{i.e.} \quad [\vec{x}]_{|\vec{e}} = \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\text{clas.}} = \underbrace{\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}}_{\text{dual}}, \tag{A.2}$$

$[\vec{x}]_{|\vec{e}}$ being the column matrix representing $\vec{x}$ relative to the basis $(\vec{e}_i)$. (Of course $x_i = x^i$ for all i.) And the column matrix $[\vec{x}]_{|\vec{e}}$ is simply named $[\vec{x}]$ if one chosen basis is imposed to all. With the sum sign:

$$\vec{x} = \underbrace{\sum_{i=1}^{n} x_i\vec{e}_i}_{\text{clas.}} = \underbrace{\sum_{i=1}^{n} x^i\vec{e}_i}_{\text{dual}} \quad (= \sum_{J=1}^{n} x_J\vec{e}_J = \sum_{\alpha=1}^{n} x^\alpha\vec{e}_\alpha). \tag{A.3}$$

(The index in a summation is a dummy index, even if you do not write the sum sign $\sum$ as can be done with Enstein's convention: $\vec{x} = \sum_{j=1}^{n} x^j\vec{e}_j =^{\text{noted}} x^j\vec{e}_j = x^i\vec{e}_i = x^J\vec{e}_J = x^\alpha\vec{e}_\alpha$.)

Example A.4 In $\vec{\mathbb{R}}^2$ with $\vec{x} = 3\vec{e}_1 + 4\vec{e}_2 = \sum_{i=1}^{2} x_i\vec{e}_i = \sum_{i=1}^{2} x^i\vec{e}_i$: We have $x_1{=}x^1{=}3$ and $x_2{=}x^2{=}4$. And $[\vec{x}]_{|\vec{e}} = 3[\vec{e}_1]_{|\vec{e}} + 4[\vec{e}_2]_{|\vec{e}} = \sum_{i=1}^{2} x_i[\vec{e}_i]_{|\vec{e}} = \sum_{i=1}^{2} x^i[\vec{e}_i]_{|\vec{e}}$. In particular, with $\delta^i_j = \delta_{ij} := \begin{Bmatrix} = 1 \text{ if } i{=}j \\ = 0 \text{ if } i{\neq}j \end{Bmatrix}$ the Kronecker symbols,

$$\vec{e}_j = \underbrace{\sum_{i=1}^{n} \delta_{ij}\vec{e}_i}_{\text{clas.}} = \underbrace{\sum_{i=1}^{n} \delta^i_j\vec{e}_i}_{\text{dual}}, \quad \text{i.e.} \quad [\vec{e}_1]_{|\vec{e}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; ..., \; [\vec{e}_n]_{|\vec{e}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{A.4}$$

that is, the components of $\vec{e}_j$ are δ_{ij} with classical notations, and δ^i_j with duality notations. And the matrices $[\vec{e}_j]_{|\vec{e}}$ mimic the use of theoretical Cartesian space $\vec{\mathbb{R}}^n = \mathbb{R} \times ... \times \mathbb{R}$ and its canonical basis. ▪

Remark A.5 The column matrix $[\vec{x}]_{|\vec{e}}$ is also called a "column vector". NB: A "column vector" is not a vector, but just a matrix (a collection of real numbers). See the change of basis formula (A.28) where the same vector is represented by two "column vectors" (two column matrices). ▪

A.3 Dual basis

Recall: Let E and F be vector spaces and $(\mathcal{F}(E;F), +, .) =^{\text{noted}} \mathcal{F}(E;F)$ be the usual real vector space of functions with the internal addition $(f,g) \to f+g$ defined by $(f+g)(x) := f(x)+g(x)$ and the external multiplication $(\lambda, f) \to \lambda.f$ defined by $(\lambda.f)(x) := \lambda(f(x))$, for all $f, g \in \mathcal{F}(E;F)$, $x \in E$, $\lambda \in \mathbb{R}$. And $\lambda.f =^{\text{noted}} \lambda f$ for all $f \in \mathcal{F}(E;F)$ and $\lambda \in \mathbb{R}$.

A.3.1 Linear forms = "Covariant vectors"

Definition A.6 The set $E^* := \mathcal{L}(E;\mathbb{R})$ of linear scalar valued functions is called the dual of E:

$$E^* := \mathcal{L}(E;\mathbb{R}) = \text{the dual of } E. \tag{A.5}$$

And a linear scalar valued function $\ell \in E^*$ is called a linear form.

More precisely, E^* as defined in (A.5) is the algebraic dual of E; To define the topological dual usually needed with L^2 functions in mechanics, E needs to be a Banach space (a vector space equipped with a norm with which E is complete), and E^* is then the set of continuous linear forms. (If E is finite dimensional then any linear form is continuous relative to any norm since all norms are equivalent in finite dimension.)

E^* is a vector space: sub-space of $(\mathcal{F}(R;\mathbb{R}),+,.)$ (trivial).

Interpretation: It answers the question: What does a function $E\to\mathbb{R}$ do? Answer: Like any function, it gives values to vectors: $\ell(\vec u)=$ the value of $\vec u$ through ℓ. That is, a $\ell\in E^*$ is a measuring tool for vectors: If $\vec u\in E$ then $\ell(\vec u)=$ real value given by ℓ.

Notation: If $\ell\in E^*$ then

$$\forall\vec u\in E,\quad \ell(\vec u)\overset{\text{noted}}{=}\ell.\vec u, \tag{A.6}$$

also written $\langle\ell,\vec u\rangle_{E^*,E}$ where $\langle .,.\rangle_{E^*,E}$ is the duality bracket: The dot in $\ell.\vec u$ is "the distributivity dot" since linearity $\ell(\vec u+\lambda\vec v)=\ell(\vec u)+\lambda\ell(\vec v)=$ distributivity $\ell.(\vec u+\lambda\vec v)=\ell.\vec u+\lambda\ell.\vec v$.

NB: The dot in $\ell.\vec u$ is ***not*** an inner dot product (since $\ell\notin E$ while $\vec u\in E$).

Definition A.7 A linear form ℓ in E^* is also called a "covariant vector"; Co-variant refers to:

1- The action of a function on a vector, cf. (A.6) (co-variant calculation), and

2- The change of coordinate formula $[\ell]_{new}=[\ell]_{|old}.P$, see (A.28) (covariant formula).

NB: E^* being a vector space, an element $\ell\in E^*$ is indeed a vector. But E^* has no existence if E has not been specified first since $E^*:=\mathcal{L}(E;\mathbb{R})$. And $\ell\in E^*$ can't be confused with a vector $\vec u\in E$ since there is no natural canonical isomorphism between E and E^* (no "intrinsic representation"), see § T.2.

Remark A.8 Misner–Thorne–Wheeler [14], box 2.1, insist: "Without it [the distinction between covariance and contravariance], one cannot know whether a vector is meant or the very different object that is a linear form." ∴

A.3.2 Covariant dual basis (= the functions that give the components of a vector)

Notation: If $\vec u_1,...,\vec u_k$ are vectors in E, then $\mathrm{Vect}\{\vec u_1,...,\vec u_k\}:=$ the vector space spanned by $\vec u_1,...,\vec u_k$.

Let E be a finite dimensional vector space, and let $(\vec e_i)_{i=1,...,n}$ be a basis in E

Definition A.9 Let $i\in[1,n]_{\mathbb{N}}$. The scalar projection on $\mathrm{Vect}\{\vec e_i\}$ parallel to $\mathrm{Vect}\{\vec e_1,...,\vec e_{i-1},\vec e_{i+1},...,\vec e_n\}$ is the linear form named $\pi_{ei}\in E^*$ with the classical notation, named $e^i\in E^*$ with the duality notation, defined by, for all i,j,

$$\begin{cases}\text{clas. not.}: & \pi_{ei}(\vec e_j)=\delta_{ij},\quad\text{i.e.}\quad \pi_{ei}.\vec e_j=\delta_{ij},\\ \text{dual not.}: & e^i(\vec e_j)=\delta^i_j,\quad\text{i.e.}\quad e^i.\vec e_j=\delta^i_j.\end{cases} \tag{A.7}$$

Thus, $\pi_{ei}=e^i$ being linear, if $\vec x=^{\text{clas.}}\sum_{i=1}^n x_i\vec e_i=^{\text{dual}}\sum_{i=1}^n x^i\vec e_i$ (classical or duality notations), then (A.7) gives

$$\pi_{ei}.\vec x\overset{\text{clas.}}{=}x_i\quad=\quad x^i\overset{\text{dual}}{=}e^i.\vec x, \tag{A.8}$$

i.e. $\pi_{ei}=e^i$ gives the i-th component of a vector $\vec x$ relative to the basis $(\vec e_i)$, see figure A.1.

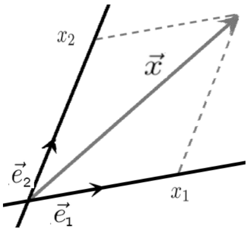

Figure A.1: Parallel projections: $\pi_{e1}(\vec x)=x_1$ and $\pi_{e2}(\vec x)=x_2$ (dual not.: $e^1(\vec x)=x^1$ and $e^2(\vec x)=x^2$).

NB: The dual basis (π_{ei}) is intrinsic to $(\vec e_i)$; And there can't be any notion of orthogonality in E here since we can't use a inner dot product: The functions $\pi_{ei}=e^i$ and vectors $\vec x$ do not belong to a same vector space.

Proposition A.10 and definition of the dual basis. *$(\pi_{ei})_{i=1,...,n}=(e^i)_{i=1,...,n}$ is a basis in E^*, called the (covariant) dual basis of the basis $(\vec e_i)$.*

Proof. If $\sum_{i=1}^n \lambda_i \pi_{ei} = 0$, then $0 = (\sum_{i=1}^n \lambda_i \pi_{ei})(\vec{e}_j) = \sum_{i=1}^n \lambda_i \pi_{ei}(\vec{e}_j) = \sum_{i=1}^n \lambda_i \delta_{ij} = \lambda_j$ for all j, thus $(\pi_{ei})_{i=1,...,n}$ is a family of n independent vectors in E^*. Then let $\ell \in E^*$ and $m = \sum_i (\ell.\vec{e}_i)\pi_{ei}$. Thus $m \in E^*$ (since E^* is a vector space), and $m(\vec{e}_j) = \sum_i (\ell.\vec{e}_i)(\pi_{ei}.\vec{e}_j) = \sum_i (\ell.\vec{e}_i)\delta_{ij} = (\ell.\vec{e}_j)$, thus $m = \ell$, thus $\ell = \sum_i (\ell.\vec{e}_i)\pi_{ei}$, thus $\mathrm{Vect}\{(\pi_{ei})_{i=1,...,n}\}$ span E^*; Thus $(\pi_{ei})_{i=1,...,n}$ is a basis in E^*; Thus $\dim E^* = n$. (Use duality notations if you prefer.) ⁂

Example A.11 Following example 1.1. The size of a child is represented on a wall by a bipoint vector $\vec{u}$. And English observer chooses the foot as unit of length, represented by a vertical bipoint vector which he names $\vec{e}$. And then defines the linear form $\pi_e : \vec{\mathbb{R}} \to \mathbb{R}$ by $\pi_e.\vec{e} = 1$. Thus π_e is a measuring instrument, which gives $s = \pi_e.\vec{u} =$ the size of the child in foot, i.e. $\vec{u} = s\vec{e}$. ⁂

Exercice A.12 Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be bases and let (π_{ai}) and (π_{bi}) be the dual bases. Let $\lambda \neq 0$. Prove:

$$\text{If } \forall i = 1,...,n,\ \vec{b}_i = \lambda \vec{a}_i, \quad \text{then} \quad \forall i = 1,...,n,\ \pi_{bi} = \frac{1}{\lambda}\pi_{ai}. \tag{A.9}$$

(With duality notations, $b^i = \frac{1}{\lambda}a^i$.)

Answer. $\pi_{bi}.\vec{b}_j = \delta_{ij} = \pi_{ai}.\vec{a}_j = \pi_{ai}.\frac{\vec{b}_j}{\lambda} = \frac{1}{\lambda}\pi_{ai}.\vec{b}_j$ for all j (since π_{ai} is linear), thus $\pi_{bi} = \frac{1}{\lambda}\pi_{ai}$, true for all i. ⁂

A.3.3 Example: aeronautical units

Example A.13 International aeronautical units: Horizontal length = nautical mile (NM), altitude = English foot (ft). Application: An air traffic controller chooses the point $O =$ the position of its control tower, and a plane p is located thanks to the bipoint vector $\vec{x} = \overrightarrow{\mathcal{O}p}$. And the traffic controller chooses $\vec{e}_1 =$ the vector of length 1 NM oriented South (first runway), $\vec{e}_2 =$ the vector of length 1 NM oriented Southwest (second runway), $\vec{e}_3 =$ the vertical vector of length 1 ft. Thus his referential is $\mathcal{R} = (\mathcal{O}, (\vec{e}_1, \vec{e}_2, \vec{e}_3))$, and his dual basis $(\pi_{e1}, \pi_{e2}, \pi_{e3})$ is defined by $\pi_{ei}(\vec{e}_j) = \delta_{ij}$ for all i, j, cf. (A.7). He writes $\vec{x} = \sum_{i=1}^n x_i \vec{e}_i \in \vec{\mathbb{R}}^n$, so that $x_1 = \pi_{e1}(\vec{x}) =$ the distance to the south in NM, $x_2 = \pi_{e2}(\vec{x}) =$ the distance to the southwest in NM, $x_3 = \pi_{e3}(\vec{x}) =$ the altitude in ft.

Here the basis $(\vec{e}_i)$ is not a Euclidean basis. This non Euclidean basis $(\vec{e}_i)$ is however vital if you take a plane. (A Euclidean basis is not essential to life...). See next remark A.14. ⁂

Remark A.14 The metre is the international unit for NASA that launched the Mars Climate Orbiter probe, and the foot is the international vertical unit for aviation; And for the Mars Climate Orbiter landing procedure, NASA (uses the metre) asked Lockheed Martin (uses the foot) to do the computation. Result? The Mars Climate Orbiter space probe burned in the Martian atmosphere because of $\lambda \sim 3$ times too high a speed during the landing procedure: One metre is $\lambda \sim 3$ times one foot, and someone forgot it... Although NASA and Lockheed Martin used a Euclidean dot product... But not the same (one based on a metre, and one based on the foot). Objectivity and covariance can be useful... ⁂

A.3.4 Matrix representation of a linear form

Let $\ell \in E^*$, let $(\vec{e}_i)$ be a basis: The components of ℓ are the n reals

$$\ell_i := \ell(\vec{e}_i) = \ell.\vec{e}_i, \quad \text{and} \quad [\ell]_{|\vec{e}} = (\,\ell_1 \quad ... \quad \ell_n\,) \tag{A.10}$$

is the row matrix of ℓ, called the matrix of ℓ relative to $(\vec{e}_i)$. Thus, if $\vec{x} \in E$ and $\vec{x} =^{\text{clas.}} \sum_{i=1}^n x_i \vec{e}_i =^{\text{dual}} \sum_{i=1}^n x^i \vec{e}_i$, then

$$\ell.\vec{x} \overset{\text{clas.}}{=} \sum_{i=1}^n \ell_i x_i \overset{\text{dual}}{=} \sum_{i=1}^n \ell_i x^i = [\ell]_{|\vec{e}}.[\vec{x}]_{|\vec{e}} \tag{A.11}$$

with usual matrix computation rules (a $1 * n$ matrix times a $n * 1$ matrix).

In particular for the dual basis $(\pi_{ei}) = (e^i)$ (classical and duality notations),

$$[\pi_{ej}]_{|\vec{e}} = [e^j]_{|\vec{e}} = (0 \ ... \ 0 \ \underbrace{1}_{j\text{th position}} \ 0 \ ... \ 0) \quad (= \text{row matrix } = [\vec{e}_j]_{|\vec{e}}^T). \tag{A.12}$$

Thus we have, with classical and duality notations,

$$\ell \overset{\text{clas.}}{=} \sum_{i=1}^n \ell_i\, \pi_{ei} \overset{\text{dual}}{=} \sum_{i=1}^n \ell_i\, e^i. \tag{A.13}$$

Remark A.15 Relative to a basis, a vector is represented by a column matrix, cf. (A.2), and a linear form by a row matrix, cf. (A.10). This enables:

• The use of matrix calculation to compute $\ell.\vec{x} = [\ell]_{|\vec{e}}.[\vec{x}]_{|\vec{e}}$, cf. (A.11), not to be confused with an inner dot product calculation $\vec{x} \bullet \vec{y}$ relative to an inner dot product in E for $\vec{x}, \vec{y} \in E$.

• Not to confuse the "nature of objects": Relative to a basis, a (contravariant) vector is a mathematical object represented by a column matrix, while a linear form (covariant vector) is a mathematical object represented by a row matrix. Cf. remark A.8. ∴

A.3.5 Example: Thermodynamic

Consider the Cartesian space $\vec{\mathbb{R}^2} = \{(T,P) \in \mathbb{R}\times\mathbb{R}\} = \{(\text{temperature,pressure})\}$. There is no meaningful inner dot product in this $\vec{\mathbb{R}^2}$: What would $\sqrt{T^2{+}P^2}$ mean (Pythagoras: Can you add Kelvin degrees and $\mathrm{kg}/(\mathrm{m}{\cdot}s^2)$? Thus, in thermodynamics, the (covariant) dual bases are the main ingredient for calculations.

E.g., in the Cartesian product $\vec{\mathbb{R}^2} = \mathbb{R} \times \mathbb{R}$ consider the basis $(\vec{E}_1{=}(1,0), \vec{E}_2{=}(0,1))$ (after a choice of temperature and pressure units); Let $\vec{X} \in \vec{\mathbb{R}^2}$, $\vec{X} = T\vec{E}_1 + P\vec{E}_2 =^{\text{noted}} (T,P)$, and let $(\pi_{E1}, \pi_{E2}) = (E^1, E^2) =^{\text{noted}} (dT, dP)$ be the (covariant) dual basis. The first principle of thermodynamics tells that the density α of internal energy is an exact differential form: $\exists U \in C^1(\vec{\mathbb{R}^2}; \mathbb{R})$ s.t. $\alpha = dU$. So, at any $\vec{X}_0 = (T_0, P_0)$,

$$\alpha(\vec{X}_0) = dU(\vec{X}_0) = \frac{\partial U}{\partial T}(\vec{X}_0)\,dT + \frac{\partial U}{\partial P}(\vec{X}_0)\,dP \quad \text{and} \quad [dU(\vec{X}_0)]_{|\vec{E}} = \begin{pmatrix} \frac{\partial U}{\partial T}(\vec{X}_0) & \frac{\partial U}{\partial P}(\vec{X}_0) \end{pmatrix} \tag{A.14}$$

(row matrix). And we have the first order Taylor expansion in the vicinity of $\vec{X}_0$,

$$\begin{aligned} U(\vec{X}_0 + \delta\vec{X}) &= U(\vec{X}_0) + dU(\vec{X}_0).\delta\vec{X} + o(\delta\vec{X}) \\ &= U(T_0,P_0) + \delta T\frac{\partial U}{\partial T}(T_0,P_0) + \delta P\frac{\partial U}{\partial T}(T_0,P_0) + o((\delta T, \delta P)). \end{aligned} \tag{A.15}$$

Matrix computation: Column matrices for vectors, row matrices for linear forms:

$$[\vec{E}_1]_{|\vec{E}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [\vec{E}_2]_{|\vec{E}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [\vec{X}_0]_{|\vec{E}} = \begin{pmatrix} T_0 \\ P_0 \end{pmatrix}, \quad [\delta\vec{X}]_{|\vec{E}} = \begin{pmatrix} \delta T \\ \delta P \end{pmatrix}, \tag{A.16}$$

$$[E^1]_{|\vec{E}} = [dT]_{|\vec{E}} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad [E^2]_{|\vec{E}} = [dP]_{|\vec{E}} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad [dU]_{|\vec{E}} = \begin{pmatrix} \frac{\partial U}{\partial T} & \frac{\partial U}{\partial P} \end{pmatrix} \tag{A.17}$$

give

$$dU(\vec{X}_0).\delta\vec{X} = \begin{pmatrix} \frac{\partial U}{\partial T}(\vec{X}_0) & \frac{\partial U}{\partial P}(\vec{X}_0) \end{pmatrix}.\begin{pmatrix} \delta T \\ \delta P \end{pmatrix} = \frac{\partial U}{\partial T}(\vec{X}_0)\delta T + \frac{\partial U}{\partial P}(\vec{X}_0)\delta P. \tag{A.18}$$

This is a "covariant calculation" (in particular no inner dot product has been used).

A.4 Einstein convention

A.4.1 Definition

When you work with components (after a choice of a basis), the goal is to visually differentiate a linear form from a vector (to visually differentiate covariance from contravariance). Framework: a finite dimension vector space E, $\dim E = n$, and duality notations.

Einstein Convention:

1. A basis in E (contravariant) is written with bottom indices: E.g., $(\vec{e}_i)$ is a basis in E.
2. A vector $\vec{x} \in E$ (contravariant) has its components relative to $(\vec{e}_i)$ (quantification) written with top indices: $\vec{x} = \sum_{i=1}^n x^i\vec{e}_i$, and is represented by the column matrix $[\vec{x}]_{|\vec{e}} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$. (Classical notations: $\vec{x} = \sum_{i=1}^n x_i\vec{e}_i$, and column matrix of x_i.)
3. A basis in $E^* = \mathcal{L}(E;\mathbb{R})$ (covariant) is written with top indices: E.g., $(e^i) \in E^{*n}$ is the dual basis of the basis $(\vec{e}_i)$. (Classical notations: (π_{ei}).)
4. A linear form $\ell \in E^*$ (covariant) has its components relative to (e^i) (quantification) written with bottom indices: $\ell = \sum_{i=1}^n \ell_i e^i$, and its matrix representation is the row matrix $[\ell]_{|\vec{e}} = \begin{pmatrix} \ell_1 & \dots & \ell_n \end{pmatrix}$.

5. You can also omit the sum sign $\sum$ when there are repeated indices at a different position; E.g. $\sum_{i=1}^n x^i \vec{e}_i =^{\text{noted}} x^i \vec{e}_i$, and $\sum_{i=1}^n L^i{}_j \vec{e}_i =^{\text{noted}} L^i_j \vec{e}_i$. In fact, before computers and word processors, to print $\sum_{i=1}^n$ was not easy. But with LaTeX this is no more a problem, so in this manuscript the sum sign $\sum$ is not omitted (and some confusions are avoided).

Remark A.16 Einstein's convention is not mandatory. E.g. Arnold doesn't use it when he doesn't need it, or when it makes reading difficult, or when it induces misunderstandings. In classical mechanics, Einstein's convention may induce more confusion than understandings, and may be misused... so it is better ***not*** to use it: Golden rule: Use classical notations when in doubt. ∴

A.4.2 Do not mistake yourself

1. Einstein's convention is just meant not to confuse a linear function with a vector.
2. It only deals with quantification relative to a basis.
3. Classical notations are as good as duality notations, even you are told that classical notations cannot detect obvious errors in component manipulations... But duality notations can be misused in classical mechanics (cf. the paradigmatic example of the vectorial dual basis, correctly treated at § F.7); And thus add confusion to the confusion.
4. The convention does not admit shortcuts; E.g. with a metric: $g(\vec{u},\vec{v}) = \sum_{i,j=1}^n g_{ij} u^i v^j$ shows the observer dependence on a choice of a basis thanks to the g_{ij}; And even if $g_{ij} = \delta_{ij}$ you can***not*** write $g(\vec{u},\vec{v}) = \sum_{i,j=1}^n u^i v^j$: You have to write $g(\vec{u},\vec{v}) = \sum_{i,j=1}^n g_{ij} u^i v^j$: Unmissable in physics because you need to see the metric and bases in use.
5. Golden rule: Return to classical notations if in doubt. (Einstein's convention can add confusions, untruths, misinterpretations, absurdities, misuses...)

A.5 Change of basis formulas

E being a finite dimension vector space, $\dim E = n$, let $(\vec{e}_{old,i})$ and $(\vec{e}_{new,i})$ be two bases in E, and let $(\pi_{old,i})$ and $(\pi_{new,i})$ be the dual bases in E^*, written (e^i_{old}) and (e^i_{new}) with duality notations.

A.5.1 Change of basis endomorphism and transition matrix

Definition A.17 The change of basis endomorphism $\mathcal{P} \in \mathcal{L}(E;E)$ from $(\vec{e}_{old,i})$ to $(\vec{e}_{new,i})$ is the endomorphism (= the linear map $E \to E$) defined by, for all $j \in [1,n]_{\mathbb{N}}$,

$$\boxed{\mathcal{P}.\vec{e}_{old,j} = \vec{e}_{new,j}}. \tag{A.19}$$

Definition A.18 The transition matrix from $(\vec{e}_{old,i})$ to $(\vec{e}_{new,i})$ is the matrix $P := [\mathcal{P}]_{|\vec{e}_{old}} = [P_{ij}]$ of the endomorphism $\mathcal{P}$ relative to the basis $(\vec{e}_{old,i})$, i.e. defined by, for all j,

$$\vec{e}_{new,j} = \mathcal{P}.\vec{e}_{old,j} = \sum_{i=1}^n P_{ij}\,\vec{e}_{old,i}, \quad \text{i.e.} \quad [\vec{e}_{new,j}]_{|\vec{e}_{old}} = P.[\vec{e}_{old,j}]_{|\vec{e}_{old}} = \begin{pmatrix} P_{1j} \\ \vdots \\ P_{nj} \end{pmatrix}, \tag{A.20}$$

i.e., $[\vec{e}_{new,j}]_{|\vec{e}_{old}}$ is the j-th column of $P = [\mathcal{P}]_{|\vec{e}_{old}}$. Duality notations: $\vec{e}_{new,j} = \sum_{i=1}^n P^i{}_j\,\vec{e}_{old,i}$ and $P := [\mathcal{P}]_{|\vec{e}_{old}} = [P^i{}_j]$.

Apart from the classical and notations, you may find other "component type" notations:

$$\vec{e}_{new,j} = \sum_{i=1}^n P_{ij}\,\vec{e}_{old,i} = \sum_{i=1}^n (P_j)_i\,\vec{e}_{old,i} = \sum_{i=1}^n P^i{}_j\,\vec{e}_{old,i} = \sum_{i=1}^n (P_j)^i\,\vec{e}_{old,i}, \tag{A.21}$$

i.e. $P_{ij} = (P_j)_i = P^i{}_j = (P_j)^i$ are four notations for the i-th component of $\vec{e}_j$, i.e.

$$[\vec{e}_{new,j}]_{|\vec{e}_{old}} = \begin{pmatrix} P_{1j} \\ \vdots \\ P_{nj} \end{pmatrix} = \begin{pmatrix} (P_j)_1 \\ \vdots \\ (P_j)_n \end{pmatrix} = \begin{pmatrix} P^1{}_j \\ \vdots \\ P^n{}_j \end{pmatrix} = \begin{pmatrix} (P_j)^1 \\ \vdots \\ (P_j)^n \end{pmatrix} \quad (= \text{the } j\text{-th column of } P). \tag{A.22}$$

A.5.2 Inverse of the transition matrix

The inverse endomorphism $\mathcal{Q} := \mathcal{P}^{-1} \in \mathcal{L}(E;E)$, cf. (A.19), is given by, for all $j \in [1,n]_{\mathbb{N}}$,

$$\vec{e}_{old,j} = \mathcal{Q}.\vec{e}_{new,j} \quad (= \mathcal{P}^{-1}.\vec{e}_{new,j}), \tag{A.23}$$

i.e. $\mathcal{Q}$ is change of basis endomorphism from $(\vec{e}_{new,i})$ to $(\vec{e}_{old,i})$. And $Q := [\mathcal{Q}]_{|\vec{e}_{new}} = [Q_{ij}]$ is the transition matrix from $(\vec{e}_{new,i})$ to $(\vec{e}_{old,i})$:

$$\vec{e}_{old,j} = \sum_{i=1}^{n} Q_{ij}\vec{e}_{new,i}, \quad [\vec{e}_{old,j}]_{|\vec{e}_{new}} = \begin{pmatrix} Q_{1j} \\ \vdots \\ Q_{nj} \end{pmatrix}. \tag{A.24}$$

i.e. $\mathcal{Q}$ is change of basis endomorphism from $(\vec{e}_{new,i})$ to $(\vec{e}_{old,i})$.

Use other notation if you prefer: $Q_{ij} = (Q_j)_i = Q^i{}_j = (Q_j)^i$

Proposition A.19

$$Q = P^{-1}. \tag{A.25}$$

Proof. $\vec{e}_{new,j} = \mathcal{P}.\vec{e}_{old,j} = \sum_{i=1}^n P_{ij}\vec{e}_{old,i} = \sum_{i=1}^n P_{ij}(\sum_{k=1}^n Q_{ki}\vec{e}_{new,k}) = \sum_{k=1}^n(\sum_{i=1}^n Q_{ki}P_{ij})\vec{e}_{new,k} = \sum_{k=1}^n (Q.P)_{kj}\vec{e}_{new,k}$ for all j, thus $(Q.P)_{kj} = \delta_{kj}$ for all j,k. Hence $Q.P = I$, i.e. (A.25). ∴

Exercice A.20 Prove $\left\{ \begin{array}{l} [\mathcal{P}]_{|\vec{e}_{old}} = [\mathcal{P}]_{|\vec{e}_{new}} = P, \\ [\mathcal{Q}]_{|\vec{e}_{new}} = [\mathcal{Q}]_{|\vec{e}_{old}} = Q, \end{array} \right\}$, i.e.

$$\left\{ \begin{array}{l} \mathcal{P}.\vec{e}_{new,j} = \sum_{i,j=1}^n P_{ij}\vec{e}_{new,i} \quad (= \sum_{i,j=1}^n P^i{}_j\vec{e}_{new,i} = \sum_{i,j=1}^n (P_j)^i\vec{e}_{new,i}), \\ \mathcal{Q}.\vec{e}_{old,j} = \sum_{i,j=1}^n Q_{ij}\vec{e}_{old,i} \quad (= \sum_{i,j=1}^n Q^i{}_j\vec{e}_{old,i} = \sum_{i,j=1}^n (Q_j)^i\vec{e}_{old,i}). \end{array} \right. \tag{A.26}$$

Answer. $Z = [Z_{ij}] = [\mathcal{P}]_{|\vec{e}_{new}}$ means $\mathcal{P}.\vec{e}_{new,j} = \sum_i Z_{ij}\vec{e}_{new,i}$, i.e. $\vec{e}_{new,j} = \mathcal{Q}.(\sum_{i=1}^n Z_{ij}\vec{e}_{new,i}) = \sum_{i=1}^n Z_{ij}\mathcal{Q}.\vec{e}_{new,i} = \sum_{i=1}^n Z_{ij}(\sum_{k=1}^n Q_{ki}\vec{e}_{new,k}) = \sum_{k=1}^n(\sum_{i=1}^n Q_{ki}Z_{ij})\vec{e}_{new,k} = \sum_{k=1}^n (Q.Z)_{kj}\vec{e}_{new,k}$ for all j, thus $(Q.Z)_{kj} = \delta_{kj}$ for all j,k, thus $Q.Z = I$, thus $Z = P$. Idem for Q, thus (A.26). ∴

Remark A.21 $P^T \neq P^{-1}$ in general. E.g., $(\vec{e}_{old,i}) = (\vec{a}_i)$ is a foot-built Euclidean basis, and $(\vec{e}_{new,i}) = (\vec{b}_i)$ is a metre-built Euclidean basis, and $\vec{b}_i = \lambda\vec{a}_i$ for all i (the basis are "aligned"), so $P = \lambda I$; Thus $P^T = \lambda I$ and $P^{-1} = \frac{1}{\lambda}I \neq P^T$, since $\lambda = \frac{1}{0.3048} \neq 1$. Thus it is essential not to confuse P^T and P^{-1} (not to confuse covariance with contravariance), cf. e.g. the Mars Climate Orbiter crash (remark A.14). ∴

A.5.3 Change of dual basis

Proposition A.22 *$(\pi_{new,i})$ and $(\pi_{old,i})$ being the dual bases of $(\vec{e}_{new,i})$ and $(\vec{e}_{old,i})$, for all $i \in [1,n]_{\mathbb{N}}$,*

$$\pi_{new,i} = \sum_{j=1}^{n} Q_{ij}\pi_{old,i}, \quad \text{and} \quad [\pi_{new,i}]_{|\vec{e}_{old}} = (\, Q_{i1} \;\; \ldots \;\; Q_{in} \,) \quad (i\text{-th row of } Q), \tag{A.27}$$

to compare with (A.20) (matrices of linear forms are row matrices).

Duality notations: $e^i_{new} = \sum_{j=1}^n Q^i{}_j e^j_{old}$ and $[e^i_{new}]_{|\vec{e}_{old}} = (\, Q^i{}_1 \;\; \ldots \;\; Q^i{}_n \,)$.

Proof. $\pi_{new,i}(\vec{e}_{old,k}) \overset{(A.24)}{=} \pi_{new,i}(\sum_j Q_{jk}\vec{e}_{new,j}) = \sum_j Q_{jk}\,\pi_{new,i}(\vec{e}_{new,j}) = \sum_j Q_{jk}\,\delta_{ij} = Q_{ik}$, and $\sum_j Q_{ij}\pi_{old,j}(\vec{e}_{old,k}) = \sum_j Q_{ij}\delta_{jk} = Q_{ik}$, true for all i,k, thus $\pi_{new,i} = \sum_j Q_{ij}$, i.e. (A.27) ∴

A.5.4 Change of coordinate system for vectors and linear forms

Proposition A.23 *Let $\vec{x} \in E$ and $\ell \in E^*$. Then*

$$\begin{array}{ll} \bullet\; [\vec{x}]_{|\vec{e}_{new}} = P^{-1}.[\vec{x}]_{|\vec{e}_{old}} & \text{(contravariance formula for vectors: between column matrices)}, \\ \bullet\; [\ell]_{|\vec{e}_{new}} = [\ell]_{|\vec{e}_{old}}.P & \text{(covariance formula for linear forms: between row matrices)}. \end{array} \tag{A.28}$$

And the scalar value $\ell.\vec{x}$ is computed indifferently with one or the other basis (objective result):

$$\ell.\vec{x} = [\ell]_{|\vec{e}_{old}}.[\vec{x}]_{|\vec{e}_{old}} = [\ell]_{|\vec{e}_{new}}.[\vec{x}]_{|\vec{e}_{new}}. \tag{A.29}$$

Proof. Let $\vec{x} = \sum_j x_j \vec{e}_{old,j} = \sum_i y_i \vec{e}_{new,i}$. We have $\vec{x} = \sum_j x_j \vec{e}_{old,j} = \sum_j x_j (\sum_{i=1}^n Q_{ij} \vec{e}_{new,i}) = \sum_{ij} Q_{ij} x_j \vec{e}_{new,i}$, thus $y_i = \sum_j Q_{ij} x_j$ for all i, thus (A.28)$_1$.

And $\ell = \sum_j m_j \pi_{new,j} = \sum_i \ell_i \pi_{old,i} =^{(A.27)} \sum_{ij} \ell_i P_{ij} \pi_{new,j}$ gives $m_j = \sum_i \ell_i P_{ij}$ for all j, thus (A.28)$_2$. (Use duality notations if you prefer.)

Thus $[\ell]_{|\vec{e}_{new}}.[\vec{x}]_{|\vec{e}_{new}} = ([\ell]_{|\vec{e}_{old}}.P).(P^{-1}.[\vec{x}]_{|\vec{e}_{old}}) = [\ell]_{|\vec{e}_{old}}.[\vec{x}]_{|\vec{e}_{old}}$, hence (A.29). ∴

Notation: (A.28) and $\vec{x} = \sum_j x_j \vec{e}_{old,j} = \sum_i y_i \vec{e}_{new,i}$ give $y_i = \sum_{j=1}^n Q_{ij} x_j$, which means: y_i is the function defined by $y_i(x_1, ..., x_n) = \sum_{j=1}^n Q_{ij} x_j$, thus $Q_{ij} = \frac{\partial y_i}{\partial x_j}(x_1, ..., x_n)$; Similarly with P_{ij}; Which is written

$$Q_{ij} = \frac{\partial y_i}{\partial x_j}, \quad \text{and} \quad P_{ij} = \frac{\partial x_i}{\partial y_j}. \tag{A.30}$$

(Use duality notations if you prefer, e.g. $Q^i{}_j = \frac{\partial y^i}{\partial x^j}$.)

A.6 Bidual basis (and contravariance)

Definition A.24 The dual of E^* is $E^{**} := (E^*)^* = \mathcal{L}(E^*; \mathbb{R})$ and is named the bidual of E.

E^{**} is also called the space of contravariant vectors = the space of directional derivatives.

Definition A.25 Let $(\vec{e}_i)$ be a basis in E, let (π_{ei}) be its dual basis (basis in E^*). The dual basis (∂_i) of (π_{ei}) is called the bidual basis of $(\vec{e}_i)$. (Duality notations: $(\pi_{ei}) = (e^i)$.)

(The notation ∂_i refers to the derivation in the direction $\vec{e}_i$: $\partial_i(df(\vec{x})) = df(\vec{x}).\vec{e}_i = \frac{\partial f}{\partial x_i}(\vec{x})$, see § S.1.)

Thus, the linear form $\partial_i \in E^{**} = \mathcal{L}(E^*; \mathbb{R})$ are characterized by, for all j,

$$\partial_i.\pi_{ej} = \delta_{ij} = \pi_{ej}.\vec{e}_i, \quad \text{so} \quad \ell = \sum_{i=1}^n \ell_i \pi_{ei} \quad \text{iff} \quad \ell_i = \partial_i.\ell \ \ (= \ell.\vec{e}_i). \tag{A.31}$$

Indeed, $\partial_i(\ell) = \partial_i(\sum_{j=1}^n \ell_j \pi_{ej}) = \sum_{j=1}^n \ell_j \partial_i(\pi_{ej}) = \sum_{j=1}^n \ell_j \delta_{ij} = \ell_i$. (Duality notation: $\partial_i.e^j = \delta_i^j = e^j.\vec{e}_i$ and $\ell = \sum_{i=1}^n \ell_i e^i$.)

Remark: With the natural canonical isomorphism $\mathcal{J} : \left\{ \begin{array}{ll} E & \to E^{**} = \mathcal{L}(E^*; \mathbb{R}) \\ \vec{u} & \to \mathcal{J}(\vec{u}), \ \text{ where } \ \mathcal{J}(\vec{u}).\ell := \ell.\vec{u}, \ \ \forall \ell \in E^* \end{array} \right\}$ see (T.9), we can identify $\vec{u}$ and $\mathcal{J}(\vec{u})$ (observer independent identification), thus $\partial_i = \mathcal{J}(\vec{e}_i) =^{\text{noted}} \vec{e}_i$, and (A.31) reads (usual notation in differential geometry) $\vec{e}_i.\pi_{ej} = \delta_{ij}$ and $\ell_i = \vec{e}_i.\ell$.

A.7 Bilinear forms

A.7.1 Definition

Let E and F be vector spaces.

Definition A.26 • A bilinear form is a 2-multilinear form $\beta(\cdot,\cdot) : \left\{ \begin{array}{ll} E \times F & \to \mathbb{R} \\ (\vec{u}, \vec{w}) & \to \beta(\vec{u}, \vec{w}) \end{array} \right\}$.

So, $\beta(\vec{u}_1 + \lambda \vec{u}_2, \vec{w}) = \beta(\vec{u}_1, \vec{w}) + \lambda \beta(\vec{u}_2, \vec{w})$ and $\beta(\vec{u}, \vec{w}_1 + \lambda \vec{w}_2) = \beta(\vec{u}, \vec{w}_1) + \lambda \beta(\vec{u}, \vec{w}_2)$ for all $\vec{u}, \vec{u}_1, \vec{u}_2 \in E$, $\vec{w}, \vec{w}_1, \vec{w}_2 \in F$, $\lambda \in \mathbb{R}$.

- $\mathcal{L}(E, F; \mathbb{R})$ is the set of bilinear forms $E \times F \to \mathbb{R}$.
- If $(\ell, m) \in E^* \times F^*$, then the bilinear form $\ell \otimes m \in \mathcal{L}(E, F; \mathbb{R})$ is defined by

$$(\ell \otimes m)(\vec{u}, \vec{w}) = \ell(\vec{u}) m(\vec{w}) \quad (= (\ell.\vec{u})(m.\vec{w})) \tag{A.32}$$

for all $(\vec{u}, \vec{w}) \in E \times F$, and is called an elementary bilinear form.

A.7.2 The transposed of a bilinear form

(Warning: Not to be confused with the subjective definition of a transposed of a linear map which requires inner dot products to be defined, see e.g. (A.54).)

Definition A.27 If $\beta \in \mathcal{L}(E, F; \mathbb{R})$ then its transposed is the bilinear form $\beta^T \in \mathcal{L}(F, E; \mathbb{R})$ defined by, for all $(\vec{w}, \vec{u}) \in F \times E$,

$$\beta^T(\vec{w}, \vec{u}) = \beta(\vec{u}, \vec{w}). \tag{A.33}$$

(This definition is observer independent: no basis or inner dot product is required in this definition.)

A.7.3 Symmetric and definite positive bilinear forms

Definition A.28 Here $F=E$ (no choice), and $\beta \in \mathcal{L}(E,E;\mathbb{R})$.

- β is semi-positive, iff for all $\vec{u} \in E$,

$$\beta(\vec{u},\vec{u}) \geq 0. \tag{A.34}$$

- β is definite positive, iff for all $\vec{u} \neq \vec{0}$,

$$\beta(\vec{u},\vec{u}) > 0. \tag{A.35}$$

- β is symmetric iff $\beta^T = \beta$, i.e., for all $\vec{u},\vec{v} \in E$,

$$\beta(\vec{u},\vec{v}) = \beta(\vec{v},\vec{u}). \tag{A.36}$$

A.7.4 Inner dot product, and metric

Definition A.29 • An "inner dot product" (or "scalar inner dot product", or "inner scalar product", or "inner product") in a vector space E is a bilinear form $\beta =^{\text{noted}} g =^{\text{noted}} g(\cdot,\cdot) \in \mathcal{L}(E,E;\mathbb{R})$ which is symmetric and definite positive. And then (for inner dot products)

$$g(\cdot,\cdot) \overset{\text{noted}}{=} (\cdot,\cdot)_g \overset{\text{noted}}{=} \cdot \underset{g}{\bullet} \cdot, \quad \text{i.e.} \quad g(\vec{u},\vec{w}) = (\vec{u},\vec{w})_g \overset{\text{noted}}{=} \vec{u} \underset{g}{\bullet} \vec{w}, \quad \forall \vec{u},\vec{w} \in E. \tag{A.37}$$

- Then two vectors $\vec{u},\vec{w} \in E$ are $(\cdot,\cdot)_g$-orthogonal iff $(\vec{u},\vec{w})_g = 0$.
- And the associated norm with $(\cdot,\cdot)_g$ is the function $||.||_g : E \to \mathbb{R}_+$ given by, for all $\vec{u} \in E$,

$$||\vec{u}||_g = \sqrt{(\vec{u},\vec{u})_g}. \tag{A.38}$$

(To prove that it is a norm, use the Cauchy–Schwarz inequality (A.39).)

• An "semi-inner dot product" $(\cdot,\cdot)_g$ (or "semi-scalar inner dot product") in a vector space E is a bilinear form $\beta =^{\text{noted}} g(\cdot,\cdot) \in \mathcal{L}(E,E;\mathbb{R})$ which is symmetric and semi-positive. And the associated semi-norm is given by (A.38).

Proposition A.30 *(Cauchy–Schwarz inequality.)* $(\cdot,\cdot)_g$ *being an inner dot product in* E*,*

$$\forall \vec{u},\vec{w} \in E, \quad |(\vec{u},\vec{w})_g| \leq ||\vec{u}||_g ||\vec{w}||_g. \tag{A.39}$$

And $|(\vec{u},\vec{w})_g| = ||\vec{u}||_g ||\vec{w}||_g$ *iff* $\vec{u}$ *and* $\vec{w}$ *are parallel.*

Proof. Let $p(\lambda) = ||\vec{u}+\lambda\vec{w}||_g^2 = (\vec{u}+\lambda\vec{w},\vec{u}+\lambda\vec{w})_g$, so $p(\lambda) = a\lambda^2 + b\lambda + c$ where $a = ||\vec{w}||_g^2$, $b = 2(\vec{u},\vec{w})_g$ and $c = ||\vec{u}||_g^2$. With $p(\lambda) \geq 0$ (since$(\cdot,\cdot)_g$ is positive), we get $b^2 - 4ac \geq 0$, thus (A.39), and $p(\lambda) = 0$ iff $\vec{u}+\lambda\vec{w} = 0$. ∴

Definition A.31 (Metric.) With $\mathbb{R}^n$ our usual affine geometric space, $n = 1$, 2 or 3, and $\vec{\mathbb{R}}^n =$ the usual associated vector space made of bipoint vectors. Let $\Omega \subset \mathbb{R}^n$ be open in $\mathbb{R}^n$. A metric in Ω is a C^∞ function $g : \left\{ \begin{array}{ll} \Omega & \to \mathcal{L}(\vec{\mathbb{R}}^n,\vec{\mathbb{R}}^n;\mathbb{R}) \\ p & \to g(p) \overset{\text{noted}}{=} g_p \end{array} \right\}$ such that g_p is an inner dot product in $\vec{\mathbb{R}}^n$ at each $p \in \Omega$.

Particular Case: When the g_p is independent of p (general case in continuum mechanics), a metric is simply called a inner dot product (e.g. a Euclidean metric is called a Euclidean dot product).

(In a differentiable manifold Ω, a metric is a C^∞ $\binom{0}{2}$ tensor g s.t. $g(p)$ is an inner dot product at each $p \in \Omega$. A Riemannian metric is a metric s.t. $g(p)$ is a Euclidean dot product at each $p \in \Omega$.)

A.7.5 Quantification: Matrice $[\beta_{ij}]$ and tensorial representation

$\dim E = n$, $\dim F = m$, $\beta \in \mathcal{L}(E,F;\mathbb{R})$, $(\vec{a}_i)$ is a basis in E which dual basis is (π_{ai}), $(\vec{b}_i)$ is a basis in F which dual basis is (π_{bi}). (With duality notations, $(\pi_{ai}) = (a^i)$ and $(\pi_{bi}) = (b^i)$.)

Definition A.32 The components of $\beta \in \mathcal{L}(E,F;\mathbb{R})$ relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$ are the nm reals

$$\beta_{ij} := \beta(\vec{a}_i,\vec{b}_j), \quad \text{and} \quad [\beta]_{|\vec{a},\vec{b}} = [\beta_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \tag{A.40}$$

is the matrix of β relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$, simply written $[\beta_{ij}]$ if the bases are implicit.

And if $F=E$ and $(\vec{b}_i) = (\vec{a}_i)$ then $[\beta]_{|\vec{a},\vec{b}} =^{\text{noted}} [\beta]_{|\vec{a}}$.

Proposition A.33 *A bilinear form $\beta \in \mathcal{L}(E,F;\mathbb{R})$ is known as soon as the nm scalars $\beta_{ij} = \beta(\vec{a}_i, \vec{b}_j)$ are known, and, for all $(\vec{u},\vec{w}) \in E \times F$,*

$$\beta(\vec{u},\vec{w}) = [\vec{u}]_{|\vec{a}}{}^T.[\beta]_{|\vec{a},\vec{b}}.[\vec{w}]_{|\vec{b}}, \quad \text{written} \quad \boxed{\beta(\vec{u},\vec{w}) = [\vec{u}]^T.[\beta].[\vec{w}]}, \tag{A.41}$$

so

$$\beta = \sum_{i=1}^{n}\sum_{j=1}^{m}\beta_{ij}\pi_{ai} \otimes \pi_{bj}, \tag{A.42}$$

and a basis in $\mathcal{L}(E,F;\mathbb{R})$ is made of the nm functions $\pi_{ai} \otimes \pi_{bj}$, and $\dim \mathcal{L}(E,F;\mathbb{R}) = nm$. (Duality notations: $\beta = \sum_{i=1}^n \sum_{j=1}^m \beta_{ij} a^i \otimes b^j$.)

Proof. β being bilinear, $\vec{u} = \sum_{i=1}^n u_i \vec{a}_i$ and $\vec{w} = \sum_{j=1}^n w_j \vec{b}_j$ give $\beta(\vec{u},\vec{w}) = \sum_{i,j=1}^n u_i w_j \beta(\vec{a}_i,\vec{b}_j) = \sum_{i,j=1}^n u_i \beta_{ij} w_j = ([\vec{u}]_{|\vec{a}})^T.[\beta]_{|\vec{a},\vec{b}}.[\vec{w}]_{|\vec{b}}$, thus (A.41). In particular, if the β_{ij} are known, then b is known. And $(\pi_{ai} \otimes \pi_{bj})(\vec{a}_k, \vec{b}_\ell) =^{(A.32)} (\pi_{ai}.\vec{a}_k)(\pi_{bj}.\vec{b}_\ell) = \delta_{ik}\delta_{j\ell}$ (all the elements of the matrix $[\pi_{ai} \otimes \pi_{bj}]_{|\vec{a},\vec{b}}$ are zero except the element at the intersection of row i and column j which is equal to 1). And $(\pi_{ai} \otimes \pi_{bj})(\vec{u},\vec{w}) =^{(A.32)} (\pi_{ai}.\vec{u})(\pi_{bj}.\vec{w}) = u_i w_j$, thus $\beta(\vec{u},\vec{w}) = \sum_{i,j=1}^n \beta_{ij} u_i w_j = \sum_{i,j=1}^n \beta_{ij} (\pi_{ai} \otimes \pi_{bj})(\vec{u},\vec{w})$, thus $\beta := \sum_{i,j=1}^n \beta_{ij}(\pi_{ai} \otimes \pi_{bj})$, thus the $\pi_{ai} \otimes \pi_{bj}$ span $\mathcal{L}(E,F;\mathbb{R})$. And $\sum_{ij} \lambda_{ij}(\pi_{ai} \otimes \pi_{bj}) = 0$ implies $0 = (\sum_{ij} \lambda_{ij}(\pi_{ai} \otimes \pi_{bj}))(\vec{a}_k,\vec{b}_\ell) = \sum_{ij} \lambda_{ij}(\pi_{ai} \otimes \pi_{bj})(\vec{a}_k,\vec{b}_\ell) = \lambda_{k\ell} = 0$ for all k,ℓ; Thus the $\pi_{ai} \otimes \pi_{bj}$ are independent. Thus $(\pi_{ai} \otimes \pi_{bj})$ is a basis in $\mathcal{L}(E,F;\mathbb{R})$ and $\dim(\mathcal{L}(E,F;\mathbb{R})) = nm$. (Duality notations: $\beta(\vec{u},\vec{w}) = \sum_{i,j=1}^n \beta_{ij} u^i w^j$ and $\beta := \sum_{i,j=1}^n \beta_{ij}\, a^i \otimes b^j$.) ∴

Example A.34 $\dim E = \dim F = 2$. $[\beta]_{|\vec{a},\vec{b}} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ means $\beta(\vec{a}_1,\vec{b}_1) = \beta_{11} = 1$, $\beta(\vec{a}_1,\vec{b}_2) = \beta_{12} = 2$, $\beta(\vec{a}_2,\vec{b}_1) = \beta_{21} = 0$, $\beta(\vec{a}_2,\vec{b}_2) = \beta_{22} = 3$. And $\beta_{12} = [\vec{a}_1]_{|\vec{a}}^T.[\beta]_{|\vec{a},\vec{b}}.[\vec{b}_2]_{|\vec{b}} = \begin{pmatrix} 1 & 0 \end{pmatrix}.\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2$. ∴

Exercice A.35 Let $\beta \in \mathcal{L}(E,E;\mathbb{R})$, let $(\vec{a}_i)$ and $(\vec{b}_i)$ be two bases in A, and let $\lambda \in \mathbb{R}^*$. Prove:

$$\text{if, } \forall i \in [1,n]_{\mathbb{N}}, \ \vec{b}_i = \lambda \vec{a}_i, \quad \text{then} \quad [\beta]_{|\vec{b}} = \lambda^2 [\beta]_{|\vec{a}}. \tag{A.43}$$

(A change of unit, e.g. from foot to metre, has a "big" influence on the matrix.)

Answer. $\vec{b}_i = \lambda \vec{a}_i$ give $\beta(\vec{b}_i,\vec{b}_j) = \beta(\lambda\vec{a}_i,\lambda\vec{a}_j) = \lambda^2\beta(\vec{a}_i,\vec{a}_j)$ (bilinearity), thus $[\beta]_{|\vec{b}} = \lambda^2[\beta]_{|\vec{a}}$. ∴

Exercice A.36 Prove

$$[\beta^T]_{\vec{b},\vec{a}} = ([\beta]_{\vec{a},\vec{b}})^T, \quad \text{written} \quad [\beta^T] = [\beta]^T. \tag{A.44}$$

Answer. Let$[\beta]_{\vec{a},\vec{b}} = [\beta_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$ and $[\beta^T]_{\vec{b},\vec{a}} = [\gamma_{ij}]_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$. We have $\gamma_{ij} = \beta^T(\vec{b}_i,\vec{a}_j) = \beta(\vec{a}_j,\vec{b}_i) = \beta_{ji}$, qed. ∴

A.8 Linear maps

A.8.1 Definition

Let E and F be vector spaces.

Definition A.37 • A function $L : E \to F$ is linear iff $L(\vec{u}_1 + \lambda\vec{u}_2) = L(\vec{u}_1) + \lambda L(\vec{u}_2)$ for all $\vec{u}_1, \vec{u}_2 \in E$ and all $\lambda \in \mathbb{R}$ (distributivity type relation). And (distributivity notation):

$$L(\vec{u}) \overset{\text{noted}}{=} L.\vec{u}, \quad \text{so} \quad L(\vec{u}_1 + \lambda\vec{u}_2) = L.(\vec{u}_1 + \lambda\vec{u}_2) = L.\vec{u}_1 + \lambda L.\vec{u}_2. \tag{A.45}$$

NB: This dot notation $L.\vec{u}$ is a linearity notation (distributivity type notation); It is an "outer" dot product between a (linear) function and a vector; It is ***not*** an "inner" dot product since L and $\vec{u}$ don't belong to a same space. It is ***not*** a matrix product since no basis has been introduced yet (no quantification has been done yet).

• $\mathcal{L}(E;F)$ is the set of linear maps $E \to F$ (vector space, subspace of $(\mathcal{F}(E;F),+,.)$).

• If $F = E$ then a linear map $L \in \mathcal{L}(E;E)$ is called an endomorphism in E.

(If $F = \mathbb{R}$ then a linear map $E \to \mathbb{R}$ is called a linear form, and $E^* := \mathcal{L}(E;\mathbb{R})$ is the dual of E.)

Vocabulary: Let $L_i(E;E)$ be the space of linear invertible linear maps. If E is a finite dimension vector space, $\dim E = n$, then, in algebra, the set $(L_i(E;E),\circ)$ of linear maps equipped with the composition rule is named $GL_n(E)$ = "the linear group" (it is indeed a group, easy check). And the "linear group" of $n*n$ invertible matrices is $GL_n(\mathcal{M}_n) := (L_i(\mathcal{M}_n;\mathcal{M}_n),.)$, i.e. $L_i(\mathcal{M}_n;\mathcal{M}_n)$ with the matrix product rule.

Exercice A.38 (Math exercise.) Let $E = (E, ||.||_E)$ and $F = (F, ||.||_F)$ be Banach spaces, and let $L_{ic}(E;F)$ be the space of invertible linear continuous maps $E \to F$, with its usual norm $||L|| = \sup_{||\vec{x}||_E=1} ||L.\vec{x}||_F$. Let $Z : \left\{ \begin{array}{ll} L_{ic}(E;F) & \to L_{ic}(E;F) \\ L & \to L^{-1} \end{array} \right\}$. Prove $dZ(L).M = -L^{-1} \circ M \circ L^{-1}$, for all $M \in L_{ic}(E;F)$. (Recall: In finite dimension, a linear map is always continuous.)

Answer. Consider $\lim_{h\to 0} \frac{Z(L+hM)-Z(L)}{h} = \lim_{h\to 0} \frac{(L+hM)^{-1}-L^{-1}}{h}$ ($=^{\text{noted}}$ $dZ(L).M$ if the limit exists). With $N = L^{-1}.M$ we have $L + hM = L(I + hN)$, and $(I + hN)$ is invertible as soon as $||hN|| < 1$, i.e. $h < \frac{1}{||N||} = \frac{1}{||L^{-1}.M||}$, its inverse being $I - hN + h^2N - ...$ (Neumann serie); Thus $I + hN = I - hN + o(h)$, and $(L + hM)^{-1} = (I + hN)^{-1}.L^{-1} = (I - hN + o(h)).L^{-1} = L^{-1} - hN.L^{-1} + o(h)$. Thus $\frac{(L+hM)^{-1}-L^{-1}}{h} = \frac{L^{-1}-hN.L^{-1}+o(h)-L^{-1}}{h} = -N.L^{-1} + o(1) \longrightarrow_{h\to 0} -N.L^{-1}$. ∎

A.8.2 Quantification: Matrices $[L_{ij}] = [L^i{}_j]$

$\dim E = n$, $\dim F = m$, $L \in \mathcal{L}(E;F)$, $(\vec{a}_i)$ is a basis in E which dual basis is (π_{ai}), $(\vec{b}_i)$ is a basis in F which dual basis is (π_{bi}). (With duality notations, $(\pi_{ai}) = (a^i)$ and $(\pi_{bi}) = (b^i)$.)

Definition A.39 The components of a linear map $L \in \mathcal{L}(E;F)$ relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$ are the nm reals named L_{ij} (classical notation) $= L^i{}_j$ (duality notation), which are the components of the vectors $L.\vec{a}_j$ relative to the basis $(\vec{b}_i)$. That is:

$$\left\{ \begin{array}{l} \text{clas. not. : } L.\vec{a}_j = \sum_{i=1}^{m} L_{ij}\vec{b}_i, \\ \text{dual not. } \ L.\vec{a}_j = \sum_{i=1}^{m} L^i{}_j\vec{b}_i, \end{array} \right\}, \quad \text{i.e.} \quad [L.\vec{a}_j]_{|\vec{b}} \overset{\text{clas.}}{=} \begin{pmatrix} L_{1j} \\ \vdots \\ L_{mj} \end{pmatrix} \overset{\text{dual}}{=} \begin{pmatrix} L^1{}_j \\ \vdots \\ L^m{}_j \end{pmatrix}. \tag{A.46}$$

And

$$[L]_{|\vec{a},\vec{b}} \overset{\text{clas.}}{=} [L_{ij}]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \overset{\text{dual}}{=} [L^i{}_j]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \tag{A.47}$$

is the matrix of L relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$ (so $[L.\vec{a}_j]_{|\vec{b}}$ is the j-th column of $[L]_{|\vec{a},\vec{b}}$).

Particular case: If $E = F$ (so L is an endomorphism) and if $(\vec{b}_i) = (\vec{a}_i)$ then $[L]_{|\vec{a},\vec{a}} =^{\text{noted}} [L]_{|\vec{a}}$.

Example A.40 $n = m = 2$. $[L]_{|\vec{a},\vec{b}} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ means $L.\vec{a}_1 = \vec{b}_1$ and $L.\vec{a}_2 = 2\vec{b}_1 + 3\vec{b}_2$ (column reading). Here L_{11}=1, L_{12}=2, L_{21}=0, L_{22}=3 (duality notations: $L^1{}_1$=1, $L^1{}_2$=2, $L^2{}_1$=0, $L^2{}_2$=3). ∎

And L being linear, for all $\vec{u} \in E$, $\vec{u} = \sum_{j=1}^n u_j\vec{a}_j = \sum_{j=1}^n u^j\vec{a}_j$, we get, thanks to linearity,

$$L.\vec{u} \overset{\text{clas.}}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} L_{ij}u_j\vec{b}_i \overset{\text{dual}}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} L^i{}_j u^j\vec{b}_i, \quad \text{i.e.} \quad \boxed{[L.\vec{u}]_{|\vec{b}} = [L]_{|\vec{a},\vec{b}}.[\vec{u}]_{|\vec{a}}}. \tag{A.48}$$

Shortened notation: $[L.\vec{u}] = [L].[\vec{u}]$ when the bases are implicit.

Proposition A.41 *A linear map $L \in \mathcal{L}(E;F)$ is known as soon as the n vectors $L.\vec{a}_j$ are known, $j \in [1,n]_{\mathbb{N}}$. And the linear maps $\mathcal{L}_{ij} \in \mathcal{L}(E;F)$ defined by $\mathcal{L}_{ij}.\vec{a}_\ell = \delta_{j\ell}\vec{b}_i$ (all the elements of the matrix $[\mathcal{L}_{ij}]_{|\vec{a},\vec{b}}$ vanish except the element at the intersection of row i and column j which is equal to 1), for $i,\ell = 1,...,n$ and $j = 1,...,m$, constitute a basis $\in \mathcal{L}(E;F)$. (Duality notations: $\mathcal{L}_{ij} =^{noted} \mathcal{L}_i{}^j$, and $\mathcal{L}_i{}^j.\vec{a}_\ell = \delta^j_\ell\vec{b}_i$.) So, $\dim(\mathcal{L}(E;F)) = nm$.*

Proof. $\vec{u} \in E$ and $\vec{u} = \sum_k u_j\vec{a}_j$ give $L.\vec{u} = \sum_j u_jL.\vec{a}_j$, since L is linear. Thus L is known iff the n vectors $L.\vec{a}_j$ are known for all $j = 1,...,n$; And $L.\vec{a}_k = \sum_i L_{ik}\vec{b}_i$ together with $\sum_{ij} L_{ij}\mathcal{L}_{ij}.\vec{a}_k = \sum_{ij} L_{ij}\delta_{jk}\vec{b}_i = \sum_i L_{ik}\vec{b}_i$, for all k, thus $L = \sum_{ij} L_{ij}\mathcal{L}_{ij}$, thus the $\mathcal{L}_{ij}$ span $\mathcal{L}(E;F)$. And $\sum_{i=1}^m\sum_{j=1}^n\lambda_{ij}\mathcal{L}_{ij} = 0$ implies, for all ℓ, $\vec{0} = \sum_{i=1}^m\sum_{j=1}^n\lambda_{ij}\mathcal{L}_{ij}.\vec{a}_\ell = \sum_{i=1}^m\sum_{j=1}^n\lambda_{ij}\delta_{j\ell}\vec{b}_i = \sum_{i=1}^m\lambda_{i\ell}\vec{b}_i$, thus $\lambda_{i\ell} = 0$, for all i and ℓ. Thus the $\mathcal{L}_{ij}$ are independent. Thus $(\mathcal{L}_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$ is a basis in $\mathcal{L}(E;F)$. ∎

Exercice A.42 If $L \in \mathcal{L}(E;E)$ (endomorphism), if $(\vec{a}_i)$ is a basis in E, prove:

$$\text{if } \lambda \in \mathbb{R}^* \text{ and } \vec{b}_i = \lambda \vec{a}_i \ \forall i \in [1,n]_{\mathbb{N}}, \quad \text{then} \quad [L]_{|\vec{b}} = [L]_{|\vec{a}}, \tag{A.49}$$

i.e., a change of unit has not influence on the matrix of an endomorphism. Check with the change of basis formulas. **NB:** To compare with (A.43): Covariance and contravariance should ***not*** be confused.

Answer. Let $L.\vec{a}_j = \sum_{i=1}^n L_{aij}\vec{a}_i$ and $L.\vec{b}_j = \sum_{i=1}^n L_{bij}\vec{b}_i$. Then $\sum_{i=1}^n L_{bij}\vec{b}_i = L.\vec{b}_j = L.(\lambda\vec{a}_j) = \lambda L.\vec{a}_j = \lambda\sum_{i=1}^n L_{aij}\vec{a}_i = \lambda\sum_{i=1}^n L_{aij}\frac{\vec{b}_i}{\lambda} = \sum_{i=1}^n L_{aij}\vec{b}_i$, thus $L_{bij} = L_{aij}$.

Change of basis formula: $[L]_{|\vec{b}} = P^{-1}.[L]_{|\vec{a}}.P$ with $P = \lambda I$ here. ∎

A.8.3 Trace of an endomorphism

Let E be a vector space, $\dim E = n$. Let $\vec{u} \in E$ and $\ell \in E^*$ and call $\mathcal{L}_{\vec{w},\ell} \in \mathcal{L}(E;E)$ the endomorphism, called an elementary endomorphism, defined by

$$\mathcal{L}_{\vec{w},\ell}.\vec{u} := \vec{w}(\ell.\vec{u}) = (\ell.\vec{u})\vec{w}. \tag{A.50}$$

Definition A.43 The trace of the endomorphism $L_{\vec{w},\ell}$ is the real

$$\mathrm{Tr}(\mathcal{L}_{\vec{w},\ell}) := \ell.\vec{w}. \tag{A.51}$$

And the trace operator is the linear map $\mathrm{Tr} : \left\{\begin{matrix} \mathcal{L}(E;E) & \to \mathbb{R} \\ L & \to \mathrm{Tr}(L) \end{matrix}\right\}$ defined on elementary endomorphisms $\ell \otimes \vec{w}$ by (A.51).

Proposition A.44 *Let $L \in \mathcal{L}(E;E)$. The real $\mathrm{Tr}(L)$ is objective (is intrinsic to L), i.e. is independent of any basis in E. And (quantification) if $(\vec{e}_i)$ is a basis and $L.\vec{e}_j = \sum_{i=1}^n L_{ij}\vec{e}_i$ for all j, then*

$$\mathrm{Tr}(L) = \sum_{i=1}^n L_{ii} \ (\in \mathbb{R}), \tag{A.52}$$

i.e., $\mathrm{Tr}(L)$ is the trace of the matrix $[L]_{|\vec{e}}$. (Duality notations $L.\vec{e}_j = \sum_{i=1}^n L^i{}_j\vec{e}_i$ and $\mathrm{Tr}(L) = \sum_{i=1}^n L^i{}_i$.)

Proof. $\mathrm{Tr}(\mathcal{L}_{\vec{w},\ell}) := \ell.\vec{w}$ is a real that can be considered by any observer, and which value is the same for all observers, cf. (A.29): It is objective.

Let $L \in \mathcal{L}(E;E)$. Let $(\vec{a}_i)$ be a basis and (π_{ai}) be its (covariant) dual basis, and $L.\vec{a}_j = \sum_i L_{ij}\vec{a}_i$, i.e. $[L]_{[\vec{a}} = [L_{ij}]$. And we have $(\sum_{ik} L_{ik}\mathcal{L}_{\vec{a}_i,\pi_{ak}}).\vec{a}_j = \sum_{ik} L_{ik}\mathcal{L}_{\vec{a}_i,\pi_{ak}}.\vec{a}_j =^{(A.50)} \sum_{ik} L_{ik}\vec{a}_i(\pi_{ak}.\vec{a}_j) = \sum_{ik} L_{ik}\vec{a}_i\delta_{kj} = \sum_i L_{ij}\vec{a}_i$, thus $L = \sum_{ij} L_{ij}\mathcal{L}_{\vec{a}_i,\pi_{aj}}$ (sum of elementary endomorphisms), thus, Tr being linear $\mathrm{Tr}(L) = \sum_{ij} L_{ij}\mathrm{Tr}\mathcal{L}_{\vec{a}_i,\pi_{aj}} = \sum_{ij} L_{ij}\delta_{ji} = \sum_i L_{ii}$, thus (A.52). ∎

Exercice A.45 Check with the change of basis formula that $\mathrm{Tr}(L)$ is an invariant (the same value for all observers).

Answer. Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be two bases, $P = [P_{ij}]$ be the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$, $Q = P^{-1}$, $[L]_{[\vec{a}} = [(L_a)_{ij}]$, $[L]_{[\vec{b}} = [(L_b)_{ij}]$. We have $[L]_{[\vec{b}} =^{(A.104)} P^{-1}.[L]_{[\vec{a}}.P$, i.e. $(L_b)_{ij} = \sum_{k\ell} Q_{ik}(L_a)_{k\ell}P_{\ell j}$, thus $\sum_i (L_b)_{ii} = \sum_{ik\ell} Q_{ik}(L_a)_{k\ell}P_{\ell i} = \sum_{k\ell}(P.Q)_{\ell k}(L_a)_{k\ell} = \sum_{k\ell}\delta_{\ell k}(L_a)_{k\ell} = \sum_k (L_a)_{kk}$, qed. ∎

Alternative definition with one-one tensors: see § Q.6.

A.9 Transposed matrix

The definition can be found in any elementary books, e.g., Strang [18]: If $M = [M_{ij}]_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ is an $m * n$ matrix then its transposed is the $n * m$ matrix $M^T = [(M^T)_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$ defined by

$$(M^T)_{ij} := M_{ji} \tag{A.53}$$

(exchange rows and columns). E.g., $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ gives $M^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, and $(M^T)_{12}$=M_{21}=3.

And M is symmetric iff $M^T = M$ (this requires m=n).

And $M.N = [\sum_k M_{ik}N_{kj}]_{\substack{i \\ j}}$ gives $(M.N)^T = [\sum_k M_{jk}N_{ki}]_{\substack{i \\ j}} = [\sum_k (N^T)_{ik}(M^T)_{kj}]_{\substack{i \\ j}} = N^T.M^T$.

Exercice A.46 Prove: If M is an $n * n$ invertible matrix then M^T is invertible and $(M^T)^{-1} = (M^{-1})^T$ ($=^{\text{noted}}$ M^{-T}); And if moreover M is symmetric, then M^{-1} is symmetric.

Answer. $M.M^{-1} = I$ gives $(M^{-1})^T.M^T = I^T = I$, thus M^T is invertible with $(M^T)^{-1} = (M^{-1})^T$. Moreover if $M = M^T$ then $M^{-1} = (M^{-1})^T$. ∎

A.10 A transposed endomorphism: depends on a chosen inner dot product

Not to be confused with the transposed of a matrix, cf. (A.53). And not to be confused with the transposed of a bilinear form (observer independent), cf. (A.33); In particular, a transposed of a linear map depends on the observer who use it (depends on the choice of an inner dot product).

A.10.1 Definition (requires an inner dot product: Not objective)

Let E be a finite dimensional vector space equipped with an inner dot product $g(\cdot,\cdot) = (\cdot,\cdot)_g$.

Definition A.47 The transpose of an endomorphism $L \in \mathcal{L}(E;E)$ relative to $(\cdot,\cdot)_g$ is the endomorphism $L_g^T \in \mathcal{L}(E;E)$ defined by

$$\forall \vec{x},\vec{y} \in E, \quad (L_g^T.\vec{y},\vec{x})_g = (\vec{y},L.\vec{x})_g, \quad \text{i.e.} \quad (L_g^T.\vec{y}) \bullet_g \vec{x} = \vec{y} \bullet_g (L.\vec{x}). \tag{A.54}$$

(It depends on $(\cdot,\cdot)_g$, see (A.59).) If $(\cdot,\cdot)_g$ is an imposed Euclidean dot product (isometric framework) then $L_g^T =^{\text{noted}} L^T$, thus $(L^T.\vec{y},\vec{x})_g = (\vec{y},L.\vec{x})_g$, i.e. $(L^T.\vec{y}) \bullet \vec{x} = \vec{y} \bullet (L.\vec{x})$.

Exercice A.48 (Math exercise.) The existence and uniqueness of L_g^T is e.g. proved with a basis in E when E is finite dimensional, see next § A.10.2. More general proof: Prove: If $(E,(\cdot,\cdot)_g)$ is an infinite dimensional Hilbert space and if $L \in \mathcal{L}(E;E)$ is continuous, then L_g^T exists, is unique, and is continuous (apply the Riesz representation theorem F.1).

Answer. Let $\vec{y} \in E$, then let $\ell_{\vec{y}g} : \vec{x} \in E \to \ell_{\vec{y}g}(\vec{x}) := (\vec{y},L.\vec{x})_g \in \mathbb{R}$. $\ell_{\vec{y}g}$ is linear (trivial since L is linear and $(\cdot,\cdot)_g$ is bilinear) and continuous: $|\ell_{\vec{y}g}.\vec{x}| \le ||\vec{y}||_g ||L.\vec{x}||_g \le ||\vec{y}||_g ||L||\,||\vec{x}||_g$ gives $||\ell_{\vec{y}g}||_{E^*} \le ||L||\,||\vec{y}||_g < \infty$; Let $\vec{\ell}_{\vec{y}g} \in E$ be the $(\cdot,\cdot)_g$-Riesz representation of $\ell_{\vec{y}g} \in E^*$: $\ell_{\vec{y}g}.\vec{x} = (\vec{\ell}_{\vec{y}g},\vec{x})_g$ for all $\vec{x}$, with $||\vec{\ell}_{\vec{y}g}||_g = ||\ell_{\vec{y}g}||_{E^*}$; We have thus defined $L_g^T : \vec{y} \in E \to L_g^T(\vec{y}) := \vec{\ell}_{\vec{y}g} \in E$; with $(L_g^T(\vec{y}),\vec{x})_g = (\vec{\ell}_{\vec{y}g},\vec{x})_g = \ell_{\vec{y}g}.\vec{x} = (\vec{y},L.\vec{x})_g$, thus L_g^T is linear (since $(\cdot,\cdot)_g$ is bilinear) and continuous: $||L_g^T.\vec{y}||_g = ||\vec{\ell}_{\vec{y}g}||_g = ||\ell_{\vec{y}g}||_{E^*} \le ||L||\,||\vec{y}||_g$ gives $||L_g^T|| \le ||L||_{\mathcal{L}(E;E)} < \infty$. ∎

Remark A.49 Recall: The transposed β^T of a bilinear form β is objective, cf. (A.33): We don't need any tool like an inner dot product to define β^T. Not to be confused with: The transposed $L_g^T =^{\text{noted}} L^T$ of a linear map L is subjective: It depends on a choice of an inner dot products $(\cdot,\cdot)_g$ by an observer. In particular it is ***dangerous*** to represent a linear map in a basis with its "bilinear tensorial representation" when dealing with the transposed: $L \in \mathcal{L}(E;F)$ is naturally canonically represented by the bilinear form $\beta_L \in \mathcal{L}(F^*,E;\mathbb{R})$, and thus $(\beta_L)^T \in \mathcal{L}(E,F^*;\mathbb{R})$; And

$$L.\vec{a}_j = \sum_{i=1}^n L^i{}_j \vec{b}_i \;\text{ gives }\; \beta_L = \sum_{i,j=1}^n L^i{}_j \vec{b}_i \otimes a^j, \;\text{ thus }\; (\beta_L)^T \overset{(A.33)}{=} \sum_{i,j=1}^n L^j{}_i a^i \otimes \vec{b}_j, \tag{A.55}$$

while $L^T \in \mathcal{L}(F;E)$ is naturally canonically represented by the bilinear form $\beta_{(L^T)} \in \mathcal{L}(E^*,F;\mathbb{R})$, and

$$L^T.\vec{b}_j = \sum_{i=1}^n (L^T)^i{}_j \vec{a}_i \;\text{ gives }\; b_{(L^T)} = \sum_{i,j=1}^n (L^T)^i{}_j \vec{a}_i \otimes b^j, \;\text{ thus }\; \boxed{\beta_{(L^T)} \ne (\beta_L)^T} \tag{A.56}$$

for two reasons: 1- $\vec{a}_i \otimes b^j \ne a^i \otimes \vec{b}_j$, and 2- $L^T := L_{gh}^T$ depends on chosen inner dot products $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ by observers in E and F, see (A.73): $(L^T)^i{}_j =^{(A.73)} \sum_{k,\ell=1}^n ([g]^{-1})_{ik} L^\ell{}_k\, h_{\ell j}$; While $(\beta_L)^T$ is independent of any inner dot products. (In fact $(\beta_L)^T \in \mathcal{L}(F^*,E;\mathbb{R})$ is the tensorial representation of the adjoint $L^* \in \mathcal{L}(F^*;E^*)$ of L: With $L^*.b^j = \sum_{i=1}^n (L^*)_i{}^j a^i$ we get $\widetilde{(L^*)} = \sum_{i,j=1}^n (L^*)_i{}^j a^i \otimes \vec{b}_j = (\beta_L)^T$, see (A.83).)

So in continuum mechanics it is strongly advised ***not*** to use the tensorial notation for linear maps when dealing with transposed (e.g. when using F^T the transposed of the deformation gradient). ∎

A.10.2 Quantification with bases

Let $(\vec{e}_i)$ be a basis in E, let $g_{ij} := g(\vec{e}_i,\vec{e}_j)$, so $[g]_{|\vec{e}} := [g_{ij}] =^{\text{noted}} [g]$, and let (classical notation)

$$L.\vec{e}_j = \sum_{i=1}^n L_{ij}\vec{e}_i, \quad L_g^T.\vec{e}_j = \sum_{i=1}^n (L_g^T)_{ij}, \quad \text{i.e.} \quad [L]_{|\vec{e}} = [L_{ij}] \overset{\text{noted}}{=} [L], \quad [L_g^T]_{|\vec{e}} = [(L_g^T)_{ij}] \overset{\text{noted}}{=} [L_g^T]. \tag{A.57}$$

(A.54) gives $[\vec{x}]^T.[g].[L_g^T.\vec{y}] = [L.\vec{x}]^T.[g].[\vec{y}]$ for all $\vec{x},\vec{y}$, thus

$$[g].[L_g^T] = [L]^T.[g], \quad \text{i.e.} \quad \sum_{k=1}^{n} g_{ik}(L_g^T)_{kj} = \sum_{k=1}^{n} L_{ki}\, g_{kj} \tag{A.58}$$

i.e.,

$$\boxed{[L_g^T] = [g]^{-1}.[L]^T.[g]}, \quad \text{i.e.} \quad (L_g^T)_{ij} = \sum_{k,\ell=1}^{n} ([g]^{-1})_{ik} L_{\ell k} g_{\ell j}. \tag{A.59}$$

To compare with (A.56). If and only if $(\vec{e}_i)$ is $(\cdot,\cdot)_g$-orthonormal then $[g] = [\delta_{ij}]$ and $(L_g^T)_{ij} = L_{ji}$. With duality notations, $L.\vec{e}_j = \sum_{i=1}^n L^i{}_j \vec{e}_i$, $L_g^T.\vec{e}_j = \sum_{i=1}^n (L_g^T)^i{}_j$, $[L]_{|\vec{e}} = [L^i{}_j]$, $[L_g^T]_{|\vec{e}} = [(L_g^T)^i{}_j]$, and

$$\sum_{k=1}^{n} g_{ik}(L_g^T)^k{}_j = \sum_{k=1}^{n} L^k{}_i\, g_{kj}, \quad \text{i.e.} \quad (L_g^T)^i{}_j = \sum_{k,\ell=1}^{n} ([g]^{-1})_{ik} L^\ell{}_k\, g_{\ell j}. \tag{A.60}$$

Remark A.50 The last equation $(A.60)_2$ is also written

$$(L_g^T)^i{}_j = \sum_{k,\ell=1}^{n} g^{ik} L^\ell{}_k\, g_{\ell j} \quad \text{when} \quad ([g]_{\vec{e}})^{-1} = [g_{ij}]^{-1} \overset{\text{noted}}{=} [g^{ij}]. \tag{A.61}$$

Don't be fooled by the notation g^{ij}, defined by $[g^{ij}] := [g_{ij}]^{-1}$. (It is also the short notation for $(g^\sharp)^{ij}$, see (F.32).) Use classical notations to avoid misuses and misinterpretations. ⁂

Remark A.51 A bilinear form $\beta \in \mathcal{L}(E,E;\mathbb{R})$ satisfies $[\beta^T] = [\beta]^T$. A linear endomorphism $L \in \mathcal{L}(E;E)$ satisfies $[L_g^T] = [g]^{-1}.[L]^T.[g] \neq [L]^T$ in general (e.g. take $[L] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $[g] = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$). So do not confuse a bilinear on E (objective) with a linear endomorphism on E (subjective). ⁂

Exercice A.52 In $\vec{\mathbb{R}^2}$, let $(\vec{e}_1,\vec{e}_2)$ be a basis. Let $L \in \mathcal{L}(\vec{\mathbb{R}^2};\vec{\mathbb{R}^2})$ be defined by $[L]_{|\vec{e}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Find two inner dot products $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ in $\vec{\mathbb{R}^2}$ such that $L_g^T \neq L_h^T$ (a transposed endomorphism is not unique, is not intrinsic to L, since it depends on a choice of an inner dot product by an observer).

Answer. Calculations with (A.58):

Choose $(\cdot,\cdot)_g$ given by $[g]_{|\vec{e}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = [I]$. Thus $[L_g^T]_{|\vec{e}} = [I].[L]_{|\vec{e}}.[I] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; So $L_g^T = L$.

Choose $(\cdot,\cdot)_h$ given by $[h]_{|\vec{e}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Thus $[L_h^T]_{|\vec{e}} = [h]_{|\vec{e}}^{-1}.[L]_{|\vec{e}}.[h]_{|\vec{e}} = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}$; So $L_h^T \neq L$.

Thus $L_h^T \neq L_g^T$, e.g., $\vec{e}_2 = L_g^T.\vec{e}_1 \neq L_h^T.\vec{e}_1 = \frac{1}{2}\vec{e}_2$. ⁂

Exercice A.53 Prove: If L is invertible then L_g^T is invertible, and $(L_g^T)^{-1} = (L^{-1})_g^T$ (written L_g^{-T}).

Answer. Suppose: $\exists \vec{y} \in E$, $\vec{y} \neq \vec{0}$, s.t. $L_g^T.\vec{y} = 0$. L being invertible, $\exists! \vec{x} \in E$ s.t. $L.\vec{x} = \vec{y}$, with $\vec{x} \neq \vec{0}$ since $\vec{y} \neq \vec{0}$ and L is linear; And $L_g^T.\vec{y} = 0$ gives $L_g^T.L.\vec{x} = 0$, thus $(L_g^T.L.\vec{x},\vec{x})_g = 0$, thus $||L.\vec{x}||_g^2 = 0$, thus $L.\vec{x} = 0$, thus $\vec{x} = 0$ since L is linear bijective; Absurd. Thus $\text{Ker}(L_g^T) = \{\vec{0}\}$, thus L_g^T is invertible since it is an endomorphism. And $(L_g^T.(L^{-1})_g^T.\vec{x},\vec{y})_g \overset{(A.54)}{=} ((L^{-1})_g^T.\vec{x}, L.\vec{y})_g \overset{(A.54)}{=} (\vec{x},(L^{-1}).L.\vec{y})_g = (\vec{x},\vec{y})_g = (L_g^T.(L_g^T)^{-1}.\vec{x},\vec{y})_g$, true $\forall \vec{x},\vec{y}$, thus $L_g^T.(L^{-1})_g^T = L_g^T.(L_g^T)^{-1}$, thus $(L^{-1})_g^T = (L_g^T)^{-1}$ since L_g^T is invertible. ⁂

Exercice A.54 Special case of proportional inner dot products $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$: $\exists \lambda > 0$ s.t. $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$. Prove: $L_a^T = L_b^T$: Two proportional inner dot products give the same transposed endomorphism.

Answer. $(L_b^T.\vec{y},\vec{x})_b = (\vec{y},L.\vec{x})_b = \lambda^2(\vec{y},L.\vec{x})_a = \lambda^2(L_a^T.\vec{y},\vec{x})_a = (L_a^T.\vec{y},\vec{x})_b$, for all $\vec{x},\vec{y}$, so $L_b^T = L_a^T$. ⁂

A.10.3 Symmetric endomorphism

Definition A.55 An endomorphism $L \in \mathcal{L}(E;E)$ is $(\cdot,\cdot)_g$-symmetric iff $L_g^T = L$:

$$L\ (\cdot,\cdot)_g\text{-symmetric} \iff L_g^T = L \iff (L.\vec{x},\vec{y})_g = (\vec{x},L.\vec{y})_g, \quad \forall \vec{x},\vec{y} \in E. \tag{A.62}$$

Remark A.56 The symmetric character of an endomorphism L is not intrinsic to the endomorphism: It depends on $(\cdot,\cdot)_g$; See exercise A.52 where L is $(\cdot,\cdot)_g$-symmetric while it is ***not*** $(\cdot,\cdot)_h$-symmetric. ⁂

A.10.4 The general flat $^\flat$ notation for an endomorphism: Relative to a $(\cdot,\cdot)_g$

Let $(\cdot,\cdot)_g$ be an inner dot product in a vector space E, and let $L \in \mathcal{L}(E;E)$ (a C^0 endomorphism).

Definition A.57 The bilinear form $L^\flat_g \in \mathcal{L}(E,E;\mathbb{R})$ which is $(\cdot,\cdot)_g$-associated to the endomorphism $L \in \mathcal{L}(E;E)$ is defined by, for all $\vec{u},\vec{w} \in E$,

$$L^\flat_g(\vec{u},\vec{w}) := (\vec{u}, L.\vec{w})_g. \tag{A.63}$$

($L^\flat_g$ depends on a choice of a $(\cdot,\cdot)_g$.) We have thus defined the $(\cdot,\cdot)_g$-dependent operator:

$$(.)^\flat_g = \mathcal{J}_g(.) : \begin{cases} \mathcal{L}(E;E) & \to \mathcal{L}(E,E;\mathbb{R}) \\ \qquad L & \to \mathcal{J}_g(L) := L^\flat_g, \end{cases} \tag{A.64}$$

If $(\cdot,\cdot)_g$ is imposed, then $L^\flat_g =^{\text{noted}} L^\flat$.

(The bilinearity of $L^\flat_g$ is trivial since L is linear and $(\cdot,\cdot)_g$ is bilinear, and the bilinear form $L^\flat_g$ continuous as soon as L and $(\cdot,\cdot)_g$ are since $|L^\flat_g(\vec{u},\vec{v})| \le ||g||\,||L.\vec{u}||\,||\vec{v}|| \le (||g||\,||L||)\,||\vec{u}||\,||\vec{v}||$.)

Proposition A.58 *With the natural canonical isomorphism $L \in \mathcal{L}(E;E) \simeq T_L \in \mathcal{L}(E^*,E;\mathbb{R})$ given by $T_L(\ell,\vec{w}) = \ell.L.\vec{w}$, and with $T_L =^{noted} L$, the function $(.)^\flat_g$ is the change of contravariance to covariance mapping given by*

$$L^\flat_g = g.L. \tag{A.65}$$

Proof. Recall: The contraction of an elementary $\binom{0}{2}$ tensor $\ell_1 \otimes \ell_2$ with an elementary $\binom{1}{1}$ tensor $\vec{v}\otimes\ell_3$ is the $\binom{0}{2}$ tensor $(\ell_1\otimes\ell_2).(\vec{v}\otimes\ell_3) := (\ell_2.\vec{v})\ell_1\otimes\ell_3$. And the contraction on any tensors is the bilinear map defined on elementaty tensors. So, with a basis $(\vec{e}_i)$ in E and its dual basis (e^i) in E^*, if $g = \sum_{ij} g_{ij} e^i \otimes e^j$ and $L = \sum_{ij} L^i{}_j \vec{e}_i \otimes e^j$ then $g.L = \sum_{ijk} g_{ik} L^k{}_j \vec{e}_i \otimes e^j$. Thus $(g.L)(\vec{u},\vec{w}) = \sum_{ij} u^i w^j (g.L)(\vec{e}_i,\vec{e}_j) = \sum_{ijk} u^i w^j g_{ik} L^k{}_j = \sum_{ik} u^i g_{ik} (L.\vec{w})^k = \sum_{ik} u^i g_{ik} (L.\vec{w})^k = g(\vec{u}, L.\vec{w}) = L^\flat_g(\vec{u},\vec{w})$. ∴

Quantification: Let $(\vec{e}_i)$ be a basis in E, and, with duality notations motivated by the flat notation "i top changed into i bottom" in the components $L^i{}_j$ of L, let $g_{ij} := g(\vec{e}_i,\vec{e}_j)$, $L.\vec{e}_j = \sum_{i=1}^n L^i{}_j \vec{e}_i$ and $L^\flat_{g,ij} = L^\flat_g(\vec{e}_i,\vec{e}_j)$, i.e. with tensorial notations for calculations

$$g = \sum_{ij} g_{ij} e^i \otimes e^j, \quad L = \sum_{ij} L^i{}_j \vec{e}_i \otimes e^j, \quad L^\flat_g = \sum_{ij} L^\flat_{g,ij} e^i \otimes e^j. \tag{A.66}$$

So $[g]_{|\vec{e}} = [g_{ij}]$, $[L]_{|\vec{e}} = [L^i{}_j]$ and $[L^\flat_g]_{|\vec{e}} = [L^\flat_{g,ij}]$. Then (A.65) gives (or see next exercise)

$$\boxed{[L^\flat_g] = [g].[L]}. \tag{A.67}$$

Exercice A.59 Prove (A.67) with components.

Answer. With (A.63) we get $L^\flat_{g,ij} = L^\flat_g(\vec{e}_i,\vec{e}_j) = (\vec{e}_i, L.\vec{e}_j)_g = (\vec{e}_i, \sum_k L^k{}_j \vec{e}_k)_g = \sum_k L^k{}_j g_{ik} = ([g].[L])_{ij}$. ∴

Remark A.60 A change of variance, here from the $\binom{1}{1}$ type tensor L to the $\binom{0}{2}$ tensor $L^\flat_g$, is necessarily observer dependent: There is no natural canonical isomorphism between a vector space E and its dual E^*, see § T.2. Details: Here fix $\vec{w}$ and write $\ell_{g,\vec{w}}(\vec{u}) = (\vec{u}, L.\vec{w})_g$ ($= L^\flat_g(\vec{u},\vec{w})$); Thus $\ell_{g,\vec{w}} \in E^*$ is the $(\cdot,\cdot)_g$-representation function (linear form) of the vector $L.\vec{w}$, i.e. $\ell_{g,\vec{w}} = \vec{R}_g(L.\vec{w})$ where $\vec{R}_g$ is the $(\cdot,\cdot)_g$-Riesz-representation operator (the change of variance operator, see (F.3). ∴

A.11 A transposed of a linear map: depends on chosen inner dot products

This paragraph is needed to define the transposed of the deformation gradient.

Not to be confused with the transposed of a matrix, cf. (A.53). And not to be confused with the objective transposed of a bilinear form, cf. (A.33); E.g., a transposed of a linear map is ***not*** objective.

A.11.1 Definition (subjective)

$(E,(\cdot,\cdot)_g)$ and $(F,(\cdot,\cdot)_h)$ are Hilbert spaces, and $L\in\mathcal{L}(E;F)$ (which is supposed to be continuous if E and F are infinite dimensional). E.g., $E=\vec{\mathbb{R}}^n_{t_0}$, $F=\vec{\mathbb{R}}^n_t$, $L=d\Phi^{t_0}_t(P)\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ = the deformation gradient, cf. (4.1), $(\cdot,\cdot)_g$ is the foot built Euclidean dot product chosen by the observer who made the measurements at t_0, $(\cdot,\cdot)_h$ is the metre built Euclidean dot product chosen by the observer who makes the measurements at t.

Definition A.61 The transposed of $L\in\mathcal{L}(E;F)$ relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ is the linear map $L^T_{gh}\in\mathcal{L}(F;E)$ defined by, for all $(\vec{x},\vec{y})\in E\times F$,

$$(L^T_{gh}.\vec{y},\vec{x})_g=(\vec{y},L.\vec{x})_h, \tag{A.68}$$

where we used the dot notation $L^T_{gh}(\vec{y})=^{\text{noted}}L^T_{gh}.\vec{y}$ since L^T_{gh} is linear. This defines the map

$$(.)^T_{gh}:\begin{cases}\mathcal{L}(E;F) & \to\mathcal{L}(F;E)\\ \qquad L & \to(.)^T_{gh}(L):=L^T_{gh}\end{cases} \tag{A.69}$$

NB: So a linear map has an infinite number of transposed (it depends on inner dot products).
Notation: If $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ are imposed then $L^T_{gh}=^{\text{noted}}L^T$.
And if $F=E$ and $(\cdot,\cdot)_h=(\cdot,\cdot)_g$ then $L^T_{gh}=L^T_g$, see § A.10.

A.11.2 Quantification with bases

Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be bases in E and F, let $g_{ij}:=g(\vec{a}_i,\vec{a}_j)$, $h_{ij}:=h(\vec{b}_i,\vec{b}_j)$, $[g]_{|\vec{a}}=[g_{ij}]$, $[h]_{|\vec{b}}=[h_{ij}]$, and let (classical notation)

$$\begin{aligned}&L.\vec{a}_j=\sum_{i=1}^m L_{ij}\vec{b}_i,\quad\text{i.e.}\quad [L]_{|\vec{a},\vec{b}}=[L_{ij}]\overset{\text{noted}}{=}[L],\\ &L^T_{gh}.\vec{b}_j=\sum_{i=1}^n (L^T_{gh})_{ij}\vec{a}_i,\quad\text{i.e.}\quad [L^T_{gh}]_{|\vec{b},\vec{a}}=[(L^T_{gh})_{ij}]\overset{\text{noted}}{=}[L^T_{gh}].\end{aligned} \tag{A.70}$$

(A.68) gives $[\vec{x}]^T_{|\vec{a}}.[g]_{|\vec{a}}.[L^T_{gh}.\vec{y}]_{|\vec{y}}=([L.\vec{x}]_{|\vec{b}})^T.[h]_{|\vec{b}}.[\vec{y}]_{|\vec{b}}$ for all $\vec{x},\vec{y}$, thus, $[g]_{|\vec{a}}.[L^T_{gh}]_{|\vec{b},\vec{a}}=([L]_{|\vec{a},\vec{b}})^T.[h]_{|\vec{b}}$ and $[L^T_{gh}]_{|\vec{b},\vec{a}}=[g]^{-1}_{|\vec{a}}.([L]_{|\vec{a},\vec{b}})^T.[h]_{|\vec{b}}$. Shortened notation:

$$[g].[L^T]=[L]^T.[h],\quad\text{i.e.}\quad\sum_{k=1}^n g_{ik}(L^T_{gh})_{kj}=\sum_{k=1}^m L_{ki}\,h_{kj}, \tag{A.71}$$

i.e.

$$\boxed{[L^T]=[g]^{-1}.[L]^T.[h]},\quad\text{i.e.}\quad (L^T_{gh})_{ij}=\sum_{k=1}^n\sum_{\ell=1}^m([g]^{-1})_{ik}L_{\ell k}h_{\ell j}. \tag{A.72}$$

With duality notations, $L.\vec{e}_j=\sum_{i=1}^n L^i{}_j\vec{e}_i$, $[L]_{|\vec{e}}=[L^i{}_j]$, $L^T_{gh}.\vec{e}_j=\sum_{i=1}^n(L^T_{gh})^i{}_j$, $[L^T_{gh}]_{|\vec{e}}=[(L^T_{gh})^i{}_j]$, and

$$\sum_{k=1}^n g_{ik}(L^T_{gh})^k{}_j=\sum_{k=1}^n L^k{}_i\,h_{kj},\quad\text{i.e.}\quad (L^T_{gh})^i{}_j=\sum_{k,\ell=1}^n([g]^{-1})_{ik}L^\ell{}_k\,h_{\ell j}\quad(\overset{\text{noted}}{=}\sum_{k,\ell=1}^n(g^{ik}L^\ell{}_k\,h_{\ell j}). \tag{A.73}$$

(Be careful with the notation $([g]^{-1})_{ik}=^{\text{noted}}g^{ij}$, see remark A.50.)

Exercice A.62 Prove: If L is invertible then $(L^T_{gh})^{-1}=(L^{-1})^T_{hg}$.

Answer. $(L^T_{gh}.(L^{-1})^T_{hg}.\vec{x},\vec{y})_g=((L^{-1})^T_{hg}.\vec{x},L.\vec{y})_h=(\vec{x},L^{-1}.L.\vec{y})_g=(\vec{x},\vec{y})_g=(L^T_{gh}.(L^T_{gh})^{-1}.\vec{x},\vec{y})_g$, true $\forall\vec{x},\vec{y}$. ∎

A.11.3 Deformation gradient symmetric: Absurd

The symmetry of a linear map $L\in\mathcal{L}(E;F)$ is a nonsense if $E\neq F$.

E.g.: The gradient of deformation $F^{t_0}_t(p_{t_0})=d\Phi^{t_0}_t(p_{t_0})=^{\text{noted}}F\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ cannot be symmetric since $F^T\in\mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_{t_0})$. Idem for the first Piola–Kirchhoff tensor $\mathit{I\!K}^{t_0}_t$, which motivates the introduction of the symmetric second Piola–Kirchhoff tensor $\mathit{S\!K}^{t_0}_t$, see Marsden–Hughes [12] or § M.2.3.

A.11.4 Isometry

Definition A.63 A linear map $L \in \mathcal{L}(E;F)$ is an isometry relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ iff

$$\forall \vec{x},\vec{y} \in E, \quad (L.\vec{x},L.\vec{y})_h = (\vec{x},\vec{y})_g, \quad \text{i.e.} \quad L^T_{gh} \circ L = I_E \text{ (identity in } E). \tag{A.74}$$

In particular, an endomorphism $L \in \mathcal{L}(E;E)$ is a $(\cdot,\cdot)_g$-isometry iff

$$\forall \vec{x},\vec{y} \in E, \quad (L.\vec{x},L.\vec{y})_g = (\vec{x},\vec{y})_g, \quad \text{i.e.} \quad L^T_g \circ L = I_E. \tag{A.75}$$

Thus, if $L \in \mathcal{L}(E;F)$ is an isometry and $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis, then $(L.\vec{e}_i)$ is a $(\cdot,\cdot)_h$-orthonormal basis, since $(L.\vec{e}_i,L.\vec{e}_j)_h = (\vec{e}_i,\vec{e}_j)_g = \delta_{ij}$ for all i,j.

Exercice A.64 Let $\vec{f} : E \to F$. Prove:

$$\text{if } \vec{f} \text{ is an isometry then } \vec{f} \text{ is linear.} \tag{A.76}$$

Answer. Let $(\vec{e}_i)$ be a $(\cdot,\cdot)_g$-orthonormal basis; Thus $(\vec{f}(\vec{e}_i))$ is a $(\cdot,\cdot)_h$-orthonormal basis (since $\vec{f}$ is an isometry). Thus, if $\vec{x} = \sum_{i=1}^n x_i\vec{e}_i$ then $\vec{f}(\vec{x}) \overset{b.o.n.}{=} \sum\limits_{i=1}^n (\vec{f}(\vec{x}),\vec{f}(\vec{e}_i))_h \vec{f}(\vec{e}_i) \overset{hyp.}{=} \sum\limits_{i=1}^n (\vec{x},\vec{e}_i)_g \vec{f}(\vec{e}_i) \overset{b.o.n.}{=} \sum\limits_{i=1}^n x_i\vec{f}(\vec{e}_i)$, thus $\vec{f}(\vec{x}+\lambda\vec{y}) = \sum\limits_{i=1}^n (x_i+\lambda y_i)\vec{f}(\vec{e}_i) = \sum\limits_{i=1}^n x_i\vec{f}(\vec{e}_i) + \lambda\sum\limits_{i=1}^n y_i\vec{f}(\vec{e}_i) = \vec{f}(\vec{x}) + \lambda\vec{f}(\vec{y})$, thus $\vec{f}$ is linear. ∴

Exercice A.65 $\mathbb{R}^n$ is an affine space, $\vec{\mathbb{R}}^n$ is the usual associated vector space, and $(\cdot,\cdot)_g$ is an inner dot product in $\vec{\mathbb{R}}^n$. Definition: A distance-preserving function $f : p \in \mathbb{R}^n \to f(p) \in \mathbb{R}^n$ is a function s.t.

$$||\overrightarrow{f(p)f(q)}||_g = ||\overrightarrow{pq}||_g, \quad \forall p,q \in \mathbb{R}^n. \tag{A.77}$$

Prove: If f is a distance-preserving function, then f is affine.

Answer. Let $O \in \mathbb{R}^n$ (an origin) and $\vec{f} : \vec{x} = \overrightarrow{Op} \in \vec{\mathbb{R}}^n \to \vec{f}(\vec{x}) := \overrightarrow{f(O)f(p)}$ (vectorial associated function). Let $\vec{x} = \overrightarrow{Op}$ and $\vec{y} = \overrightarrow{Oq}$. Then the remarkable identity $2(\vec{f}(\vec{x}),\vec{f}(\vec{y}))_g = ||\vec{f}(\vec{x})||_g^2 + ||\vec{f}(\vec{y})||_g^2 - ||\vec{f}(\vec{x})-\vec{f}(\vec{y})||_g^2$ gives $2(\vec{f}(\vec{x}),\vec{f}(\vec{y}))_g = ||\vec{f}(\vec{x})||_g^2 + ||\vec{f}(\vec{y})||_g^2 - ||\overrightarrow{f(q)f(p)}||_g^2 = ||\vec{f}(\vec{x})||_g^2 + ||\vec{f}(\vec{y})||_g^2 - ||\overrightarrow{qp}||_g^2 = ||\vec{x}||_g^2 + ||\vec{y}||_g^2 - ||\vec{x}-\vec{y}||_g^2 = 2(\vec{x},\vec{y})_g$, thus $\vec{f}$ is an isometry, thus $\vec{f}$ is linear cf. (A.76), thus f is affine since $f(p) = f(O) + \vec{f}(\overrightarrow{Op})$. ∴

A.12 The adjoint of a linear map (objective)

(For mathematicians; May produce misunderstandings, misuses, problematic mechanical interpretations.)

No inner dot product is required here: A linear map L has only one adjoint L^* (intrinsic to L); While L has many transposed $L^T = L^T_{gh}$ which depend on inner dot products.

A.12.1 Definition

E and F are vector spaces, and $E^* = \mathcal{L}(E;\mathbb{R})$ and $F^* = \mathcal{L}(F;\mathbb{R})$ are the dual spaces (made of linear continuous forms). (If E and F are finite dimensional, the continuity is always satisfied.)

Definition A.66 Let $L \in \mathcal{L}(E;F)$ (linear and continuous); Its adjoint is the linear map $L^* \in \mathcal{L}(F^*;E^*)$ canonically defined by

$$L^* : \begin{cases} F^* & \to E^* \\ m & \to L^*(m) := m \circ L, \end{cases} \tag{A.78}$$

i.e., for all $(\vec{x},m) \in E \times F^*$,

$$(L^*(m))(\vec{x}) := m(L(\vec{x})). \tag{A.79}$$

(The adjoint L^* cannot be confused with a transposed L^T which requires inner dot products, cf. (A.68).)

The linearity of L^* is trivial, thus, together with the linearity of m and L, we can use the dot notation:

$$L^*.m := m.L, \quad \text{and} \quad (L^*.m).\vec{x} := m.L.\vec{x}. \tag{A.80}$$

And $||L^*.m||_{E^*} = ||m.L||_{E^*} \le ||m||_{F^*}||L||_{\mathcal{L}(E;F)}$ gives $||L^*||_{\mathcal{L}(F^*;E^*)} \le ||L||_{\mathcal{L}(E;F)} < \infty$, thus L^* is continuous (when L is).

A.12.2 Quantification

E and F are finite dimensional, $\dim E = n$, $\dim F = m$, and $(\vec a_i)$ and $(\vec b_i)$ are bases in E and F. Let $[L]_{|\vec a,\vec b} =^{\text{noted}} [L]$, $[L^*]_{|b,a} =^{\text{noted}} [L^*]$, $[m]_{|b} =^{\text{noted}} [m]$ and $[\vec x]_{|\vec a} =^{\text{noted}} [\vec x]$ be the matrices relative to the chosen bases: (A.80) gives $([L^*].[m].[\vec x] = [m].[L].[\vec x]$ for all $\vec x \in E$ and $m \in F^*$, thus, for all $m \in F^*$ (recall that $[m]$ is a line matrix), thus $[L^*].[m]^T = ([L]^T.[m]^T$, thus

$$\boxed{[L^*] = [L]^T} \quad \text{(transposed matrix).} \tag{A.81}$$

(Full notation: $[L^*]_{|b,a} = ([L]_{|\vec a,\vec b})^T$.)

Details: With the dual bases (π_{ai}) and (π_{bi}), with $L.\vec a_j = \sum_{i=1}^m L_{ij}\vec b_i$, i.e. $[L]_{|\vec a,\vec b} = [L_{ij}]_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$, and with $L^*.\pi_{bj} = \sum_{i=1}^n (L^*)_{ij}\pi_{ai}$, i.e. $[L^*]_{|b,a} = [(L^*)_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$, (A.80) gives, for all $(i,j) \in [1,n]_{\mathbb N} \times [1,m]_{\mathbb N}$,

$$(L^*.\pi_{bj}).\vec a_i = \pi_{bj}.(L.\vec a_i), \quad \text{thus} \quad \boxed{(L^*)_{ij} = L_{ji}} \quad \text{and} \quad [L^*] = [L]^T. \tag{A.82}$$

Duality notations (warning: can be misused): $L.\vec a_j = \sum_{i=1}^m L^i{}_j\vec b_i$, i.e. $[L]_{|\vec a,\vec b} = [L^i{}_j]_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$, and $L^*.b^j = \sum_{i=1}^n (L^*)_i{}^j a^i$, i.e. $[L^*]_{|b,a} = [((L^*)_i{}^j]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$, thus, for all $(i,j) \in [1,n]_{\mathbb N} \times [1,m]_{\mathbb N}$,

$$(L^*.b^j).\vec a_i = b^j.(L.\vec a_i), \quad \text{thus} \quad (L^*)_i{}^j = L^j{}_i \quad \text{and} \quad [L^*] = [L]^T. \tag{A.83}$$

(Recall: Use classical notations if in doubt, or, preferably, don't use duality notations here.)

Remark A.67 Reminder: The transposed b^T of a bilinear b form is intrinsic to b, and the adjoint L^* of a linear map L is intrinsic to L; But a transposed L^T of a linear form L is ***not*** intrinsic to the linear form (it depends on chosen inner dot products): Watch out for the (unfortunate) vocabulary "transpose"! ∴

A.12.3 Relation with the transposed when inner dot products are introduced

let $L \in \mathcal L(E;F)$. We need inner dot products $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ in E and F to define $L^T = L^T_{gh}$. To have a functional relation between L^* and L^T_{gh}, we use the $(\cdot,\cdot)_g$-Riesz representation mapping $\vec R_g : \left\{ \begin{array}{ll} E^* & \to E \\ \ell & \to \vec R_g(\ell) = \vec\ell_g \end{array} \right\}$, where $\ell.\vec x = (\vec\ell_g, \vec x)_g$ for all $\vec x \in E$, see (F.3); idem with F.

Let $L \in \mathcal L(E;F)$ (continuous). For all $\vec x \in E$ and all $m \in F^*$ we have

$$(L^*.m).\vec x \overset{(A.79)}{=} m.(L.\vec x), \quad \text{thus} \quad (\vec R_g(L^*.m), \vec x)_g = (\vec R_h(m), L.\vec x)_h, \tag{A.84}$$

thus $((\vec R_g \circ L^*).m), \vec x)_g = ((L^T_{gh} \circ \vec R_h).m, L.\vec x)_g$. Thus $\vec R_g \circ L^* = L^T_{gh} \circ \vec R_h$, i.e.

$$\boxed{L^T_{gh} = \vec R_g \circ L^* \circ (\vec R_h)^{-1}} \quad \text{i.e.} \quad \begin{tikzcd} E & F \arrow[l, "L^T_{gh}"'] \\ E^* \arrow[u, "\vec R_g"] & F^* \arrow[l, "L^*"] \arrow[u, "\vec R_h"'] \end{tikzcd} \quad \text{is a commutative diagram.} \tag{A.85}$$

Exercice A.68 From (A.85), recover (A.71), i.e. $[L^T_{gh}] = [g]^{-1}.[L]^T.[h]$.

Answer. $[L^T_{gh}] =^{(A.85)} [\vec R_g].[L^*].[\vec R_h]^{-1} =^{(F.6)} [g]^{-1}.[L]^T.[h]$. ∴

A.13 Tensorial representation of a linear map

A.13.1 A tensorial representation

Consider the natural canonical isomorphism (between linear maps $E \to F$ and bilinear forms $F^* \times E \to \mathbb R$)

$$\widetilde{\mathcal J} : \left\{ \begin{array}{rl} \mathcal L(E;F) & \to \mathcal L(F^*, E; \mathbb R) \\ L & \to \beta_L = \widetilde{\mathcal J}(L) \end{array} \right\} \quad \text{where} \quad \beta_L(m, \vec u) := m.(L.\vec u), \quad \forall (m, \vec u) \in F^* \times E, \tag{A.86}$$

see § T.4. And β_L is also named L for calculations purposes, see (A.89).

(NB: It can be dangerous to substitute L with β_L, see e.g. § A.7.2.)

Quantification: Let $(\vec{a}_i)_{i=1,...,n}$ be a basis in E, $(\vec{b}_i)_{i=1,...,m}$ be a basis in F which dual basis is (π_{bi}), $L \in \mathcal{L}(E;F)$. Then

$$\beta_L(\pi_{bi}, \vec{a}_i) = \pi_{bi}.L.\vec{a}_i. \tag{A.87}$$

Thus, if

$$L.\vec{a}_j = \sum_{i=1}^{m} L_{ij}\vec{b}_i \quad \text{then} \quad \beta_L = \sum_{i=1}^{m}\sum_{j=1}^{n} L_{ij}\vec{b}_i \otimes \pi_{aj} \tag{A.88}$$

Indeed, $(\sum_{ij} L_{ij}\vec{b}_i \otimes \pi_{aj})(\pi_{bk}, \vec{a}_\ell) = \sum_{ij} L_{ij}(\vec{b}_i \otimes \pi_{aj})(\pi_{bk}, \vec{a}_\ell) = \sum_{ij} L_{ij}(\vec{b}_i.\pi_{bk})(\pi_{aj}.\vec{a}_\ell) = \sum_{ij} L_{ij}(\vec{b}_i.\pi_{bk})(\pi_{aj}.\vec{a}_\ell) = \sum_{ij} L_{ij}\delta_{ki}\delta_{j\ell} = L_{k\ell} = \pi_{bk}.L.\vec{a}_\ell$, so (A.87) gives (A.88).

Duality notations: $L.\vec{a}_j = \sum_{i=1}^{m} L^i{}_j\vec{b}_i$ and $\beta_L = \sum_{i=1}^{m}\sum_{j=1}^{n} L^i{}_j\vec{b}_i \otimes a^j$.

Contraction rule. If you write $L = \sum_{i=1}^{m}\sum_{j=1}^{n} L_{ij}\vec{b}_i \otimes \pi_{aj}$ $(\simeq \beta_L)$, then the vector $L.\vec{u} \in F$ is computed thanks to the "contraction rule":

$$L.\vec{u} = (\sum_{i=1}^{m}\sum_{j=1}^{n} L_{ij}\vec{b}_i \otimes \underbrace{\pi_{aj}).\vec{u}}_{\text{contraction}} := \sum_{i=1}^{m}\sum_{j=1}^{n} L_{ij}\vec{b}_i(\pi_{aj}.\vec{u}) = \sum_{i=1}^{m}\sum_{j=1}^{n} L_{ij}u_j\vec{b}_i. \tag{A.89}$$

(With duality notations: $L.\vec{u} = (\sum_{i=1}^{m}\sum_{j=1}^{n} L^i{}_j\vec{b}_i \otimes \underbrace{a^j).\vec{u}}_{\text{contraction}} = \sum_{i=1}^{m}\sum_{j=1}^{n} L^i{}_j\vec{b}_i(a^j.\vec{u}) = \sum_{i=1}^{m}\sum_{j=1}^{n} L^i{}_j u^j\vec{b}_i.$)

Remark A.69 Warning: The bilinear form β_L should not be confused with the linear map L: The domain of definition of β_L is $F^* \times E$, and β_L acts on the two objects ℓ (linear form) and $\vec{u}$ (vector) to get a ***scalar*** result; While the domain of definition of L is E, and L acts one object $\vec{u}$ to get a ***vector*** result. However, you can use the tensorial notation for L... only to calculate $L.\vec{u}$ with (A.89). ⁂

A.13.2 Warning: Confusion between transposed and adjoint

The transposed $L^T \in \mathcal{L}(F;E)$ of a linear map $L \in \mathcal{L}(E;F)$ needs inner dot products to be defined, cf (A.68): It is ***not*** intrinsic to L, ***not*** objective ; While the transposed $b^T \in \mathcal{L}(B,A;\mathbb{R})$ of a bilinear form $b \in \mathcal{L}(A,B;\mathbb{R})$ is intrinsic to L (it does not need inner dot products to be defined).

So if you represent a linear map $L \in \mathcal{L}(E;F)$ by its tensorial representation $\beta_L \in \mathcal{L}(F^*,E;\mathbb{R})$, cf. (A.88), then

1- you know the transposed $(\beta_L)^T$ (given by $(\beta_L)^T(\vec{w},\vec{u}) = \beta_L(\vec{u},\vec{w})$),

2- but you ***cannot*** deduce the transposed $L^T \in \mathcal{L}(F;E)$ from $(\beta_L)^T$ (i.e., to start with $(\beta_L)^T$ is misleading): You need to choose inner dot products, and then use the formula $(L^T.\vec{y},\vec{x})_g = (\vec{y},L.\vec{x})_h$ where $L^T := L^T_{gh}$ to get $[L^T] = [g]^{-1}.[L]^T.[h]$ (and $[L^T] \neq [L]^T$ in general).

3- In particular: If $L \in \mathcal{L}(E;E)$ is symmetric (relative to the chosen inner dot products), then $\beta_L \in \mathcal{L}(E^*,E;\mathbb{R})$ is never symmetric because $E^* \neq E$! (Recall: there is no natural canonical isomorphism between E and E^*.)

A.14 Change of basis formulas for bilinear forms and linear maps

A.14.1 Notations for transitions matrices for bilinear forms and linear maps

Let A and B be finite dimension vector spaces, $\dim A = n$, $\dim B = m$. (E.g. application to the change of basis formula for the deformation gradient $A{=}\vec{\mathbb{R}}^n_{t_0} \to B{=}\vec{\mathbb{R}}^n_t$.)

Let $(\vec{a}_{old,i})$ and $(\vec{a}_{new,i})$ be two bases in A, and $(\vec{b}_{old,i})$ and $(\vec{b}_{new,i})$ be two bases in B. Let $\mathcal{P}_A$ and $\mathcal{P}_B$ be the change of basis endomorphisms from old to new bases, and $P_A := [\mathcal{P}_A]_{|\vec{a}_{old}} = [P_{Aij}]$ and $P_B := [\mathcal{P}_B]_{|\vec{b}_{old}} = [P_{Bij}]$ be the associated transition matrices, and $Q_A = P_A{}^{-1}$ and $Q_B = P_B{}^{-1}$:

$$\begin{aligned} \vec{a}_{new,j} &= \mathcal{P}_A.\vec{a}_{old,i} = \sum_{i,j=1}^{n} P_{Aij}\vec{a}_{old,i}, \quad \pi_{anew,j} = \sum_{i=1}^{n} Q_{Aij}\pi_{aold,i}, \\ \vec{b}_{new,j} &= \mathcal{P}_B.\vec{b}_{old,i} = \sum_{i,j=1}^{m} P_{Bij}\vec{b}_{old,i}, \quad \pi_{bnew,j} = \sum_{i,j=1}^{n} Q_{Bij}\pi_{bold,i}. \end{aligned} \tag{A.90}$$

Duality notations: $\vec{a}_{new,j} = \sum_{i=1}^{n} P_A{}^i{}_j\vec{a}_{old,i}$ and $a^i_{new} = \sum_{j=1}^{n} Q_A{}^i{}_j a^j_{old}$ and $\vec{b}_{new,j} = \sum_{i=1}^{n} P_B{}^i{}_j\vec{b}_{old,i}$ and $b^i_{new} = \sum_{j=1}^{n} Q_B{}^i{}_j b^j_{old}$.

A.14.2 Change of coordinate system for bilinear forms $\in \mathcal{L}(A,B;\mathbb{R})$

Let $g \in \mathcal{L}(A,B;\mathbb{R})$, and, for all $(i,j) \in [1,n]_{\mathbb{N}} \times [1,m]_{\mathbb{N}}$,

$$g(\vec{a}_{old,i},\vec{b}_{old,j}) = M_{ij}, \quad g(\vec{a}_{new,i},\vec{b}_{new,j}) = N_{ij}, \quad \text{i.e.} \quad \begin{cases} [g]_{|olds} = M = [M_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}, \\ [g]_{|news} = N = [N_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}. \end{cases} \tag{A.91}$$

Proposition A.70 *Change of basis formula:*

$$[g]_{|news} = {P_A}^T.[g]_{|olds}.P_B, \quad \text{i.e.} \quad N = {P_A}^T.M.P_B. \tag{A.92}$$

In particular, if $A = B$ and $(\vec{a}_{old,i}) = (\vec{b}_{old,i})$ and $(\vec{a}_{new,i}) = (\vec{b}_{new,i})$, then $P_A = P_B =^{noted} P$, and

$$\boxed{[g]_{|new} = P^T.[g]_{old}.P}, \quad \text{i.e.} \quad N = P^T.M.P. \tag{A.93}$$

Proof. $N_{ij} = g(\vec{a}_{new,i},\vec{b}_{new,j}) = \sum_{k\ell} {P_A}^k{}_i {P_B}^\ell{}_j g(\vec{a}_{old,k},\vec{b}_{old,\ell}) = \sum_{k\ell} {P_A}^k{}_i M_{k\ell} {P_B}^\ell{}_j = \sum_{k\ell} ({P_A}^T)^i{}_k M_{k\ell} {P_B}^\ell{}_j$. ∴

Exercice A.71 Prove (objective result):

$$g(\vec{u},\vec{w}) = [\vec{u}]^T_{|\vec{a}_{new}}.[g]_{|news}.[\vec{w}]_{|\vec{b}_{new}} = [\vec{u}]^T_{|\vec{a}_{old}}.[g]_{|olds}.[\vec{w}]_{|\vec{b}_{old}}. \tag{A.94}$$

Answer. $[\vec{u}]^T_{|\vec{a}_{new}}.[g]_{|news}.[\vec{w}]_{|\vec{b}_{new}} = ({P_A}^{-1}.[\vec{u}]_{|\vec{a}_{old}})^T.({P_A}^T.[g]_{|olds}.P_B).({P_B}^{-1}.[\vec{w}]_{|\vec{b}_{old}})$. ∴

A.14.3 Change of coordinate system for bilinear forms $\in \mathcal{L}(A^*,B^*;\mathbb{R})$

Let $z \in \mathcal{L}(A^*,B^*;\mathbb{R})$, and, for all $(i,j) \in [1,n]_{\mathbb{N}} \times [1,m]_{\mathbb{N}}$,

$$z(a^i_{old},b^j_{old}) = M^{ij}, \quad z(a^i_{new},b^j_{new}) = N^{ij}, \quad \text{i.e.} \quad \begin{cases} [z]_{|olds} = M = [M^{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}, \\ [z]_{|news} = N = [N^{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}. \end{cases} \tag{A.95}$$

Proposition A.72 *Change of basis formula:*

$$[z]_{|news} = {P_A}^{-T}.[z]_{|olds}.{P_B}^{-1}, \quad \text{i.e.} \quad N = {P_A}^{-T}.M.{P_B}^{-1}. \tag{A.96}$$

In particular, if $A = B$ and $(\vec{a}_{old,i}) = (\vec{b}_{old,i})$ and $(\vec{a}_{new,i}) = (\vec{b}_{new,i})$, then $P_A = P_B =^{noted} P$, and

$$\boxed{[z]_{|new} = P^{-T}.[z]_{old}.P^{-1}}, \quad \text{i.e.} \quad N = P^{-T}.M.P^{-1}. \tag{A.97}$$

Proof. $N_{ij} = z(a^i_{new},b^j_{new}) = \sum_{k\ell} {Q_A}^k{}_i {Q_B}^\ell_j z(a^k_{old},b^\ell_{old}) = \sum_{k\ell} {Q_A}^k{}_i M^{k\ell} {Q_B}^\ell{}_j = \sum_{k\ell} ({Q_A}^T)^i{}_k M^{k\ell} {Q_B}^\ell{}_j$. ∴

A.14.4 Change of coordinate system for bilinear forms $\in \mathcal{L}(B^*,A;\mathbb{R})$

(Toward linear maps $L \in \mathcal{L}(A;B) \simeq \mathcal{L}(B^*,A;\mathbb{R})$ thanks to the natural canonical isomorphism.)

Let $T \in \mathcal{L}(B^*,A;\mathbb{R})$, and, for all $(i,j) \in [1,n]_{\mathbb{N}} \times [1,m]_{\mathbb{N}}$,

$$T(b^i_{old},\vec{a}_{old,j}) = M^i{}_j, \quad T(b^i_{new},\vec{a}_{new,j}) = N^i{}_j, \quad \text{i.e.} \quad \begin{cases} [T]_{|olds} = M = [M^i{}_j]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}, \\ [T]_{|news} = N = [N^i{}_j]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}. \end{cases} \tag{A.98}$$

Proposition A.73 *Change of basis formula:*

$$[T]_{|news} = {P_B}^{-1}.[T]_{|olds}.P_A, \quad \text{i.e.} \quad N = Q_A.M.P_B. \tag{A.99}$$

In particular, if $A = B$ and $(\vec{a}_{old,i}) = (\vec{b}_{old,i})$ and $(\vec{a}_{new,i}) = (\vec{b}_{new,i})$, then $P_A = P_B =^{noted} P$, and

$$\boxed{[T]_{|new} = P^{-1}.[T]_{old}.P}, \quad \text{i.e.} \quad N = P^{-1}.M.P, \quad \text{i.e.} \quad N^i{}_j = \sum_{k,\ell=1}^{n} Q^i{}_k M^k{}_\ell P^\ell{}_j. \tag{A.100}$$

Proof. $N^i{}_j = T(b^i_{new},\vec{a}_{new,j}) = \sum_{k\ell} {Q_B}^i{}_k {P_A}^\ell{}_j T(b^i_{old},\vec{a}_{old,j}) = \sum_{k\ell} {Q_B}^i{}_k M^i{}_j {P_A}^\ell{}_j$ ∴

A.14.5 Change of coordinate system for tri-linear forms $\in \mathcal{L}(A^*, A, A; \mathbb{R})$

(Toward $d^2\vec{u}$: For a vector field $\vec{u} \in \Gamma(U) \simeq T_0^1(U)$, $\vec{u}(p) \in \vec{\mathbb{R}}^n$, its differential satisfies $d\vec{u}(p) \in \mathcal{L}(\vec{\mathbb{R}}^n; \vec{\mathbb{R}}^n) \simeq \mathcal{L}(\mathbb{R}^{n*}, \vec{\mathbb{R}}^n; \mathbb{R})$, and $d^2\vec{u}(p) \in \mathcal{L}(\vec{\mathbb{R}}^n; \mathcal{L}(\vec{\mathbb{R}}^n; \vec{\mathbb{R}}^n)) \simeq \mathcal{L}(\mathbb{R}^{n*}, \vec{\mathbb{R}}^n, \vec{\mathbb{R}}^n; \mathbb{R})$, see § S.1.3.)

Consider a tri-linear form $T \in \mathcal{L}(A^*, A, A; \mathbb{R})$, and

$$M^i_{jk} = T(a^i_{old}, \vec{a}_{old,j}, \vec{a}_{old,k}), \quad N^i_{jk} = T(a^i_{new}, \vec{a}_{new,j}, \vec{a}_{new,k}), \quad \text{i.e.} \quad [T]_{|\vec{a}_{old}} = [M^i_{jk}], \quad [T]_{|\vec{a}_{new}} = [N^i_{jk}]. \tag{A.101}$$

Then

$$N^i_{jk} = \sum_{\lambda,\mu,\nu=1}^{n} Q^i_\lambda P^\mu_j P^\nu_k M^\lambda_{\mu\nu}. \tag{A.102}$$

Indeed $\sum_{\lambda\mu\nu} M^\lambda_{\mu\nu} \vec{a}_{old,\lambda} \otimes a^\mu_{old} \otimes a^\nu_{old} = \sum_{\lambda\mu\nu ijk} M^\lambda_{\mu\nu} Q^i_\lambda P^\mu_j P^\nu_k \vec{a}_{new,i} \otimes a^j_{new} \otimes a^k_{new}$. ∴

A.14.6 Change of coordinate system for linear maps $\in \mathcal{L}(A; B)$

Notation of § A.14.1. Let $L \in \mathcal{L}(A; B)$ be a linear map, and let, for all $j = 1, ..., n$,

$$\begin{cases} L.\vec{a}_{old,j} = \sum_{i=1}^{m} M_{ij} \vec{b}_{old,i} = \sum_{i=1}^{m} {M^i}_j \vec{b}_{old,i} \quad \text{i.e.} \quad [L]_{|olds} = M = [M_{ij}] = [{M^i}_j]_{\substack{i=1,\dots,m \\ j=1,\dots,n}}, \\ L.\vec{a}_{new,j} = \sum_{i=1}^{m} N_{ij} \vec{b}_{new,i} = \sum_{i=1}^{m} {N^i}_j \vec{b}_{new,i} \quad \text{i.e.} \quad [L]_{|news} = N = [N_{ij}] = [{N^i}_j]_{\substack{i=1,\dots,m \\ j=1,\dots,n}}, \end{cases} \tag{A.103}$$

with classical and duality notations.

Proposition A.74 *Change of bases formula:*

$$[L]_{|news} = {P_B}^{-1}.[L]_{|olds}.P_A, \quad \text{i.e.} \quad N = {P_B}^{-1}.M.P_A. \tag{A.104}$$

In particular, if $A = B$, if $(\vec{a}_{old,i}) = (\vec{b}_{old,i})$, $(\vec{a}_{new,i}) = (\vec{b}_{new,i})$, then $P_A = P_B =^{noted} P$ and

$$\boxed{[L]_{|new} = P^{-1}.[L]_{|old}.P}, \quad \text{i.e.} \quad N = P^{-1}.M.P, \quad \text{i.e.} \quad N_{ij} = \sum_{k,\ell=1}^{n} Q_{ik} M_{k\ell} P_{\ell j}, \tag{A.105}$$

with $Q = P^{-1}$, and with duality notations ${N^i}_j = \sum_{k\ell} {Q^i}_k {M^k}_\ell {P^\ell}_j$.

Proof. $L.\vec{a}_{new,j} = \sum_i {N^i}_j \vec{b}_{new,i} = \sum_{ik} {N^i}_j {P_B}^k{}_i \vec{b}_{old,k} = \sum_k (P_B.N)^k{}_j \vec{b}_{old,k}$ and $L.\vec{a}_{new,j} = L.(\sum_i {P_A}^i{}_j \vec{a}_{old,i}) = \sum_i {P_A}^i{}_j \sum_k {M^k}_i \vec{b}_{old,k} = \sum_k (M.P_A)^k{}_j \vec{b}_{old,k}$, for all j, thus $P_B.N = M.P_A$. ∴

Exercice A.75 Prove:

$$\ell.L.\vec{u} = [\ell]_{|\vec{b}_{new}}.[L]_{|news}.[\vec{u}]_{|\vec{a}_{new}} = [\ell]_{|\vec{b}_{old}}.[L]_{|olds}.[\vec{u}]_{|\vec{a}_{old}} \quad \text{(objective result).} \tag{A.106}$$

Answer. $[\ell]_{|\vec{b}_{new}}.[L]_{|news}.[\vec{u}]_{|\vec{a}_{new}} = ([\ell]_{|\vec{b}_{old}}.P_B).({P_B}^{-1}.[L]_{|olds}.P_A).({P_A}^{-1}.[\vec{u}]_{|\vec{a}_{old}})$. ∴

Remark A.76 Bilinear forms in $\mathcal{L}(A, A; \mathbb{R})$ and endomorphisms in $\mathcal{L}(A; A)$ behave differently: The formulas (A.93) and (A.105) should not be confused since $P^{-1} \neq P^T$ in general. E.g., if an English observer uses a Euclidean (old) basis $(\vec{a}_i) = (\vec{a}_{old,i})$ in foot, if a French observer uses a Euclidean (new) basis $(\vec{b}_i) = (\vec{a}_{new,i})$ in metre, and if (simple case) $\vec{b}_i = \lambda \vec{a}_i$ for all i (change of unit), then

$$[L]_{|new} = [L]_{|old}, \quad \text{while} \quad [g]_{|new} = \underbrace{\lambda^2}_{>10} [g]_{|old}. \tag{A.107}$$

Quite different results! I.e. $P^{-1}.[L]_{|old}.P \neq P^T.[L]_{|old}.P$ for a general change of basis. See the Mars Climate Orbiter crash, remark A.14, where someone forgot that 1 foot $\neq$ 1 metre. ∴

B Euclidean Frameworks

Time and space are decoupled (classical mechanics). $\mathbb{R}^n$ is the geometric affine space, $n = 1, 2, 3$, and $\vec{\mathbb{R}}^n$ is the associated vector space made of "bi-point vectors".

B.1 Euclidean basis

Manufacturing of a Euclidean basis.

An observer chooses a unit of measure (foot, metre, a unit of length used by Euclid, the diameter a of pipe...) and makes a "unit rod" of length 1 in this unit.

Postulate: The length of the rod does not depend on its direction in space.

• Space dimension $n=1$: This rod models a vector $\vec{e}_1$ which makes a basis $(\vec{e}_1)$ called the Euclidean basis relative to the chosen unit of measure.

• Space dimension $n \geq 2$:

- The observers makes three rods of length 3, 4 and 5, and makes a triangle (A,B,C) with A, B and C are the vertices and A not on the side on length 5.
- Pythagoras: $3^2+4^2=5^2$ gives: The triangle (A,B,C) is said to have a right angle at A.
- Two vectors $\vec{u}$ and $\vec{w}$ in $\vec{\mathbb{R}}^n$ are orthogonal iff the triangle (A,B,C) can be positioned such that $\vec{AB}$ and $\vec{AC}$ are parallel to $\vec{u}$ and $\vec{w}$.
- A basis $(\vec{e}_i)_{i=1,...,n}$ is Euclidean relative to the chosen unit of measurement iff the $\vec{e}_i$ are two to two orthogonal and their length is 1 (relative to the chosen unit).

Example B.1 An English observer defines a Euclidean basis $(\vec{a}_i)$ using the foot. A French observer defines a Euclidean basis $(\vec{b}_i)$ using the metre. We have

$$1\,\text{foot} = \mu\,\text{metre}, \quad \mu = 0.3048, \quad \text{and} \quad 1\,\text{metre} = \lambda\,\text{foot}, \quad \lambda = \frac{1}{\mu} \simeq 3.28. \tag{B.1}$$

($\mu = 0,3048$ is the official length in metre for the English foot.) E.g., the bases are "aligned" iff, for all i,

$$\vec{b}_i = \lambda \vec{a}_i \quad \text{(change of measurement unit)}, \tag{B.2}$$

thus the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$ is $P=\lambda I$, thus $P^T=P$, $P^{-1}=\frac{1}{\lambda}I$ and $P^T.P=\lambda^2 I$. ∴

Remark B.2 The bases used in practice are not all Euclidean. See example A.13, especially if you fly. ∴

B.2 Euclidean dot product

Definition B.3 An observer who has built) his Euclidean basis $(\vec{e}_i)$, cf. § B.1. The associated Euclidean dot product is the bilinear form $g(\cdot,\cdot) = (\cdot,\cdot)_g \in \mathcal{L}(\vec{\mathbb{R}}^n, \vec{\mathbb{R}}^n; \mathbb{R})$ defined by

$$(g_{ij} =) \quad g(\vec{e}_i,\vec{e}_j) = \delta_{ij}, \quad \forall i,j, \quad \text{i.e.} \quad [g]_{|\vec{e}} = [\delta_{ij}] = I. \tag{B.3}$$

In other words,

$$(\cdot,\cdot)_g := \sum_{i=1}^n \pi_{ei} \otimes \pi_{ei} = \sum_{i=1}^n e^i \otimes e^i, \tag{B.4}$$

with classical and duality notations, $(\pi_{ei}) = (e^i)$ being the dual basis of $(\vec{e}_i)$. And if you want to use the Einstein convention you have to write $(\cdot,\cdot)_g := \sum_{i,j=1}^n \delta_{ij} e^i \otimes e^j$: You cannot avoid writing $\delta_{ij} = g_{ij}$.

Thus, for all $\vec{x},\vec{y} \in \vec{\mathbb{R}}^n$, with $\vec{x} = \sum_{i=1}^n x_i \vec{e}_i$ and $\vec{y} = \sum_{i=1}^n y_i \vec{e}_i$ (classical notations),

$$(\vec{x},\vec{y})_g = \sum_{i=1}^n x_i y_i = [\vec{x}]_{|\vec{e}}^T.[\vec{y}]_{|\vec{e}}. \tag{B.5}$$

With duality notations, $\vec{x} = \sum_{i=1}^n x^i \vec{e}_i$, $\vec{y} = \sum_{i=1}^n y^i \vec{e}_i$ and $(\vec{x},\vec{y})_g = \sum_{i=1}^n x^i y^i$; And if you want to use the Einstein convention then write $(\vec{x},\vec{y})_g := \sum_{i,j=1}^n \delta_{ij} x^i y^j$: You cannot avoid writing δ_{ij}.

Definition B.4 The associated norm is $||.||_g := \sqrt{(\cdot,\cdot)_g}$, and the length of a vector $\vec{x}$ relative to the chosen Euclidean unit of measurement is $||\vec{x}||_g := \sqrt{(\vec{x},\vec{x})_g}$.

Thus with the Euclidean basis $(\vec{e}_i)$ (used to build $(\cdot,\cdot)_g$), if $\vec{x} = \sum_{i=1}^n x_i \vec{e}_i$, then $||\vec{x}||_g = \sqrt{\sum_{i=1}^n x_i^2}$ is the length of $\vec{x}$ relative to the chosen Euclidean unit of measure (Pythagoras). (With duality notations $||\vec{x}||_g = \sqrt{\sum_{i=1}^n (x^i)^2}$, and if you want to use the Einstein convention: $||\vec{x}||_g = \sqrt{\sum_{i,j=1}^n \delta_{ij} x^i x^j}$.)

Definition B.5 The angle $\theta(\vec{x},\vec{y})$ between two vectors $\vec{x},\vec{y} \in \vec{\mathbb{R}}^n - \{\vec{0}\}$ is defined by

$$\cos(\theta(\vec{x},\vec{y})) = (\frac{\vec{x}}{||\vec{x}||_g}, \frac{\vec{y}}{||\vec{y}||_g})_g. \tag{B.6}$$

(With a calculator, this formula gives $\theta(\vec{x},\vec{y}) = \arccos((\frac{\vec{x}}{||\vec{x}||_g}, \frac{\vec{y}}{||\vec{y}||_g})_g)$ a value in $[0,\pi]$.)

B.3 Change of Euclidean basis

Let $(\vec{a}_i)$ (e.g. English observer basis built with the foot) and $(\vec{b}_i)$ (e.g. French observer basis built with the metre) be Euclidean bases in $\vec{\mathbb{R}}^n$, and let $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ be the associated Euclidean dot products.

B.3.1 Two Euclidean dot products are proportional

Proposition B.6 *If $\lambda = ||\vec{b}_1||_g$, then $||\vec{b}_i||_g = \lambda$ for all $i = 1, ..., n$ (change of unit) and*

$$(\cdot,\cdot)_g = \lambda^2 (\cdot,\cdot)_h, \quad \textit{and} \quad ||.||_g = \lambda ||.||_h. \tag{B.7}$$

Proof. By definition of a Euclidean basis, the length of the rod that enabled to define $(\vec{b}_i)$ is independent of i, cf. § B.1, thus $||\vec{b}_i||_g = ||\vec{b}_1||_g$ for all i, and here $||\vec{b}_i||_g =^{\text{noted}} \lambda$. Thus $||\vec{b}_i||_g^2 = \lambda^2 = \lambda^2 ||\vec{b}_i||_h^2$ for all i, since $||\vec{b}_i||_h^2 = 1$. And if $i \neq j$ then $(\vec{b}_i, \vec{b}_j)_g = 0 = (\vec{b}_i, \vec{b}_j)_h$ since $\vec{b}_i$ and $\vec{b}_j$ form a right angle (Pythagoras), cf. (B.4). Hence $(\vec{b}_i, \vec{b}_j)_g = \lambda^2 (\vec{b}_i, \vec{b}_j)_h$ for all i, j, thus (B.7). ∎

Example B.7 Continuation of example B.1: $(\cdot,\cdot)_a = \sum_{i=1}^n a^i \otimes a^i$ is the English Euclidean dot product (foot), and $(\cdot,\cdot)_b = \sum_{i=1}^n b^i \otimes b^i$ is the French Euclidean dot product (metre). (B.7) and (B.1) give:

$$(\cdot,\cdot)_a = \lambda^2 (\cdot,\cdot)_b \quad \text{and} \quad ||.||_a = \lambda ||.||_b, \quad \text{with} \quad \lambda \simeq 3.28 \quad \text{and} \quad \lambda^2 \simeq 10.76. \tag{B.8}$$

In particular, if $\vec{w}$ is s.t. $||\vec{w}||_b = 1$ (its length is 1 metre), then $||\vec{w}||_a = \lambda$ (its length is $\lambda \simeq 3.28$ foot). ∎

B.3.2 Counterexample : non existence of a Euclidean dot product

1- Thermodynamic: Let T be the temperature and P the pressure, and consider the Cartesian vector space $\{(T,P)\} = \{(\text{temperature,pressure})\} = \mathbb{R} \times \mathbb{R}$. There is no associated Euclidean dot product: An associated norm would give $||(T,P)|| = \sqrt{T^2 + P^2} \in \mathbb{R}$ which is meaningless (incompatible dimensions). See § A.3.5.

2- Polar coordinate system $\vec{q} = (r, \theta) \in \mathbb{R} \times \mathbb{R}$: There is no Euclidean norm $\sqrt{r^2 + \theta^2}$ for $\vec{q}$ that is physically meaningful (incompatible dimensions), see example 6.11.

B.4 Euclidean transposed of the deformation gradient

Let $n \in \{1,2,3\}$ and consider a linear map $L \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ (e.g., $L = F^{t_0}_t(P)$).

Let $(\cdot,\cdot)_G$ be a Euclidean dot product in $\vec{\mathbb{R}}^n_{t_0}$ (used in the past by someone), and let $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ be Euclidean dot products in $\vec{\mathbb{R}}^n_t$ (the actual space where the results are obtained by two observers, e.g., $(\cdot,\cdot)_g$ built with a foot and $(\cdot,\cdot)_h$ built with a metre). Let L^T_{Gg} and L^T_{Gh} be the transposed of L relative to the dot products, that is, L^T_{Gg} and L^T_{Gh} in $\mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0})$ are characterized by, for all $(\vec{X}, \vec{y}) \in \vec{\mathbb{R}}^n_{t_0} \times \vec{\mathbb{R}}^n_t$, cf. (A.68),

$$(L^T_{Gg}.\vec{y}, \vec{X})_G = (L.\vec{X}, \vec{y})_g \quad \text{and} \quad (L^T_{Gh}.\vec{y}, \vec{X})_G = (L.\vec{X}, \vec{y})_h. \tag{B.9}$$

Corollary B.8

$$\textit{if} \quad (\cdot,\cdot)_g = \lambda^2 (\cdot,\cdot)_h \quad \textit{then} \quad L^T_{Gg} = \lambda^2 L^T_{Gh}. \tag{B.10}$$

NB: Do not forget λ^2, cf. remark A.14 (Mars Climate Orbiter crash).

Proof. $(L^T_{Gg}.\vec{y}, \vec{X})_G \overset{(B.9)}{=} (L.\vec{X}, \vec{y})_g \overset{(B.10)_1}{=} \lambda^2 (L.\vec{X}, \vec{y})_h \overset{(B.9)}{=} \lambda^2 (L^T_{Gh}.\vec{y}, \vec{X})_G$ for all $\vec{X} \in \vec{\mathbb{R}}^n_{t_0}$ and all $\vec{y} \in \vec{\mathbb{R}}^n_t$, thus $L^T_{Gg}.\vec{y} = \lambda^2 L^T_{Gh}.\vec{y}$ for all $\vec{y} \in \vec{\mathbb{R}}^n_t$, thus $(B.10)_2$. ∎

B.5 The Euclidean transposed for endomorphisms

Let $n \in \{1,2,3\}$ and consider an endomorphism $L \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t)$ (e.g. $L = d\vec{v}_t(p) \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t)$ the differential of the Eulerian velocity). Let $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ be dot products in $\vec{\mathbb{R}}^n$. Let L^T_g and L^T_h be the transposed of L relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$, that is, L^T_g and L^T_h in $\mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t)$ are the endomorphisms defined by, for all $\vec{x}, \vec{y} \in \vec{\mathbb{R}}^n_t$, cf. (A.54),

$$(L^T_g.\vec{y}, \vec{x})_g = (L.\vec{x}, \vec{y})_g, \quad \text{and} \quad (L^T_h.\vec{y}, \vec{x})_h = (L.\vec{x}, \vec{y})_h. \tag{B.11}$$

Corollary B.9

$$\textit{if}\quad (\cdot,\cdot)_g = \lambda^2(\cdot,\cdot)_h \quad \textit{then}\quad L_g^T = L_h^T \overset{noted}{=} L^T \quad \in \mathcal{L}(\vec{\mathbb{R}}_t^n;\vec{\mathbb{R}}_t^n) \tag{B.12}$$

(an endomorphism type relation): Thus we can speak of "the Euclidean transposed of an endomorphism".

Proof. $(L_g^T.\vec{y},\vec{x})_g \overset{(B.11)}{=} (L.\vec{x},\vec{y})_g \overset{hyp}{=} \lambda^2(L.\vec{x},\vec{y})_h \overset{(B.11)}{=} \lambda^2(L_h^T.\vec{y},\vec{x})_h \overset{hyp}{=} (L_h^T.\vec{y},\vec{x})_g$ for all $\vec{x},\vec{y} \in \vec{\mathbb{R}}^n$, thus $L_g^T.\vec{y} = L_h^T.\vec{y}$ for all $\vec{y} \in \vec{\mathbb{R}}^n$. ∎

B.6 Unit normal vector, unit normal form

The results in this § are not objective: We need a Euclidean dot product (need a unit of length: Foot? Meter?) to get a unit (Euclidean) normal vector.

B.6.1 Framework

$(\cdot,\cdot)_g$ is a Euclidean dot product (needed to define Euclidean orthonormality) and, for all $\vec{u},\vec{w} \in \vec{\mathbb{R}}^n$,

$$(\vec{u},\vec{w})_g \overset{\text{noted}}{=} \vec{u} \bullet_g \vec{w} \tag{B.13}$$

(or $=^{\text{noted}}$ $\vec{u} \bullet \vec{w}$ when one chosen Euclidean dot product is imposed to all).

Ω is a regular open bounded set in $\mathbb{R}^n$, $n = 2$ or 3, and $\Gamma := \partial\Omega$ is its regular surface (dimension $n{-}1$). If $p \in \Gamma$ then $T_p\Gamma$ is the tangent plane at p to Γ, and a basis $(\vec{\beta}_1(p),...,\vec{\beta}_{n-1}(p))$ in $T_p\Gamma$ is known (usually obtained thanks to a coordinate system describing Γ). And, to lighten the writings, $(\vec{\beta}_1(p),...,\vec{\beta}_{n-1}(p))$ is written $(\vec{\beta}_1,...,\vec{\beta}_{n-1})$.

B.6.2 Unit normal vector

Call $\vec{n}_g(p)$ the unit outward normal vector at $p \in \Gamma$ at $T_p\Gamma$ relative to $(\cdot,\cdot)_g$; So $\vec{n}_g(p) \bullet_g \vec{\beta}_i(p) = 0$ for all $i = 1,...,n{-}1$, and $||\vec{n}_g(p)||_g = 1$, i.e. $\vec{n}_g$ is defined on Γ by (up to its sign)

$$\forall i = 1,...,n{-}1,\quad \vec{\beta}_i \bullet_g \vec{n}_g = 0, \quad\text{and}\quad \vec{n}_g \bullet_g \vec{n}_g = 1 \quad (= ||\vec{n}_g||_g^2), \tag{B.14}$$

i.e., at any $p \in \Gamma$, $\vec{n}_g(p)$ is orthogonal to the hyperplane $\text{Vect}\{\vec{\beta}_1(p),...,\vec{\beta}_{n-1}(p)\}$ and $\vec{n}_g(p)$ is unitary. So $(\vec{\beta}_1(p),...,\vec{\beta}_{n-1}(p),\vec{n}_g(p))$ is a basis at p in $\vec{\mathbb{R}}^n$, written in short $(\vec{\beta}_1,...,\vec{\beta}_{n-1},\vec{n}_g)$. Drawing.

Thus, for all $\vec{w} \in \vec{\mathbb{R}}^n$, if $\vec{w} = \sum_{i=1}^{n-1} w_i\vec{\beta}_i + w_n\vec{n}_g$ (classical notations) then

$$w_n = \vec{w} \bullet_g \vec{n}_g = \text{ the normal component of } \vec{w} \text{ at } p \text{ at } \Gamma. \tag{B.15}$$

(w_n depends on $(\cdot,\cdot)_g$.) (Duality notations: $\vec{w} = \sum_{i=1}^{n-1} w^i\vec{\beta}_i + w^n\vec{n}_g$ and $w^n = \vec{w} \bullet_g \vec{n}_g$.)

Exercice B.10 Let $(\vec{a}_i)$ be a basis in $\vec{\mathbb{R}}^n$, $\vec{\beta}_j = \sum_{i=1}^n B_{ij}\vec{a}_i$ for $j = 1,...,n{-}1$, and $\vec{n}_g = \sum_{i=1}^n n_i\vec{a}_i$, and $g_{ij} = g(\vec{a}_i,\vec{a}_j)$ for all i,j. What equations satisfy the n_j? And particular case $(\vec{a}_i)$ is $(\cdot,\cdot)_g$-orthonormal?

Answer. (B.14) gives $[\vec{\beta}_i]_{|\vec{a}}^T.[g]_{|\vec{a}}.[\vec{n}_g]_{|\vec{a}} = 0$ for $i = 1,...,n{-}1$ (so $n{-}1$ equations), with $[\vec{n}_g]_{|\vec{a}}^T.[g]_{|\vec{a}}.[\vec{n}_g]_{|\vec{a}} = 1$ (so 1 equation), and $\vec{n}_g$ is obtained up to its sign.

If $(\vec{a}_i)$ is $(\cdot,\cdot)_g$-orthonormal, then $\sum_{j=1}^n B_{ij}n_j = 0$ for $j = 1,...,n{-}1$, with $\sum_{i=1}^n n_i^2 = 1$. ∎

Exercice B.11 Let $(\vec{a}_i)$ be a Euclidean basis in foot, $(\vec{b}_i)$ a Euclidean basis in metre, $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ the associated Euclidean dot products, so $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$ with $\lambda \simeq 3.28$, cf. (B.7). Let $\vec{n}_a(p)$ and $\vec{n}_b(p)$ be the corresponding unit outward normal vectors, cf. (B.14). 1- Prove (up to the sign):

$$\vec{n}_b = \lambda\vec{n}_a, \quad\text{and}\quad (\vec{w},\vec{n}_a)_a = \lambda(\vec{w},\vec{n}_b)_b \quad \forall\vec{w} \in \vec{\mathbb{R}}^n \tag{B.16}$$

2- Then let $\vec{n}_a = \sum_{i=1}^m n_{ai}\vec{a}_i$ and $\vec{n}_b = \sum_{i=1}^m n_{bi}\vec{b}_i$; Prove:

$$\text{If, } \forall i = 1,...,n,\ \vec{b}_i = \lambda\vec{a}_i \quad\text{then}\quad \forall i = 1,...,n,\ n_{ai} = n_{bi}. \tag{B.17}$$

So the vectors $\vec{n}_a$ and $\vec{n}_b$ are different ($\lambda > 1$), and their respective components are equal... relative to different bases! And of course $1 = ||\vec{n}_a||_a^2 = \sum_{i=1}^n (n_{ai})^2 = \sum_{i=1}^n (n_{bi})^2 = ||\vec{n}_b||_b^2 = 1$.

Answer. $\vec{n}_a(p) \parallel \vec{n}_b(p)$, since the vectors are Euclidean and orthogonal to $T_p\Gamma$ cf. (B.14). And $||.||_a = \lambda||.||_b$ cf. (B.8), thus $||\vec{n}_b||_b = 1 = ||\vec{n}_a||_a = \lambda||\vec{n}_a||_b = ||\lambda\vec{n}_a||_b$, so $\vec{n}_b = \pm\lambda\vec{n}_a$. And they both are outward vectors, so $\vec{n}_b = +\lambda\vec{n}_a$. Thus $(\vec{w},\vec{n}_a)_a = \lambda^2(\vec{w},\vec{n}_a)_b = \lambda^2(\vec{w},\frac{\vec{n}_b}{\lambda})_b = \lambda(\vec{w},\vec{n}_b)_b$.

And if $\vec{b}_i = \lambda\vec{a}_i$ (B.16) gives $\sum_{i=1}^n n_b^i\vec{b}_i = \lambda\sum_{i=1}^n n_a^i\vec{a}_i = \sum_{i=1}^n n_a^i(\lambda\vec{a}_i) = \sum_{i=1}^n n_a^i\vec{b}_i$, then $n_a^i = n_b^i$. ∎

B.6.3 Unit normal form $n^\flat$ associated to $\vec{n}$

(For mathematicians; May produce misunderstandings and lack of mechanical interpretations; Don't forget: $n^\flat$ is obtained after $\vec{n}$ has been defined.)

At $p \in \Gamma$, once you have computed $\vec{n}_g(p)$, you can define the associated unit normal form $n_g^\flat(p) \in \mathbb{R}^{n*}$: It is the linear form defined by $n_g^\flat(p).\vec{w} := \vec{n}_g(p) \bullet_g \vec{w}$ for all $\vec{w} \in \vec{\mathbb{R}}^n$, i.e. on Γ, for all $\vec{w} \in \vec{\mathbb{R}}^n$,

$$n_g^\flat.\vec{w} := \vec{n}_g \bullet_g \vec{w} \tag{B.18}$$

($=^{\text{noted}}$ $\vec{n} \bullet \vec{w}$ if one chosen Euclidean dot product is imposed to all). Thus $[n_g^\flat].[\vec{w}] = [\vec{n}_g]^T.[g].[\vec{w}]$.

Quantification: Let $(\vec{e}_i)$ be a basis in $\vec{\mathbb{R}}^n$; Then (B.18) gives $[n_g^\flat]_{|\vec{e}}.[\vec{w}]_{|\vec{e}} = [\vec{n}_g]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}$ simply written $[n_g^\flat].[\vec{w}] = [\vec{n}_g]^T.[g].[\vec{w}]$ if the basis $(\vec{e}_i)$ is imposed.

So, with duality notations to justify the $^\flat$ notation, with (e^i) the dual basis of $(\vec{e}_i)$, let

$$\vec{n}_g = \sum_{i=1}^n n_g^i \vec{e}_i \quad \text{and} \quad n_g^\flat = \sum_{i=1}^n n_{gi} e^i. \tag{B.19}$$

i.e. n_g^i and n_{gi} are the components of $\vec{n}_g$ and $n_g^\flat$ relative to the basis $(\vec{e}_i)$ and (e^i). Since (B.18) gives $n_g^\flat.\vec{e}_i := \vec{n}_g \bullet_g \vec{e}_i$ for all i, we get, for all i,

$$n_{ig} = \sum_{j=1}^n g_{ij} n_g^j \tag{B.20}$$

Particular case $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-Euclidean basis, then $n_{ig} = n_g^i$.

Classical notations: $\vec{n}_g = \sum_{i=1}^n (\vec{n}_g)_i \vec{e}_i$, dual basis (π_{ei}), $n_g^\flat = \sum_{i=1}^n (n_g^\flat)_i \pi_{ei}$, $(n_g^\flat)_i = \sum_{j=1}^n g_{ij} (\vec{n}_g)_j$.

NB: In physics don't forget to write the g_{ij} in (B.20) even if $g_{ij} = \delta_{ij}$, since you need to see the chosen metric and basis (and verify the Einstein convention), although (B.20) is simply written $n_i = \sum_{j=1}^n g_{ij} n^j$...

B.7 Integration by parts (Green–Gauss–Ostrogradsky)

Let Ω be a regular bounded open set in $\mathbb{R}^n$ and $\Gamma = \partial\Omega$ its frontier, let $\varphi \in C^1(\overline{\Omega};\mathbb{R})$, let $(\vec{e}_i)$ be a Euclidean basis and $(\cdot,\cdot)_g$ ites associated Euclidean dot product, let $\frac{\partial\varphi}{\partial x_i}(p) := d\varphi(p).\vec{e}_i$ (usual notation), let $\vec{n}_g(p) = \vec{n}(p) = \sum_{i=1}^n n_i(p)\vec{e}_i$ (classical notations) be the unit outward normal at $p \in \Gamma$. Then, for $i = 1, ..., n$,

$$\int_{p\in\Omega} \frac{\partial\varphi}{\partial x_i}(p)\, d\Omega = \int_{p\in\Gamma} \varphi(p) n_i(p)\, d\Gamma, \quad \text{in short} \quad \int_\Omega \frac{\partial\varphi}{\partial x_i}\, d\Omega = \int_\Gamma \varphi n_i\, d\Gamma. \tag{B.21}$$

Thus, for any $v \in C^1(\overline{\Omega};\mathbb{R})$, with φv instead of φ in (B.21), we get the integration by parts formula (Green formula):

$$\int_\Omega \frac{\partial\varphi}{\partial x_i} v\, d\Omega = -\int_\Omega \varphi \frac{\partial v}{\partial x_i}\, d\Omega + \int_\Gamma \varphi v n_i\, d\Gamma. \tag{B.22}$$

Thus, for any $\vec{v} \in C^1(\overline{\Omega};\vec{\mathbb{R}}^n)$ (vector field), with $\vec{v}(p) = \sum_{i=1}^n v_i(p)\vec{e}_i$) we get

$$\int_\Omega \frac{\partial\varphi}{\partial x_i} v_i\, d\Omega = -\int_\Omega \varphi \frac{\partial v_i}{\partial x_i}\, d\Omega + \int_\Gamma \varphi v_i n_i\, d\Gamma. \tag{B.23}$$

Thus, with the gradient vector $\vec{\text{grad}}\varphi(p) = \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}\vec{e}_i$ and with $\text{div}\vec{v} = \sum_{i=1}^n \frac{\partial v_i}{\partial x_i}$, we get the Gauss–Ostrogradsky formula:

$$\int_\Omega \vec{\text{grad}}\varphi \bullet \vec{v}\, d\Omega = -\int_\Omega \varphi\, \text{div}\vec{v}\, d\Omega + \int_\Gamma \varphi\vec{v} \bullet \vec{n}\, d\Gamma. \tag{B.24}$$

(And $\int_\Gamma \varphi\vec{v} \bullet \vec{n}\, d\Gamma$ gives the flux through Γ.)

Exercice B.12 Use the differential $d\varphi$ instead of the gradient $\vec{\text{grad}}\varphi$ (which is the $(\cdot,\cdot)_g$-Riesz representation vector of $d\varphi$) to express (B.23). Is the use of $n^\flat$ useful in that case?

Answer. $\int_\Omega d\varphi.\vec{v}\, d\Omega = -\int_\Omega \varphi\, \text{div}\vec{v}\, d\Omega + \int_\Gamma \varphi\vec{v} \bullet \vec{n}\, d\Gamma$. Since $n^\flat$ depends on $\vec{n}$ (definition), there is no reason that justifies the use of $n^\flat$ (unless you want to introduce useless notations here). ∴

C Rate of deformation tensor and spin tensor

Let $\widetilde{\Phi} : [t_1, t_2] \times \mathit{Obj} \to \mathbb{R}^n$ be a regular motion, cf. (1.5), and let $\vec{v} : \mathcal{C} \to \vec{\mathbb{R}}^n$ be the Eulerian velocity field, cf. (2.4), that is, $\vec{v}(t,p) = \frac{\partial \widetilde{\Phi}}{\partial t}(t, P_{\mathit{Obj}})$ when $p = \widetilde{\Phi}(t, P_{\mathit{Obj}})$. Its differential $d\vec{v}$ is given in (2.8).

At t, an observer chooses a unit of measurement (foot, metre...) and builds the associated Euclidean dot product $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$, cf. § B.2. (We loose the objective point of view here). And the same $(\cdot,\cdot)_g$ is used at all t.

C.1 The symmetric and antisymmetric parts of $d\vec{v}$

With the imposed chosen Euclidean dot product $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$, we can consider the transposed endomorphism $d\vec{v}_t(p)^T_g =^{\text{noted}} d\vec{v}_t(p)^T \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t)$, which is defined by, for all $\vec{w}_1, \vec{w}_2 \in \vec{\mathbb{R}}^n_t$ vectors at p,

$$(d\vec{v}_t(p)^T.\vec{w}_1, \vec{w}_2)_g = (\vec{w}_1, d\vec{v}_t(p).\vec{w}_2)_g \tag{C.1}$$

cf. § A.11. We have thus defined

$$d\vec{v}_t^T : \begin{cases} \Omega_t & \to \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t) \\ p & \to d\vec{v}_t^T(p) := d\vec{v}_t(p)^T \end{cases} \tag{C.2}$$

Other usual notations (definitions): $d\vec{v}_t(p)^T =^{\text{noted}} d\vec{v}(t,p)^T =^{\text{noted}} d\vec{v}^T(t,p)$.

Definition C.1 The (Eulerian) rate of deformation tensor, or stretching tensor, is the $(\cdot,\cdot)_g$-symmetric part of $d\vec{v}$:

$$\mathcal{D} = \frac{d\vec{v} + d\vec{v}^T}{2}, \quad \text{i.e.,} \quad \forall (t,p) \in \bigcup_{t\in\mathbb{R}} (\{t\} \times \Omega_t), \quad \mathcal{D}(t,p) = \frac{d\vec{v}(t,p) + d\vec{v}(t,p)^T}{2}. \tag{C.3}$$

The (Eulerian) spin tensor is the $(\cdot,\cdot)_g$-antisymmetric part of $d\vec{v}$:

$$\Omega = \frac{d\vec{v} - d\vec{v}^T}{2}, \quad \text{i.e.,} \quad \forall (t,p) \in \bigcup_{t\in\mathbb{R}} (\{t\} \times \Omega_t), \quad \Omega(t,p) = \frac{d\vec{v}(t,p) - d\vec{v}(t,p)^T}{2}. \tag{C.4}$$

(So $d\vec{v} = \mathcal{D} + \Omega$ with $\mathcal{D}$ the rate of deformation tensor and $\Omega = \vec{\omega}\wedge$ a rotation times a dilation, see the following.)

NB: The same notation is used for the set of points $\Omega_t = \Phi_t^{t_0}(\Omega_{t_0}) \subset \mathbb{R}^n$ and for the spin tensor $\Omega_t = \frac{d\vec{v}_t - d\vec{v}_t^T}{2}$: The context removes ambiguities.

C.2 Quantification with a basis

With a basis $(\vec{e}_i)$ in $\vec{\mathbb{R}}^n_t$, (C.1) gives

$$[g]_{|\vec{e}}.[d\vec{v}^T]_{|\vec{e}} = [d\vec{v}]^T_{|\vec{e}}.[g]_{|\vec{e}}, \quad \text{and} \quad [d\vec{v}^T]_{|\vec{e}} = [g]^{-1}_{|\vec{e}}.[d\vec{v}]^T_{|\vec{e}}.[g]_{|\vec{e}}. \tag{C.5}$$

In particular, if $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis, then $[d\vec{v}^T]_{|\vec{e}} = [d\vec{v}]^T_{|\vec{e}}$ (orthonormal basis case). Thus for the endomorphisms $\mathcal{D}$ and Ω, and with the above Euclidean framework and its Euclidean orthonormal basis, we have $\mathcal{D}.\vec{e}_j = \sum_{i=1}^n \mathcal{D}_{ij}\vec{e}_i$ and $\Omega.\vec{e}_j = \sum_{i=1}^n \Omega_{ij}\vec{e}_i$ with $\mathcal{D}_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$ and $\Omega_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})$, that is,

$$[\mathcal{D}]_{|\vec{e}} = \frac{[d\vec{v}]_{|\vec{e}} + [d\vec{v}]^T_{|\vec{e}}}{2} \quad \text{and} \quad [\Omega]_{|\vec{e}} = \frac{[d\vec{v}]_{|\vec{e}} - [d\vec{v}]^T_{|\vec{e}}}{2} \quad \text{(Euclidean framework).} \tag{C.6}$$

Duality notations: $\mathcal{D}.\vec{e}_j = \sum_{i=1}^n \mathcal{D}^i_j \vec{e}_i$, $\mathcal{D}^i_j = \frac{1}{2}(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i})$ and $\Omega.\vec{e}_j = \sum_{i=1}^n \Omega^i{}_j \vec{e}_i$, $\Omega^i{}_j = \frac{1}{2}(\frac{\partial v^i}{\partial x^j} - \frac{\partial v^j}{\partial x^i})$, so with $\mathcal{D}^i_j = \mathcal{D}^j_i$ and $\Omega^i{}_j = -\Omega^j{}_i$.

D Interpretation of the rate of deformation tensor

We are interested in the evolution of the deformation gradient $F(t) := F^{t_0}_{p_{t_0}}(t)$ along the trajectory of a particle P_{Obj} which was at p_{t_0} at t_0. So:

Let $\vec{A} = \vec{a}(t_0, p_{t_0})$ and $\vec{B} = \vec{b}(t_0, p_{t_0})$ be vectors at t_0 at p_{t_0} in Ω_{t_0}, and consider their push-forwards by the flow $\Phi^{t_0}_t$ (the transported vectors), i.e. the vectors at t at $p(t) = \Phi^{t_0}_{p_{t_0}}(t)$ given by

$$\vec{a}(t, p(t)) := F(t).\vec{A} \quad \text{and} \quad \vec{b}(t, p(t)) := F(t).\vec{B}. \tag{D.1}$$

see (4.3) and figure 4.1. They define the function

$$(\vec{a}, \vec{b})_g : \begin{cases} \qquad \mathcal{C} & \to \mathbb{R} \\ (t, p_t) & \to (\vec{a}, \vec{b})_g(t, p_t) := (\vec{a}(t, p_t), \vec{b}(t, p_t))_g. \end{cases} \tag{D.2}$$

Proposition D.1 *The rate of deformation tensor* $\mathcal{D} = \frac{d\vec{v} + d\vec{v}^T}{2}$ *gives (half) the evolution rate between two vectors deformed by the flow, that is, along trajectories,*

$$\frac{D(\vec{a}, \vec{b})_g}{Dt} = 2(\mathcal{D}.\vec{a}, \vec{b})_g. \tag{D.3}$$

Proof. $f(t) := (\vec{a}(t, p(t)), \vec{b}(t, p(t)))_g = (F(t).\vec{A}, F(t).\vec{B})_g$ gives

$$f'(t) = (F'(t).\vec{A}, F(t).\vec{B})_g + (F(t).\vec{A}, F'(t).\vec{B})_g. \tag{D.4}$$

And $F'(t) = d\vec{v}(t, p(t)).F(t)$, cf. (3.33). Thus, with $\vec{a}(t, p(t)) = F(t).\vec{A}$ and $\vec{b}(t, p(t)) = F(t).\vec{B}$,

$$\begin{aligned} f'(t) &= (d\vec{v}(t, p(t)).F(t).\vec{A}, F(t).\vec{B})_g + (F(t).\vec{A}, d\vec{v}(t, p(t)).F(t).\vec{B})_g \\ &= (d\vec{v}(t, p(t)).\vec{a}(t, p(t)), \vec{b}(t, p(t)))_g + (\vec{a}(t, p(t)), d\vec{v}(t, p(t)).\vec{b}(t, p(t)))_g \\ &= ((d\vec{v}(t, p(t)) + d\vec{v}(t, p(t))^T).\vec{a}(t, p(t)), \vec{b}(t, p(t)))_g, \end{aligned} \tag{D.5}$$

i.e. (D.3), since $f(t) = (\vec{a}, \vec{b})_g(t, p(t))$ gives $f'(t) = \frac{D(\vec{a}, \vec{b})_g}{Dt}(t, p(t))$. ∴

E Rigid body motions and the spin tensor

Choose a Euclidean dot product $(\cdot, \cdot)_g$ (required to characterize a rigid body motion).

Result: A rigid body motion is a motion whose Eulerian velocity satisfies $d\vec{v} + d\vec{v}^T = 0$, i.e., $\mathcal{D} = 0$ (Eulerian approach independent of any initial time t_0 chosen by some observer).

But the usual classical introduction to rigid body motion relies on some initial time t_0 (Lagrangian approach). So, to begin with, let us do it with the Lagrangian approach. Recall: T the first order Taylor expansion of $\Phi^{t_0}_t$ in the vicinity of a $p_{t_0} \in \Omega_{t_0}$ is

$$\Phi^{t_0}_t(q_{t_0}) = \Phi^{t_0}_t(p_{t_0}) + F^{t_0}_t(p_{t_0}).\overrightarrow{p_{t_0}q_{t_0}} + o(\overrightarrow{p_{t_0}q_{t_0}}). \tag{E.1}$$

E.1 Affine motions and rigid body motions

E.1.1 Affine motions

Definition E.1 Φ^{t_0} is an affine motion (understood "affine motion in space") iff $\Phi^{t_0}_t$ is an "affine motion", i.e. iff $\Phi^{t_0}_t$ is a C^1 diffeomorphism (in space), and (E.1) reads, for all $p_{t_0}, q_{t_0} \in \Omega_{t_0}$ and all $t \in [t_1, t_2]$,

$$\Phi^{t_0}_t(q_{t_0}) = \Phi^{t_0}_t(p_{t_0}) + F^{t_0}_t(p_{t_0}).\overrightarrow{p_{t_0}q_{t_0}}. \tag{E.2}$$

Marsden–Hughes notations: $\Phi(Q) = \Phi(P) + F(P).\overrightarrow{PQ}$.

Proposition E.2 and definition. *If* Φ^{t_0} *is an affine motion, then* $F^{t_0}_t(p_{t_0})$ *is independent of* p_{t_0}*, i.e., for all* $t \in]t_1, t_2[$ *and all* $p_{t_0} \in \Omega_{t_0}$ *and all* $q_{t_0} \in \Omega_{t_0}$*,*

$$F^{t_0}_t(p_{t_0}) = F^{t_0}_t(q_{t_0}) \overset{noted}{=} F^{t_0}_t. \tag{E.3}$$

And then $dF^{t_0}_t(p_{t_0}) = 0$*, i.e.* $d^2\Phi^{t_0}_t(p_{t_0}) = 0$*. And for all* $t \in]t_1, t_2[$*,* Φ^t *is an affine motion: For all* $\tau \in]t_1, t_2[$ *and all* $p_t, q_t \in \Omega_t$*,*

$$\Phi^t_\tau(q_t) = \Phi^t_\tau(p_t) + F^t_\tau.\overrightarrow{p_tq_t}. \tag{E.4}$$

And $\widetilde{\Phi}$ *is said to be an affine motion (understood "affine motion in space").*

Proof. $q_{t_0} = p_{t_0} + \overrightarrow{p_{t_0}q_{t_0}}$ gives $\Phi_t^{t_0}(q_{t_0}) = \Phi_t^{t_0}(p_{t_0} + \overrightarrow{p_{t_0}q_{t_0}}) = \Phi_t^{t_0}(p_{t_0}) + d\Phi_t^{t_0}(p_{t_0}).\overrightarrow{p_{t_0}q_{t_0}}$, and, similarly, $\Phi_t^{t_0}(p_{t_0}) = \Phi_t^{t_0}(q_{t_0} + \overrightarrow{q_{t_0}p_{t_0}}) = \Phi_t^{t_0}(q_{t_0}) + d\Phi_t^{t_0}(q_{t_0}).\overrightarrow{q_{t_0}p_{t_0}}$. Thus (addition) $\Phi_t^{t_0}(q_{t_0}) + \Phi_t^{t_0}(p_{t_0}) = \Phi_t^{t_0}(p_{t_0}) + \Phi_t^{t_0}(q_{t_0}) + (d\Phi_t^{t_0}(p_{t_0}) - d\Phi_t^{t_0}(q_{t_0})).\overrightarrow{p_{t_0}q_{t_0}}$, thus $(d\Phi_t^{t_0}(p_{t_0}) - d\Phi_t^{t_0}(q_{t_0})).\overrightarrow{p_{t_0}q_{t_0}} = 0$, true for all p_{t_0}, q_{t_0}, thus $d\Phi_t^{t_0}(p_{t_0}) - d\Phi_t^{t_0}(q_{t_0}) = 0$, i.e. (E.3).

Thus $d^2\Phi_t^{t_0}(p_{t_0}).\vec{u}_{t_0} = \lim_{h\to0} \frac{d\Phi_t^{t_0}(p_{t_0}+h\vec{u}_{t_0}) - d\Phi_t^{t_0}(p_{t_0})}{h} = \lim_{h\to0} \frac{d\Phi_t^{t_0} - d\Phi_t^{t_0}}{h} = 0$ for all p_{t_0} and all $\vec{u}_{t_0}$, thus $d^2\Phi_t^{t_0}(p_{t_0}) = 0$ for all p_{t_0}, thus $d^2\Phi_t^{t_0} = 0$.

And (5.17) gives $(\Phi_\tau^t \circ \Phi_t^{t_0})(p_{t_0}) = \Phi_\tau^{t_0}(p_{t_0})$, thus, with $p_t = \Phi_t^{t_0}(p_{t_0})$, we get $d\Phi_\tau^t(p_t).d\Phi_t^{t_0}(p_{t_0}) = d\Phi_\tau^{t_0}(p_{t_0})$, thus $d\Phi_\tau^t(p_t) = d\Phi_\tau^{t_0}(p_{t_0}).d\Phi_t^{t_0}(p_{t_0})^{-1}$, and (E.2) gives

$$d\Phi_\tau^t(p_t) = d\Phi_\tau^{t_0}.{d\Phi_t^{t_0}}^{-1} \overset{\text{noted}}{=} d\Phi_\tau^t \quad \text{(independent of } p_t), \tag{E.5}$$

thus (E.4). ∴

Corollary E.3 *If $\widetilde{\Phi}$ is affine then, $\vec{v}_t$ is affine for all t, and $\vec{V}_t^{t_0}$ is affine for all t_0, t, i.e., for all $p_t \in \Omega_t$ we have $d\vec{v}_t(p_t) = d\vec{v}_t$ (independent of p_t), and for all $p_{t_0} \in \Omega_{t_0}$ we have $d\vec{V}_t^{t_0}(p_{t_0}) =^{noted} d\vec{V}_t^{t_0}$ (independent of p_{t_0}): For all $q_t \in \Omega_t$ and all $q_{t_0} \in \Omega_{t_0}$,*

$$\begin{cases} \bullet \;\; \vec{v}_t(q_t) = \vec{v}_t(p_t) + d\vec{v}_t.\overrightarrow{p_tq_t}, \\ \bullet \;\; \vec{V}_t^{t_0}(q_{t_0}) = \vec{V}_t^{t_0}(p_{t_0}) + d\vec{V}_t^{t_0}.\overrightarrow{p_{t_0}q_{t_0}}. \end{cases} \tag{E.6}$$

Proof. (E.2) gives $\Phi^{t_0}(t, q_{t_0}) = \Phi^{t_0}(t, p_{t_0}) + F^{t_0}(t).\overrightarrow{p_{t_0}q_{t_0}}$, and the derivation in time gives $(E.6)_2$, then $(E.6)_1$ thanks to $p_t = \Phi_t^{t_0}(p_{t_0})$, $q_t = \Phi_t^{t_0}(q_{t_0})$ and $\overrightarrow{p_{t_0}q_{t_0}} = (F_t^{t_0})^{-1}.\overrightarrow{p_tq_t}$, cf. (E.2). ∴

Example E.4 In $\mathbb{R}^2$, with a basis $(\vec{E}_1, \vec{E}_2)$ in $\vec{\mathbb{R}}^n_{t_0}$ and a basis $(\vec{e}_1, \vec{e}_2) \in \vec{\mathbb{R}}^n_t$, then $F_t^{t_0}$ given by $[F_t^{t_0}]_{|\vec{E},\vec{e}} = \begin{pmatrix} 1+t & 2t^2 \\ 3t^3 & e^t \end{pmatrix}$ derives from the affine motion $[\overrightarrow{\Phi_t^{t_0}(p_{t_0})\Phi_t^{t_0}(q_{t_0})}]_{|\vec{e}} = \begin{pmatrix} 1+t & 2t^2 \\ 3t^3 & e^t \end{pmatrix}.[\overrightarrow{p_{t_0}q_{t_0}}]_{|\vec{E}}$. ∴

E.1.2 Rigid body motion

A Euclidean dot product $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$ is chosen, the same at all time t. Let $\Phi := \Phi_t^{t_0}$ and $F := F_t^{t_0}$. Recall: If $P \in \Omega_{t_0}$ and $p = \Phi(P)$ $(\in \Omega_t)$ then the transposed of the linear map $F(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ relative to $(\cdot,\cdot)_g$ is the linear map $F^T(p) := F(P)^T \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0})$ defined by

$$F^T(p) := F(P)^T : \begin{cases} \vec{\mathbb{R}}^n_t \to \vec{\mathbb{R}}^n_{t_0} \\ \vec{w}_p \to F^T(p).\vec{w}_p \quad \text{s.t.} \quad (F^T(p).\vec{w}_p, \vec{U}_P)_g = (\vec{w}_p, F(P).\vec{U}_P)_g, \;\; \forall \vec{U}_P \in \vec{\mathbb{R}}^n_{t_0}. \end{cases} \tag{E.7}$$

We have thus defined the function $F^T : \Omega_t \to \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0})$.

Particular case: For an affine motion, since F is independent of P, we get F^T is independent of p.

Definition E.5 A rigid body motion is an affine motion $\widetilde{\Phi}$ such that, for all $t_0, t \in \mathbb{R}$, $P \in \Omega_{t_0}$, $\vec{U}_P, \vec{W}_P \in \vec{\mathbb{R}}^n_{t_0}$, and with $p = \Phi_t^{t_0}(P)$,

$$(F.\vec{U}_P, F.\vec{W}_P)_g = (\vec{U}_P, \vec{W}_P)_g, \quad \text{i.e.} \quad (F^T.F.\vec{U}_P, \vec{W}_P)_g = (\vec{U}_P, \vec{W}_P)_g, \quad \text{i.e.} \quad \boxed{F^T.F = I}. \tag{E.8}$$

(Angles and lengths are unchanged.)

In other words, with the Cauchy strain tensor $C \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ defined by $C = F^T.F$, the motion is rigid iff it is affine and

$$\boxed{C = I}, \quad \text{i.e.} \quad \boxed{F^{-1} = F^T}. \tag{E.9}$$

Proposition E.6 *If Φ^{t_0} is a rigid body motion, if $(\vec{A}_i)$ is a $(\cdot,\cdot)_g$-Euclidean basis in $\vec{\mathbb{R}}^n_{t_0}$, if $P \in \Omega_{t_0}$, if $t \in [t_0, T]$ and $p = \Phi_t^{t_0}(P)$, and if $\vec{a}_i(t,p) = F^{t_0}(t,P).\vec{A}_i$ for all i, then $\vec{a}_i(t,p) =^{noted} \vec{a}_{i,t}$ is independent of p, and $(\vec{a}_{i,t})$ is a $(\cdot,\cdot)_g$-Euclidean basis with the same orientation than $(\vec{A}_i)$ for all t.*

Proof. $\Phi_t^{t_0}$ is affine, thus, for all t, P, $F_t^{t_0}(P) = F_t^{t_0}$ (independent of P), thus $\vec{a}_{i,t}(p) = F_t^{t_0}.\vec{A}_i \in \vec{\mathbb{R}}_t^n$ is independent of p, this at all t. Let t be fixed and $\vec{a}_{i,t} =^{\text{noted}} \vec{a}_i$ $(= F.\vec{A}_j)$. We get $(\vec{a}_i, \vec{a}_j)_g = (F.\vec{A}_i, F.\vec{A}_j)_g = (F^T.F.\vec{A}_i, \vec{A}_j)_g = (I.\vec{A}_i, \vec{A}_j)_g = (\vec{A}_i, \vec{A}_j)_g = \delta_{ij}$ for all i, j, thus $(\vec{a}_i)$ is $(\cdot,\cdot)_g$-orthonormal basis. And $\det(\vec{a}_1, ..., \vec{a}_n) = \det(F.\vec{A}_1, ..., F.\vec{A}_n) = \det(F)\det(\vec{A}_1, ..., \vec{A}_n) = \det(F)$ since $(\vec{A}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis. And, $\Phi_t^{t_0}$ being a diffeomorphism, $t \to \det(F_t^{t_0})$ is continuous, does not vanish, moreover with $\det(F_{t_0}^{t_0}) = \det(I) = 1 > 0$; Thus $\det(F_t^{t_0}) > 0$ for all t, hence $\det(\vec{a}_1, ..., \vec{a}_n) > 0$: The bases have the same orientation. ∴

Example E.7 In $\mathbb{R}^2$, a rigid body motion is given by $F_t^{t_0} = \begin{pmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{pmatrix}$ with θ a regular function s.t. $\theta(t_0) = 0$. ∴

Exercice E.8 Let $\widetilde{\Phi}$ be a rigid body motion. Prove

$$(F^T)'(t) = (F'(t))^T, \quad \text{and} \quad F^T.F' \text{ is antisymmetric.} \tag{E.10}$$

Answer. Let $t \in \mathbb{R}$, $p(t) = \Phi_t^{t_0}(P)$, $\vec{U}, \vec{W} \in \vec{\mathbb{R}}_{t_0}^n$ and $\vec{w}(t, p(t)) = F(t).\vec{W}$. And recall that the function $F^T : t \to F^T(t)$ is defined (as usual) by $F^T(t) := (F(t))^T$. We have $(F(t)^T.\vec{w}(t, p(t)), \vec{U})_g = (\vec{w}(t, p(t)), F(t).\vec{U})_g$. Thus $((F^T)'(t).\vec{w}(t, p(t)) + F^T(t).\frac{D\vec{w}}{Dt}(t, p(t)), \vec{U})_g = (\frac{D\vec{w}}{Dt}(t, p(t)), F(t).\vec{U})_g + (\vec{w}(t, p(t)), F'(t).\vec{U})_g$, which simplifies into $((F^T)'(t).\vec{w}(t, p(t)), \vec{U})_g = (\vec{w}(t, p(t)), F'(t).\vec{U})_g = ((F'(t))^T.\vec{w}(t, p(t)), \vec{U})_g$, thus $(F^T)'(t) = (F'(t))^T$.

And (E.8) reads $F^T(t).F(t) = I_{t_0}$, thus $(F^T)'(t).F(t) + F^T(t).F'(t) = 0$, thus $(F')^T(t).F(t) + F^T(t).F'(t) = 0$, thus $F^T.F'$ is antisymmetric. ∴

E.1.3 Alternative definition of a rigid body motion: $d\vec{v} + d\vec{v}^T = 0$

The stretching tensor $\mathcal{D}_t = \frac{d\vec{v}_t + d\vec{v}_t^T}{2}$ and the spin tensor $\Omega_t = \frac{d\vec{v}_t - d\vec{v}_t^T}{2}$ have been defined in (C.3)-(C.4).

Proposition E.9 *If $\widetilde{\Phi}$ is a rigid body motion, cf. (E.8), then the endomorphism $d\vec{v}_t \in \mathcal{L}(\vec{\mathbb{R}}_t^n; \vec{\mathbb{R}}_t^n)$ is antisymmetric at all t:*

$$d\vec{v}_t = \Omega_t, \quad \text{i.e.} \quad \mathcal{D}_t = 0. \tag{E.11}$$

Conversely, if $d\vec{v}_t + d\vec{v}_t^T = 0$ at all t, then $\widetilde{\Phi}$ is a rigid body motion (here no initial time is required).

So the relation « $d\vec{v}_t + d\vec{v}_t^T = 0$ for all t » gives an equivalent definition to the definition E.5.

Proof. Let $F(t) := F_{p_{t_0}}^{t_0}(t)$ and $F^T(t) := F(t)^T$ and $V(t) := \vec{V}_{p_{t_0}}^{t_0}(t) = (\Phi_{p_{t_0}}^{t_0})'(t) = \vec{v}(t, p_t)$ (the Lagrangian and Eulerian velocities). (E.8) gives $(F.F^T)'(t) = 0 = F'(t).F(t)^T + F(t).(F^T)'(t) \overset{(E.10)}{=} F'(t).F(t)^T + (F'(t).F(t)^T)^T = dV(t).F(t)^{-1} + (dV(t).F(t)^{-1})^T \overset{(3.27)}{=} d\vec{v}(t, p_t) + d\vec{v}(t, p_t)^T$. Thus (E.11).

Conversely, suppose $d\vec{v} + d\vec{v}^T = 0$. Then (D.3) gives $\frac{D(\vec{a}, \vec{b})_g}{Dt} = 0$, thus $(\vec{a}, \vec{b})_g(t, p_t) = (\vec{a}, \vec{b})_g(t_0, p_{t_0})$ for all t, t_0 and all $p_{t_0} = \Phi_t^{t_0}(p_t)$, i.e. $(F_t^{t_0}(p_{t_0}).\vec{A}, F_t^{t_0}(p_{t_0}).\vec{B})_g = (\vec{A}, \vec{B})_g$ for all t, t_0, all p_{t_0} and all $\vec{A}, \vec{B} \in \vec{\mathbb{R}}_{t_0}^n$: Thus $\widetilde{\Phi}$ is a rigid body motion, cf (E.8). ∴

E.2 Representation of the spin tensor Ω: vectors, and pseudo-vectors

We are dealing here with concepts that are sometimes misunderstood. Framework: $\mathbb{R}^n = \mathbb{R}^3$.

E.2.1 Reminder

- The determinant $\det_{|\vec{e}}$ associated with a basis $(\vec{e}_i)$ in $\mathbb{R}^3$ is the alternating multilinear form defined by $\det_{|\vec{e}}(\vec{e}_1, \vec{e}_2, \vec{e}_3) = 1$; The algebraic volume (or signed volume) limited by three vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is $\det_{|\vec{e}}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$; And the (positive) volume is $|\det_{|\vec{e}}(\vec{u}_1, \vec{u}_2, \vec{u}_3)|$, see § K.
- Let A and B be two observers (e.g. A=English and B=French), let $(\vec{a}_i)$ be a Euclidean basis chosen by A (e.g. based on the foot), let $(\vec{b}_i)$ be a Euclidean basis chosen by B (e.g. based on the metre), see § B.1.

Let $\lambda = ||\vec{b}_1||_a > 0$ (change of unit of length coefficient). The relation between the determinants is:

$$\det_{|\vec{a}} = \pm\lambda^3 \det_{|\vec{b}} \quad\text{with}\quad \begin{cases} + \text{ if } \det_{|\vec{a}}(\vec{b}_1,\vec{b}_2,\vec{b}_3) > 0 & \text{(i.e. if the bases have the same orientation)},\\ - \text{ if } \det_{|\vec{a}}(\vec{b}_1,\vec{b}_2,\vec{b}_3) < 0 & \text{(i.e. if the bases have opposite orientation)}.\end{cases} \tag{E.12}$$

In particular, if A and B use the same unit of length (or if A uses two $(\cdot,\cdot)_g$-Euclidean basis $(\vec{a}_i)$ and $(\vec{b}_i)$), then $\lambda = 1$ and $\det_{|\vec{a}} = \pm\det_{|\vec{b}}$.

- With an imposed Euclidean dot product $(\cdot,\cdot)_g$: An endomorphism L is $(\cdot,\cdot)_g$-antisymmetric iff

$$\forall \vec{u},\vec{v},\ \ (L.\vec{u},\vec{v})_g + (\vec{u},L.\vec{v})_g = 0, \quad \text{i.e.} \quad L^T = -L. \tag{E.13}$$

E.2.2 Definition of the vector product (cross product)

Let $(\vec{e}_i)$ be a $(\cdot,\cdot)_g$-orthonormal basis, let $\vec{u},\vec{v} \in \vec{\mathbb{R}^3}$, and let $\ell_{\vec{e},\vec{u},\vec{v}} \in \mathcal{L}(\vec{\mathbb{R}^3},\mathbb{R})$ be the linear form defined by

$$\ell_{\vec{e},\vec{u},\vec{v}} : \begin{cases} \vec{\mathbb{R}^3} \to \mathbb{R} \\ \ \vec{z} \to \ell_{\vec{e},\vec{u},\vec{v}}(\vec{z}) := \det_{|\vec{e}}(\vec{u},\vec{v},\vec{z}) \end{cases} \tag{E.14}$$

(the algebraic volume of the parallelepiped limited by $\vec{u},\vec{v},\vec{z}$ in the Euclidean chosen unit).

Definition E.10 The vector product, or cross product, $\vec{u}\wedge_e\vec{v}$ of two vectors $\vec{u}$ and $\vec{v}$ is the $(\cdot,\cdot)_g$-Riesz representation vector of $\ell_{\vec{e},\vec{u},\vec{v}}$, that is, $\vec{u}\wedge_e\vec{v} \in \vec{\mathbb{R}^3}$ is characterized by $\ell_{\vec{e},\vec{u},\vec{v}}(\vec{z}) = (\vec{u}\wedge_e\vec{v},\vec{z})_g$ for all $\vec{z} \in \vec{\mathbb{R}^3}$, cf. (F.2), i.e.

$$\forall \vec{z} \in \vec{\mathbb{R}^3}, \quad \boxed{(\vec{u}\wedge_e\vec{v},\vec{z})_g = \det_{|\vec{e}}(\vec{u},\vec{v},\vec{z})}. \tag{E.15}$$

NB: $\vec{u}\wedge_e\vec{v}$ depends on $(\cdot,\cdot)_g$ since we need a $(\cdot,\cdot)_g$-Euclidean basis $(\vec{e}_i)$ (and depends on the orientation of $(\vec{e}_i)$).

We have thus defined the bilinear cross product operator

$$\wedge_e : \begin{cases} \vec{\mathbb{R}^3}\times\vec{\mathbb{R}^3} \to \vec{\mathbb{R}^3} \\ \quad (\vec{u},\vec{v}) \to \wedge_e(\vec{u},\vec{v}) := \vec{u}\wedge_e\vec{v}. \end{cases} \tag{E.16}$$

(The bilinearity is trivial thanks to the multilinearity of the determinant.) If one Euclidean basis is imposed by one observer to all the other observers, then $\vec{u}\wedge_e\vec{v}$ is written $\vec{u}\wedge\vec{v}$ (non objective).

E.2.3 Calculation of the vector product

$\vec{u} = \sum_{i=1}^3 u_i\vec{e}_i$, $\vec{v} = \sum_{i=1}^3 v_i\vec{e}_i$ and (E.15) give

$$(\vec{u}\wedge_e\vec{v},\vec{e}_1)_g = \det_{|\vec{e}}(\vec{u},\vec{v},\vec{e}_1) = \det\begin{pmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 0 \\ u_3 & v_3 & 0 \end{pmatrix} = \det\begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} = u_2v_3 - u_3v_2. \tag{E.17}$$

Similar calculation for $(\vec{u}\wedge_e\vec{v},\vec{e}_2)_e$ and $(\vec{u}\wedge_e\vec{v},\vec{e}_3)_e$, thus

$$\vec{u}\wedge_e\vec{v} = \sum_{i=1}^3 (u_{i+1}v_{i+2} - u_{i+2}v_{i+1})\,\vec{e}_i, \quad \text{i.e.} \quad [\vec{u}\wedge_e\vec{v}]_{|\vec{e}} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}. \tag{E.18}$$

with the generic notation $w_4 := w_1$ and $w_5 = w_2$. (In particular $\vec{e}_i\wedge_e\vec{e}_{i+1} = \vec{e}_{i+2}$.)

Proposition E.11 *1- $\vec{u}\wedge_e\vec{v} = -\vec{v}\wedge_e\vec{u}$.*

2- $\vec{u} \parallel \vec{v}$ iff $\vec{u}\wedge_e\vec{v} = 0$.

3- If $\vec{u}$ and $\vec{v}$ are independent then $\vec{u}\wedge_e\vec{v}$ is orthogonal to the linear space Vect$\{\vec{u},\vec{v}\}$ *generated by $\vec{u}$ and $\vec{v}$.*

4- $\vec{u}\wedge_e\vec{v}$ depends on the unit of measurement and on the orientation of $(\vec{e}_i)$: If $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ are two Euclidean dot products, let $\lambda > 0$ such that $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$, and then

$$\vec{u}\wedge_a\vec{v} = \pm\lambda\vec{u}\wedge_b\vec{v}. \tag{E.19}$$

Proof. 1- $\det_{|\vec{e}}(\vec{u},\vec{v},\vec{z}) = -\det_{|\vec{e}}(\vec{v},\vec{u},\vec{z})$ (since $\det_{|\vec{e}}$ is alternated).

2- If $\vec{u} \parallel \vec{v}$ then $\det_{|\vec{e}}(\vec{u},\vec{v},\vec{z}) = 0 = (\vec{u}\wedge_e\vec{v},\vec{z})_e$, so $\vec{u}\wedge_e\vec{v} \perp_g \vec{z}$, for all $\vec{z}$. And if $\vec{u}\wedge_e\vec{v}=0$ then (E.18) gives $\vec{u}\parallel\vec{v}$.

3- If $\vec{z}\in\mathrm{Vect}\{\vec{u},\vec{v}\}$ then $\det_{|\vec{e}}(\vec{u},\vec{v},\vec{z}) = 0 = (\vec{u}\wedge_e\vec{v},\vec{z})_g$ thus $\vec{u}\wedge_e\vec{v}\perp_g\vec{z}$.

4- $(\vec{u}\wedge_a\vec{v},\vec{z})_a \overset{(E.15)}{=} \det_{|\vec{a}}(\vec{u},\vec{v},\vec{z}) \overset{(E.12)}{=} \pm\lambda^3\det_{|\vec{b}}(\vec{u},\vec{v},\vec{z}) \overset{(E.15)}{=} \pm\lambda^3(\vec{u}\wedge_b\vec{v},\vec{z})_b = \pm\lambda^3\frac{1}{\lambda^2}(\vec{u}\wedge_b\vec{v},\vec{z})_a$, true for all $\vec{z}$, thus (E.19). ∎

Exercice E.12 Prove that $\vec{u}\wedge_e\vec{v}$ is a contravariant vector.

Answer. It is a vector (Riesz representation vector) in $\vec{\mathbb{R}}^3$, so it is contravariant; Or calculation: It satisfies the contravariance change of basis formula, see (F.17). ∎

E.2.4 Antisymmetric endomorphism represented by a vector

Proposition E.13 *Let $(\vec{e}_i)$ be a chosen $(\cdot,\cdot)_g$-Euclidean basis. If an endomorphism $\Omega\in\mathcal{L}(\vec{\mathbb{R}}^3;\vec{\mathbb{R}}^3)$ is $(\cdot,\cdot)_g$-antisymmetric then there exists a unique vector $\vec{\omega}_e\in\vec{\mathbb{R}}^3$ s.t., for all $\vec{y},\vec{z}\in\vec{\mathbb{R}}^3$,*

$$(\Omega.\vec{y},\vec{z})_g = \det_{|\vec{e}}(\vec{\omega}_e,\vec{y},\vec{z}), \tag{E.20}$$

i.e., there exists a unique vector $\vec{\omega}_e\in\vec{\mathbb{R}}^3$ s.t., for all $\vec{y},\vec{z}\in\vec{\mathbb{R}}^3$,

$$\boxed{\Omega.\vec{y} = \vec{\omega}_e\wedge_e\vec{y}}, \tag{E.21}$$

And

$$[\Omega]_{|\vec{e}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \quad \textit{iff} \quad [\vec{\omega}_e]_{|\vec{e}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{E.22}$$

In particular $\Omega.\vec{\omega}_e = \vec{0}$ $(= \vec{\omega}_e\wedge_e\vec{\omega}_e)$, i.e. $\vec{\omega}_e$ is an eigenvector associated with the eigenvalue 0.

Proof. Ω is antisymmetric, thus $[\Omega]_{|\vec{e}}$ is given as in (E.22). In particular $[\Omega.\vec{e}_1]_{|\vec{e}} = [\Omega]_{|\vec{e}}.[\vec{e}_1]_{|\vec{e}} = \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}$.

Calculation of the components of $\vec{\omega}_e$ if it exists: Let $\vec{\omega} = \omega_1\vec{e}_1+\omega_2\vec{e}_2+\omega_3\vec{e}_3$; thus $[\vec{\omega}\wedge\vec{e}_1]_{|\vec{e}} = \begin{pmatrix} 0 \\ \omega_3 \\ -\omega_2 \end{pmatrix}$, cf. (E.18), thus $\omega_3 = c$ and $\omega_2 = b$; Idem with $\vec{e}_2$ so that $\omega_1 = a$. Thus if it exists $\vec{\omega}$ is unique. And $\vec{\omega}_e$ given in (E.22) satisfies (E.21): It exists. ∎

Proposition E.14 *Let $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ be two Euclidean dot products (e.g. in foot and metre), let $(\vec{a}_i)$ and $(\vec{b}_i)$ be Euclidean associated bases, let $||\vec{b}_1||_a = \lambda$ (change of unit coefficient), so $(\cdot,\cdot)_a = \lambda_2(\cdot,\cdot)_b$). Suppose $[\Omega]_{|\vec{a}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$, thus $[\vec{\omega}_a]_{|\vec{a}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, cf. (E.22). Then (change of representation vector for Ω):*

$$\begin{aligned} &\bullet\ \textit{If } (\vec{b}_i) \textit{ and } (\vec{a}_i) \textit{ have the same orientation, then } \quad \vec{\omega}_b = \lambda\vec{\omega}_a, \\ &\bullet\ \textit{If } (\vec{b}_i) \textit{ and } (\vec{a}_i) \textit{ have opposite orientation, then } \quad \vec{\omega}_b = -\lambda\vec{\omega}_a, \end{aligned} \tag{E.23}$$

E.g., if $\vec{b}_i = \lambda\vec{a}_i$ for all i (change of unit, same orientation) then $\vec{\omega}_b = \lambda\vec{\omega}_a$, and if $\vec{b}_1 = -\lambda\vec{a}_1$, $\vec{b}_2 = \lambda\vec{a}_2$, $\vec{b}_3 = \lambda\vec{a}_3$ (change of unit, opposite orientation) then $\vec{\omega}_b = -\lambda\vec{\omega}_a$.

NB: *The formula $\vec{\omega}_b = \pm\lambda\vec{\omega}_a$ is a change of vector formula, **not** a change of basis formula.*

Proof. Apply (E.19). ∎

Interpretation of $\vec{\omega}_e$: Suppose $[\Omega]_{|\vec{e}} = \alpha\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So Ω is the rotation with angle $\frac{\pi}{2}$ in the horizontal plane composed with the dilation with ratio α. And $[\vec{\omega}_e]_{|\vec{e}} = \alpha\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \alpha\,\vec{e}_3$ is orthogonal to the horizontal plane and gives the rotation axis and the dilation coefficient.

Exercice E.15 Let Ω s.t. $[\Omega]_{|\vec{e}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$ (see (E.22)). Find a direct orthonormal basis $(\vec{b}_i)$ (relative to $(\vec{e}_i)$) s.t. $[\Omega]_{|\vec{b}} = \sqrt{a^2+b^2+c^2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Answer. Let $\vec{b}_3 = \frac{\vec{\omega}_e}{||\vec{\omega}_e||_e}$, that is, $[\vec{b}_3]_{|\vec{e}} = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Then choose $\vec{b}_1 \perp \vec{b}_3$, e.g. $[\vec{b}_1]_{|\vec{e}} = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}$. Then choose $\vec{b}_2 = \vec{b}_3 \wedge_e \vec{b}_1$, that is, $[\vec{b}_2]_{|\vec{e}} = \frac{1}{\sqrt{a^2+b^2}} \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix}$. Thus $(\vec{b}_i)$ is a direct orthonormal basis, and the transition matrix from $(\vec{e}_i)$ to $(\vec{b}_i)$ is $P = \begin{pmatrix} [\vec{b}_1]_{|\vec{e}} & [\vec{b}_2]_{|\vec{e}} & [\vec{b}_3]_{|\vec{e}} \end{pmatrix}$. And $[\Omega]_{|\vec{b}} = P^{-1}.[\Omega]_{|\vec{e}}.P$ (change of basis formula), with $P^{-1} = P^T$ (change of orthonormal basis), thus $[\Omega]_{|\vec{b}} = P^T.[\Omega]_{|\vec{e}}.P$

With $[\Omega]_{|\vec{e}}.[\vec{b}_1]_{|\vec{e}} = \frac{1}{\sqrt{b^2+c^2}} \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} . \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix} = \frac{1}{\sqrt{b^2+c^2}} \begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix} = \sqrt{a^2+b^2+c^2}[\vec{b}_2]_{|\vec{e}}$ (expected), $[\Omega]_{|\vec{e}}.[\vec{b}_2]_{|\vec{e}} = \frac{1}{\sqrt{b^2+c^2}} \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} . \begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix} = \frac{1}{\sqrt{b^2+c^2}} \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} bc^2 + b(a^2+b^2) \\ -ac^2 - a(a^2+b^2) \\ abc - abc \end{pmatrix} = -\sqrt{a^2+b^2+c^2}[\vec{b}_1]_{|\vec{e}}$ (expected), and $[\Omega]_{|\vec{e}}.[\vec{b}_3]_{|\vec{e}} = [\vec{0}]$ (expected since $\vec{b}_3 \parallel \vec{\omega}_e$). Thus $[\Omega]_{|\vec{e}}.P = \sqrt{a^2+b^2+c^2} \begin{pmatrix} [\vec{b}_2]_{|\vec{e}} & -[\vec{b}_1]_{|\vec{e}} & [\vec{0}]_{|\vec{e}} \end{pmatrix}$. And $(P^T.[\Omega]_{|\vec{e}}.P)_{ij} = [\vec{b}_i]^T_{|\vec{e}}.[\Omega]_{|\vec{e}}.[\vec{b}_j]_{|\vec{e}}$ gives the result. ∎

E.2.5 Curl

Definition E.16 If $\vec{v}$ is a C^1 vector field, if $(\vec{e}_i)$ is a Euclidean basis in $\vec{\mathbb{R}^3}$, and if $\vec{v} = \sum_{i=1}^3 v^i \vec{e}_i$, then the curl (or rotational) of $\vec{v}$ relative to $(\vec{e}_i)$ is the vector field $\vec{\text{curl}}_e \vec{v} = \vec{\text{rot}}_e \vec{v}$ given by

$$\vec{\text{curl}}_e \vec{v} = \sum_{i=1}^3 (\frac{\partial v_{i+2}}{\partial x_{i+1}} - \frac{\partial v_{i+1}}{\partial x_{i+2}})\, \vec{e}_i, \quad \text{i.e.} \quad [\vec{\text{curl}}_e \vec{v}]_{|\vec{e}} = \begin{pmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{pmatrix}. \tag{E.24}$$

Proposition E.17 *Let $\Omega(t,p_t) = \frac{d\vec{v}(t,p_t) - d\vec{v}(t,p_t)^T}{2}$, and let $\vec{\omega}_e(t,p_t)$ be the associated vector relative to the Euclidean basis $(\vec{e}_i)$, cf. (E.21). Then*

$$\vec{\omega}_e = \frac{1}{2} \vec{\text{curl}}_e \vec{v}. \tag{E.25}$$

Proof. (C.6) gives $[\Omega]_{|\vec{e}} = \frac{1}{2} \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \cdot & 0 & \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} \\ \cdot & \cdot & 0 \end{pmatrix}$, with $[\Omega]_{|\vec{e}}$ antisymmetric. Thus (E.22), (E.18) and (E.24) gives (E.25). ∎

E.3 Pseudo-cross product, and pseudo-vector

Framework: $\mathcal{M}_{31}$ the space of $3*1$ matrices, so we leave the vector framework to enter the matrix world.

E.3.1 Definition

Definition E.18 A column matrice is also called a pseudo-vector, or a column vector.

Definition E.19 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \overset{\text{noted}}{=} [\vec{x}]$ and $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \overset{\text{noted}}{=} [\vec{y}]$ being two matrices in $\mathcal{M}_{31}$, their pseudo-cross product is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \overset{\circlearrowleft}{\wedge} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix} \overset{\text{noted}}{=} [\vec{x}] \overset{\circlearrowleft}{\wedge} [\vec{y}]. \tag{E.26}$$

Thus the pseudo-cross product of two pseudo-vectors is a pseudo-vector (is a matrix).

E.3.2 Antisymmetric matrix represented by a pseudo-vector

Definition E.20 Let $A = [A_{ij}] = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$ be an antisymmetric matrix ($A_{ji} = -A_{ij}$ for all i,j). The pseudo-vecteur $\overset{\circlearrowleft}{\omega}$ associated to A is the column matrix $\overset{\circlearrowleft}{\omega} := \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. So,

$$\boxed{A.[\vec{y}] = \overset{\circlearrowleft}{\omega}\overset{\circlearrowleft}{\wedge}[\vec{y}]}, \quad \text{i.e.} \quad A.\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \overset{\circlearrowleft}{\omega}\overset{\circlearrowleft}{\wedge}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \text{for all matrix } [\vec{y}] = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{E.27}$$

E.3.3 Antisymmetric endomorphism and its pseudo-vectors representations

Let $\mathbb{R}^3$ be our usual affine space, $(\cdot,\cdot)_g$ be a Euclidean dot product, and $(\vec{e}_i)$ be a $(\cdot,\cdot)_g$-Euclidean associated basis. Let Ω be an antisymmetric endomorphism relative to $(\cdot,\cdot)_g$, so $\Omega^T = -\Omega$, cf. (E.13). Thus $[\Omega]_{|\vec{e}}$ is an antisymmetric matrix. Call $\overset{\circlearrowleft}{\omega}$ the associated pseudo-vector, i.e., cf. (E.27), for all $\vec{y} \in \vec{\mathbb{R}^3}$,

$$[\Omega]_{|\vec{e}}.[\vec{y}]_{|\vec{e}} = \overset{\circlearrowleft}{\omega}\overset{\circlearrowleft}{\wedge}[\vec{y}]_{|\vec{e}}. \tag{E.28}$$

This formula is widely used in mechanics, and unfortunately sometimes noted $\Omega.\vec{y} = \vec{\omega} \wedge \vec{y}$ (!):

Be careful: (E.28) is ***not*** a vectorial formula; This is just a formula for matrix calculations which gives false result if a change of basis is considered; E.g., with $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ be a $(\cdot,\cdot)_g$-Euclidean basis, and $(\vec{b}_1, \vec{b}_2, \vec{b}_3) = (-\vec{a}_1, \vec{a}_2, \vec{a}_3)$. So $(\vec{b}_i)$ is also a $(\cdot,\cdot)_g$-Euclidean basis, but with a different orientation.

1- Vector approach: Let P be the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$, so $P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let $[\Omega]_{|\vec{a}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$. Thus, Ω being an endomorphism, the change of basis formula gives

$$[\Omega]_{|\vec{b}} = P^{-1}.[\Omega]_{|\vec{a}}.P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}.\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & c & -b \\ -c & 0 & -a \\ b & a & 0 \end{pmatrix}. \tag{E.29}$$

Thus the vectors $\vec{\omega}_a$ and $\vec{\omega}_b$ are given by (E.22):

$$[\vec{\omega}_a]_{|\vec{a}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad [\vec{\omega}_b]_{|\vec{b}} = \begin{pmatrix} a \\ -b \\ -c \end{pmatrix}, \quad \text{i.e.} \quad \left\{ \begin{aligned} \vec{\omega}_a &= a\vec{a}_1 + b\vec{a}_2 + c\vec{a}_3, \\ \vec{\omega}_b &= a\vec{b}_1 - b\vec{b}_2 - c\vec{b}_3, \end{aligned} \right\} \quad \text{thus} \quad \boxed{\vec{\omega}_b = -\vec{\omega}_a}. \tag{E.30}$$

2- Matrix approach (E.27) gives $[\Omega]_{|\vec{a}}.[\vec{y}] = \overset{\circlearrowleft}{\omega}_a\overset{\circlearrowleft}{\wedge}[\vec{y}]$ and $[\Omega]_{|\vec{b}}.[\vec{y}] = \overset{\circlearrowleft}{\omega}_b\overset{\circlearrowleft}{\wedge}[\vec{y}]$, with

$$\overset{\circlearrowleft}{\omega}_a = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \overset{\circlearrowleft}{\omega}_b = \begin{pmatrix} a \\ -b \\ -c \end{pmatrix}, \quad \text{so} \quad \boxed{\overset{\circlearrowleft}{\omega}_a \neq -\overset{\circlearrowleft}{\omega}_b}. \tag{E.31}$$

And $\overset{\circlearrowleft}{\omega}$ does not represent a single vector either, since it does not satisfy the vector change of basis formula $\overset{\circlearrowleft}{\omega}_b \neq P^{-1}.\overset{\circlearrowleft}{\omega}_a$. Thus $\overset{\circlearrowleft}{\omega}$ is not a vector (is not tensorial): It is just a matrix (called a "pseudo-vector").

E.4 Examples

E.4.1 Rectilinear motion

Let $\widetilde{\Phi} : [t_1, t_2] \times \mathit{Obj} \to \mathbb{R}^n$ be a C^1 motion. Let $t_0 \in]t_1, t_2[$ and $P_{\mathit{Obj}} \in \mathit{Obj}$.

Definition E.21 The motion of P_{Obj} is rectilinear iff, for all $t_0, t \in [t_1, t_2]$,

$$\frac{\widetilde{\Phi}_{P_{Obj}}(t) - \widetilde{\Phi}_{P_{Obj}}(t_0)}{t{-}t_0} \parallel \widetilde{\Phi}_{P_{Obj}}{}'(t_0). \tag{E.32}$$

And the motion is rectilinear uniform iff, for all $t_0, t \in [t_1, t_2]$,

$$\widetilde{\Phi}_{P_{Obj}}(t) = \widetilde{\Phi}_{P_{Obj}}(t_0) + (t{-}t_0)\,\widetilde{\Phi}_{P_{Obj}}{}'(t_0), \quad \text{i.e.} \quad p(t) = p(t_0) + (t{-}t_0)\,\vec{V}^{t_0}(t_0, p(t_0)) \tag{E.33}$$

when $p(t) = \widetilde{\Phi}(t, P_{Obj})$, that is, the trajectory is traveled at constant velocity.

E.4.2 Circular motion

Let $(\vec{E}_1, \vec{E}_2)$ be a Euclidean basis. Let $t_0 \in [t_1, t_2]$. A motion Φ^{t_0} is a circular motion iff

$$\overrightarrow{\mathcal{O}\Phi_P^{t_0}(t)} = x(t)\vec{E}_1 + y(t)\vec{E}_2, \quad [\overrightarrow{\mathcal{O}\Phi_P^{t_0}(t)}]_{|\vec{E}} = \begin{pmatrix} x(t) = a + R\cos(\theta(t)) \\ y(t) = b + R\sin(\theta(t)) \end{pmatrix}, \tag{E.34}$$

for some $R > 0$ (called the radius), some $a, b \in \mathbb{R}$, and some function $\theta : \mathbb{R} \to \mathbb{R}$. And $\begin{pmatrix} a \\ b \end{pmatrix} = \mathcal{O}_C \in \mathbb{R}^2$ is the center of the circle and $\theta(t)$ is the angle at t. And the particle P_{Obj} (s.t. $\widetilde{\Phi}(t_0, P_{Obj}) = P$) stays on the circle with center $\mathcal{O}_C$ and radius R.

The circular motion is uniforme iff, for all t, $\theta''(t) = 0$, that is, $\exists \omega_0 \in \mathbb{R}$, $\forall t \in [t_1, t_2]$, $\theta(t) = \omega_0 t$.

Notation:

$$\vec{\varphi}_P^{t_0}(t) = R\cos(\theta(t)\vec{E}_1 + R\sin(\theta(t))\vec{E}_2, \quad \text{so} \quad [\vec{\varphi}_P^{t_0}(t)]_{|\vec{E}} = \begin{pmatrix} R\cos(\theta(t)) \\ R\sin(\theta(t)) \end{pmatrix}. \tag{E.35}$$

Thus the Lagrangian velocity of a circular motion is

$$\vec{V}_P^{t_0}(t) = (\Phi_t^{t_0})'(t) = (\vec{\varphi}_P^{t_0})'(t), \quad \text{so} \quad [\vec{V}_P^{t_0}(t)]_{|\vec{E}} = R\theta'(t) \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix}, \tag{E.36}$$

and $\vec{V}_P^{t_0}(t)$ is orthogonal to $\vec{\varphi}_P^{t_0}(t)$ (the radius vector). And the Lagrangian acceleration is

$$\vec{\Gamma}_P^{t_0}(t) = R\theta''(t) \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix} + R(\theta'(t))^2 \begin{pmatrix} -\cos(\theta(t)) \\ -\sin(\theta(t)) \end{pmatrix}. \tag{E.37}$$

Consider

$$\vec{e}_r(t) = \frac{\vec{\varphi}_P^{t_0}(t)}{||\vec{\varphi}_P^{t_0}(t)||} = \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}, \quad \text{and} \quad \vec{e}_\theta(t) = \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix}, \tag{E.38}$$

thus $(\vec{e}_r(t), \vec{e}_\theta(t))$ is an orthonormal basis. Then :

$$\vec{V}_P^{t_0}(t) = R\theta'(t)\,\vec{e}_\theta(t), \tag{E.39}$$

and :

$$\vec{\Gamma}_P^{t_0}(t) = -R(\theta'(t))^2\,\vec{e}_r(t) + R\theta''(t)\,\vec{e}_\theta(t). \tag{E.40}$$

E.g., in $\mathbb{R}^3$ and a motion in he "horizontal" plane given by $(\vec{e}_1, \vec{e}_2)$, the vertical line being given by $\vec{E}_3$. Here

$$\vec{V}_P^{t_0}(t) = \vec{\omega}(t) \wedge \vec{\varphi}_P^{t_0}(t), \quad \text{where} \quad \vec{\omega}(t) = \omega(t)\vec{e}_3 \quad \text{and} \quad \omega(t) = \theta'(t). \tag{E.41}$$

And

$$\vec{\Gamma}^{t_0}(t) = \frac{d\vec{\omega}}{dt}(t) \wedge \vec{\varphi}_P^{t_0}(t) + \vec{\omega}(t) \wedge \vec{V}_P^{t_0}(t) \qquad \left(= R\frac{d\omega}{dt}(t)\,\vec{e}_\theta(t) - \omega^2(t)R\,\vec{e}_r(t)\right). \tag{E.42}$$

E.4.3 Motion of a planet (centripetal acceleration)

Illustration: *Obj* is e.g. a planet from the solar system.

Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be a Euclidean basis (e.g. fixed relative to stars an $(\vec{e}_1, \vec{e}_2)$ define the ecliptic plane), $(\cdot,\cdot)_g$ be the Euclidean associated dot product, $||.||$ the Euclidean associated norm, $\mathcal{O}$ an origin in $\mathbb{R}^3$ (e.g. the center of the Sun), and $\mathcal{R} = (\mathcal{O}, (\vec{e}_i))$.

Consider a motion $\widetilde{\Phi}$ of *Obj* in $\mathcal{R}$, cf. (1.5). Let $t_0 \in [t_1, t_1]$, and consider $\Phi^{t_0} =^{\text{noted}} \Phi$ or $\vec{\varphi}^{t_0} =^{\text{noted}} \vec{\varphi}$, cf. (3.1)-(3.4).

Definition E.22 The motion of a particle P_{Obj} is a centripetal acceleration motion iff the particle is not static and, at all time, its acceleration vector $\vec{A}(t)$ points to a fixed point F (focus).

We will take the focus F as the origin of the referential, that is, $\mathcal{O} := F$.

Thus, for all $t \in [t_1, t_2]$, $\overrightarrow{\mathcal{O}\Phi_P(t)} \parallel \vec{A}_P(t)$, that is,

$$\overrightarrow{\mathcal{O}\Phi_P(t)} \wedge \vec{A}_P(t) = \vec{0}. \tag{E.43}$$

Remark E.23 A rectilinear motion is a centripetal acceleration motion, but such a motion is usually excluded in the definition E.22. ⁂

Example E.24 The motion of a planet from the solar system is a centripetal acceleration motion: An elliptical motion of focus the center of the Sun. ⁂

Example E.25 The second Newton's law of motion $\sum \vec{f} = m\vec{\gamma}$ (Galilean referential) gives: If $\sum \vec{f}$ is, at all time, directed to a unique point F, then the motion is a centripetal acceleration motion. ⁂

Let Φ be a centripetal acceleration motion, let $\mathcal{O}$ be the focus, and let $\vec{\varphi}_P(t) := \overrightarrow{\mathcal{O}\Phi_P(t)}$. So the Lagrangian velocity and acceleration are

$$\vec{V}_P(t) = \frac{d\Phi_P}{dt}(t) = \frac{d\vec{\varphi}_P}{dt}(t), \quad \text{and} \quad \vec{A}_P(t) = \frac{d^2\Phi_P}{dt^2}(t) = \frac{d^2\vec{\varphi}_P}{dt^2}(t), \tag{E.44}$$

and $\vec{\varphi}_P(t) \wedge \vec{A}_P(t) = \vec{0}$, cf. (E.43).

Definition E.26 The areolar velocity at t is the vector

$$\vec{Z}(t) = \frac{1}{2}\vec{\varphi}_P(t) \wedge \vec{V}_P(t). \tag{E.45}$$

Proposition E.27 *If Φ is a centripetal acceleration motion, then the areolar velocity is contant, that is, $\frac{d\vec{Z}}{dt}(t) = \vec{0}$ pour tout t, so*

$$\vec{Z}(t) = \vec{Z}(t_0), \quad \forall t. \tag{E.46}$$

That is, the position vectors sweep equal areas in equal times. And $\vec{Z}(t_0) = \vec{0}$ iff Φ is a rectilinear motion.

If $\vec{Z}(t_0) \neq \vec{0}$ then :

- $\vec{\varphi}_P(t)$ and $\vec{V}_P(t)$ are orthogonal to $\vec{Z}(t_0)$ at all time t,

- The motion of the particle P_{Obj} takes place in the affine plane orthogonal to $\vec{Z}(t_0)$ passing through O.

- $\vec{V}_P(t)$ never vanishes.

Proof. (E.45) and (E.43) give $2\frac{d\vec{Z}}{dt}(t) = \frac{d\vec{\varphi}_P}{dt}(t) \wedge \vec{V}_P(t) + \vec{\varphi}(t) \wedge \frac{d\vec{V}_P}{dt}(t) = \vec{V}_P(t) \wedge \vec{V}_P(t) + \vec{\varphi}(t) \wedge \vec{A}_P(t) = \vec{0} + \vec{0}$. Thus $\vec{Z}$ is constant, $\vec{Z}(t) = \vec{Z}(t_0)$ for all t. Then, if $\vec{Z}(t_0) \neq \vec{0}$ then $\vec{Z}(t) \neq \vec{0}$ pour tout t, and

- $\vec{Z}(t) = \frac{1}{2}\vec{\varphi}_P(t) \wedge \vec{V}_P(t)$ gives that $\vec{\varphi}_P(t)$ et $\vec{V}_P(t)$ are orthogonal to $\vec{Z}(t_0)$ for all t, thus $\vec{A}_P(t)$ is orthogonal to $\vec{Z}(t_0)$, cf. (E.43).
- The Taylor expansion reads $\vec{\varphi}_P(t) = \vec{\varphi}_P(t_0) + \vec{V}_P(t_0)(t - t_0) + \int_{\tau=t_0}^{t} \vec{A}_P(\tau)(t-\tau)^2\, d\tau$, with $\vec{V}_P(t_0)$ and $\vec{A}_P(\tau) \perp \vec{Z}(t_0)$ for all τ, thus $\vec{\varphi}_P(t) - \vec{\varphi}_P(t_0) \perp \vec{Z}(t_0)$ for all τ, that is $\overrightarrow{\mathcal{O}p(t)} - \overrightarrow{\mathcal{O}P} = \overrightarrow{Pp(t)} \perp \vec{Z}(t_0)$ for all τ, Thus $p(t)$ belongs to the affine plane containing P orthogonal to $\vec{Z}(t_0)$, for all t. And $\overrightarrow{\mathcal{O}P} = \vec{\varphi}_P(t_0) \perp \vec{Z}(t_0)$, thus O belong to the same plane.
- $\vec{Z}(t) = \vec{Z}(t_0) \neq \vec{0}$ implies $\vec{V}_P(t) \neq \vec{0}$ for all t, and (E.45) gives: $(\vec{\varphi}_P(t), \vec{V}_P(t), \vec{Z}(t_0))$ is a positively-oriented basis. Since $\vec{\varphi}_P$ and $\vec{V}$ are continuous and do not vanish, since $\vec{Z}(t_0) \neq \vec{0}$, we get: P_{Obj} "turns around $\vec{Z}(t_0)$" and keeps its direction.

If $\vec{Z}(t) = \vec{0}$ then $\vec{\varphi}_P(t) \parallel \vec{V}_P(t)$ for all t, cf. (E.45), so $\vec{V}_P(t) = f(t)\vec{\varphi}_P(t)$ where f is some scalar function. And $\vec{V}_P(t) = \vec{\varphi}_P{}'(t)$ gives $\vec{\varphi}_P{}'(t) = f(t)\vec{\varphi}_P(t)$, thus $\vec{\varphi}_P(t) = \vec{\varphi}_P(t_0)e^{F(t)}$ where F is a primitive of f s.t. $F(t_0) = 0$, thus $\vec{\varphi}_P(t) \parallel \vec{\varphi}_P(t_0)$, so $\overrightarrow{\mathcal{O}\Phi_P(t)} \parallel \overrightarrow{\mathcal{O}\Phi_P(t_0)}$, for all t: The motion is rectilinear. ▞

Interpretation. (Non rectilinear motion.) The area swept by $\vec{\varphi}_P(t)$ is, at first order, the area of the triangle whose sides are $\vec{\varphi}_P(t)$ and $\vec{\varphi}_P(t+\tau)$ ("anglular sector"). So, with τ close to 0, let

$$\vec{S}_t(\tau) = \frac{1}{2}\vec{\varphi}_P(t) \wedge \vec{\varphi}_P(t+\tau), \quad \text{and} \quad S_t(\tau) = ||\vec{S}_t(\tau)||, \tag{E.47}$$

the vectorial an scalar area. With $\vec{\varphi}_P(t+\tau) = \vec{\varphi}_P(t) + \vec{V}_P(t)\tau + o(\tau)$ we get

$$\vec{S}_t(\tau) = \frac{1}{2}\vec{\varphi}_P(t) \wedge (\vec{V}_P(t)\tau + o(\tau)), \tag{E.48}$$

Since $\vec{S}_t(0) = 0$ we get $\frac{\vec{S}_t(\tau)-\vec{S}(0)}{\tau} = \frac{1}{2}\vec{\varphi}_P(t) \wedge \vec{V}_P(t) + o(1)$, then

$$\frac{d\vec{S}_t}{d\tau}(0) = \frac{1}{2}\vec{\varphi}_P(t) \wedge \vec{V}_P(t) = \vec{Z}(t) = \vec{Z}(t_0), \tag{E.49}$$

thanks to (E.46), thus

$$\frac{d\vec{S}_t}{d\tau}(0) = \frac{d\vec{S}_{t_0}}{d\tau}(0), \qquad \forall t \in [t_0, T], \tag{E.50}$$

that is, the rate of variation of $\vec{S}_t$ is constant. And with $||\vec{S}_t(\Delta\tau)||^2 = (\vec{S}_t(\Delta\tau), \vec{S}_t(\Delta\tau))$ we get

$$\frac{d||\vec{S}_t||^2}{d\tau}(\Delta\tau) = 2(\frac{d\vec{S}_t}{d\tau}(\Delta\tau), \vec{S}_t(\Delta\tau)), \tag{E.51}$$

so, since $\vec{S}_t(0) = 0$,

$$\frac{d||\vec{S}_t||^2}{d\tau}(0) = 0. \tag{E.52}$$

Therefore the function $t \to ||\vec{S}_t(0)||^2 = S_t(0)^2$ is constant, thus $t \to S_t(0)$ est constant, and $\frac{dS_t}{d\tau}(0)$ is constant.

Exercice E.28 Give a parametrization of the swept area, and redo the calculations.

Answer. Let

$$r(t) = ||\vec{\varphi}_P(t)||, \quad \theta(t) = \widehat{p(t)OP} \quad \text{(angle)}, \tag{E.53}$$

then

$$\vec{\varphi}_P(t) = \begin{pmatrix} r(t)\cos(\theta(t)) \\ r(t)\sin(\theta(t)) \\ 0 \end{pmatrix}. \tag{E.54}$$

Thus

$$\vec{V}_P(t) = \begin{pmatrix} r'(t)\cos(\theta(t) - r(t))\theta'(t)\sin(\theta(t)) \\ r'(t)\sin(\theta(t) + r(t))\theta'(t)\cos(\theta(t)) \\ 0 \end{pmatrix}. \tag{E.55}$$

With (E.45) we get

$$\vec{Z}(t) = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ r^2(t)\theta'(t) \end{pmatrix}, \quad \text{with} \quad r^2(t)\theta'(t) = r^2(t_0)\theta'(t_0) \quad \text{(constant)}, \tag{E.56}$$

cf. (E.46). A parametrization of the swept area is then

$$\vec{\mathcal{A}} : \left\{ \begin{array}{rl} [0,1] \times [t_0, T] & \to \mathbb{R}^3 \\ (\rho, t) & \to \vec{\mathcal{A}}(\rho, t) \end{array} \right\}, \qquad \vec{\mathcal{A}}(\rho, t) = \begin{pmatrix} \rho\, r(t)\cos(\theta(t)) \\ \rho\, r(t)\sin(\theta(t)) \\ 0 \end{pmatrix}. \tag{E.57}$$

Therefore, the tangent associated vectors are

$$\frac{\partial\vec{\mathcal{A}}}{\partial\rho}(\rho, t) = \begin{pmatrix} r(t)\cos(\theta(t)) \\ r(t)\sin(\theta(t)) \\ 0 \end{pmatrix}, \qquad \frac{\partial\vec{\mathcal{A}}}{\partial t}(\rho, t) = \begin{pmatrix} \rho r'(t)\cos(\theta(t) - \rho r(t))\theta'(t)\sin(\theta(t)) \\ \rho r'(t)\sin(\theta(t) + \rho r(t))\theta'(t)\cos(\theta(t)) \\ 0 \end{pmatrix}, \tag{E.58}$$

hence the vectorial and scalare element areas are

$$d\vec{\sigma} = (\frac{\partial \vec{\mathcal{A}}}{\partial \rho} \wedge \frac{\partial \vec{\mathcal{A}}}{\partial t})d\rho dt = \begin{pmatrix} 0 \\ 0 \\ \rho r^2 \theta' \, d\rho dt \end{pmatrix}, \qquad d\sigma = \rho r^2 \theta' \, d\rho d\theta. \tag{E.59}$$

Therefore the area between t_0 and t is

$$\mathcal{A}(t) = \mathcal{A}(t_0) + \int_{\rho=0}^{1} \int_{\tau=t_0}^{t} \rho r^2(\tau)\theta'(\tau)\, d\rho d\tau = \frac{1}{2} \int_{\tau=t_0}^{t} r(\tau)^2 \theta'(\tau)\, d\tau. \tag{E.60}$$

Hence

$$\mathcal{A}'(t) = r(t)^2\theta'(t) = r(t_0)^2\theta'(t_0) \quad (= \text{constant} = ||\vec{Z}(t_0)||), \tag{E.61}$$

cf. (E.56). ∴

Exercice E.29 Prove the Binet formulas (non rectilinear central motion):

$$V_P(t)^2 = Z_0^2 \Big(\frac{1}{r^2} + (\frac{d\frac{1}{r}}{d\theta})^2 \Big)(t), \qquad \vec{\Gamma}_P(t) = -\frac{Z_0^2}{r^2} \Big(\frac{1}{r} + \frac{d^2 \frac{1}{r}}{d\theta^2} \Big)(t)\, \vec{e}_r(t), \tag{E.62}$$

for the energy and the acceleration.

Answer. Proposition E.27 tells that Φ is a planar motion. With (E.53) and $\vec{e}_r(t) = \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}$ we have $\vec{\varphi}(t) = r(t)\vec{e}_r(t)$ (in the plane). Let $\vec{e}_\theta(t) = \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix}$, thus

$$\vec{V}(t) = \frac{dr}{dt}(t)\vec{e}_r(t) + r(t)\frac{d\vec{e}_r}{dt}(t) = r'(t)\vec{e}_r(t) + r(t)\theta'(t)\vec{e}_\theta(t).$$

And $\vec{e}_r(t) \perp \vec{e}_\theta(t)$ gives

$$V^2(t) = (r'(t))^2 + (r(t)\theta'(t))^2.$$

Since $\theta'(t) \neq 0$ for all t (non rectilinear central motion) Let $s(\theta(t)) = r(t)$. Let us suppose that θ is C^1, thus $\theta' > 0$ or $\theta' < 0$, and $\theta : t \to \theta(t)$ defines a change of variable. And

$$r'(t) = s'(\theta(t))\theta'(t).$$

And (E.61) and $\theta'(t) = \frac{Z_0}{r^2(t)}$ give

$$V^2(t(\theta)) = (s'(\theta))^2 \frac{Z_0^2}{r^4(t)} + r^2(t)\frac{Z_0^2}{r^4(t)} = Z_0^2 (\frac{(s'(\theta))^2}{s^4(\theta)} + \frac{1}{s^2(\theta)}) = Z_0^2 [\Big(\frac{d\frac{1}{s}}{d\theta}(\theta) \Big)^2 + \frac{1}{s^2(\theta)}].$$

Thus $r(t) = s(\theta)$ and $\frac{dr}{d\theta} := \frac{ds}{d\theta}$ give the first Binet formula. Then

$$\vec{\Gamma}(t) = r''(t)\vec{e}_r(t) + r'(t)\frac{d\vec{e}_r}{dt}(t) + (r'(t)\theta'(t) + r(t)\theta''(t))\vec{e}_\theta(t) + r(t)\theta'(t)\frac{d\vec{e}_\theta}{dt}(t),$$

with $\frac{d\vec{e}_r}{dt} \parallel \vec{e}_\theta$, and $\frac{d\vec{e}_\theta}{dt}(t) = -\theta'(t)\vec{e}_r(t)$, and $\vec{e}_\theta \perp \vec{\Gamma}$ (central motion), we get

$$\vec{\Gamma}(t) = (r''(t) - r(t)(\theta'(t))^2)\vec{e}_r(t).$$

And

$$r'(t) = s'(\theta)\theta'(t) = s'(\theta)\frac{Z_0}{r^2(t)} = Z_0 \frac{s'(\theta)}{s^2(\theta)} = -Z_0 \frac{d\frac{1}{s}}{d\theta}(\theta),$$

thus

$$r''(t) = -Z_0 \frac{d^2 \frac{1}{s}}{d\theta^2}(\theta)\, \theta'(t) = -\frac{Z_0^2}{r^2(t)} \frac{d^2 \frac{1}{s}}{d\theta^2}(\theta),$$

which is the second Binet formula. ∴

F Riesz representation theorem

F.1 The Riesz representation theorem

Framework: $(E,(\cdot,\cdot)_g)$ is Hilbert space, i.e. E is a vector space equipped with an inner dot product $(\cdot,\cdot)_g$ such that, with the associated norm defined by$||\vec{v}||_g := \sqrt{(\vec{v},\vec{v})_g}$, $(E,||.||_g)$ is a complete space. And $E^* = \mathcal{L}(E;\mathbb{R})$ is the space of the linear and continuous forms on E (the space of linear "measuring tools") equipped with its norm $||\ell||_{E^*} := \sup\limits_{||\vec{x}||_g=1} |\ell.\vec{x}| < \infty$.

• We have the easy statement:

$$\forall \vec{v}\in E \text{ (vector)},\ \exists! v_g \in E^* \text{ (linear continuous form)} \quad \text{s.t.} \quad v_g.\vec{x} = (\vec{v},\vec{x})_g,\ \ \forall \vec{x}\in E, \tag{F.1}$$

and $||v_g||_{E^*} = ||\vec{v}||_g$. (Usual notation in finite dimension: $v_g.\vec{x} = \vec{v} \bullet_g \vec{x}$, or simply $v.\vec{x} = \vec{v}\bullet\vec{x}$ if a chosen $(\cdot,\cdot)_g$ is imposed to all observers.)

Indeed: Define $v_g : E \to \mathbb{R}$ by $v_g(\vec{x}) = (\vec{v},\vec{x})_g$ for all $\vec{x}\in E$; The definition domain of v_g is E and v_g is trivially linear; And the Cauchy–Schwarz inequality gives $|v_g(\vec{x})| = |(\vec{v},\vec{x})_g| \le ||\vec{v}||_g\, ||\vec{x}||_g$ for all $\vec{x}\in E$, thus $||v_g||_{E^*} \le ||\vec{v}||_g < \infty$, thus v_g is continuous; And $|v_g(\vec{v})| = |(\vec{v},\vec{v})_g| = ||\vec{v}||_g\, ||\vec{v}||_g$, thus $||v_g||_{E^*} \ge ||\vec{v}||_g$, thus $||v_g||_{E^*} = ||\vec{v}||_g$.

• The Riesz representation theorem concerns the converse: If you choose an inner dot product $(\cdot,\cdot)_g$ in E (e.g. English of French), then you can represent a "measuring instrument" $\ell\in E^*$ by a vector $\vec{\ell}_g \in E$:

Theorem F.1 (Riesz representation theorem, and definition) $(E,(\cdot,\cdot)_g)$ *being a Hilbert space,*

$$\forall \ell \in E^* \textit{ (linear continuous form)},\ \exists! \vec{\ell}_g \in E \textit{ (vector)} \quad \textit{s.t.} \quad \ell.\vec{x} = (\vec{\ell}_g,\vec{x})_g,\ \ \forall \vec{x}\in E, \tag{F.2}$$

and $||\vec{\ell}_g||_g = ||\ell||_{E^*}$. *And* $\vec{\ell}_g$ *is called the* $(\cdot,\cdot)_g$*-Riesz representation vector of* ℓ *(depends on* g*).*

(Usual notation in finite dimension: $v_g.\vec{x} = \vec{v}\bullet_g\vec{x}$*, or simply* $v.\vec{x} = \vec{v}\bullet\vec{x}$ *if a chosen* $(\cdot,\cdot)_g$ *is imposed to all observers.)*

Proof. Easy in finite dimension: With a basis $(\vec{e}_i)$, if $[\ell]_{|\vec{e}} = (\,\ell_1 \ \ \dots \ \ \ell_n\,)$ (row matrix since ℓ is a linear form) then (F.2) gives $[\ell]_{|\vec{e}}.[\vec{x}]_{|\vec{e}} = [\vec{\ell}_g]_{\vec{e}}^T.[g]_{|\vec{e}}.[\vec{x}]_{|\vec{e}}$, thus $[\vec{\ell}_g]_{\vec{e}} = [g]_{|\vec{e}}^{-1}.[\ell]_{|\vec{e}}^T$ (column matrix), thus $\vec{\ell}_g$.

General case (e.g. with $E = L^2(\Omega)$ and the finite element method): If $\ell = 0$ then $\vec{\ell}_g = \vec{0}$ (trivial). Suppose $\ell \ne 0$: Thus $\text{Ker}\ell = \ell^{-1}(\{0\}) \ne \{\vec{0}\}$ (the kernel). If $\dim E = 1$, it is trivial (exercise). Suppose $\dim E \ge 2$. Since ℓ is continuous, its kernel $\text{Ker}\ell = \ell^{-1}(\{0\})$ is closed in E. Thus, if $\vec{x}\in E$, then its $(\cdot,\cdot)_g$-orthogonal projection $\vec{x}_0 \in \text{Ker}\ell$ on $\text{Ker}\ell$ exists, is unique, and is given by: $\forall \vec{y}_0 \in \text{Ker}\ell,\ (\vec{x}-\vec{x}_0,\vec{y}_0)_g = 0$. (So $\vec{x}-\vec{x}_0 \perp_g \text{Ker}\ell$.) Choose a $\vec{x} \notin \text{Ker}\ell$ (possible since $\ell\ne 0$), and let $\vec{n} := \frac{\vec{x}-\vec{x}_0}{||\vec{x}-\vec{x}_0||_g}$; So $\vec{n}$ is a $(\cdot,\cdot)_g$-orthonormal vector to $\text{Ker}\ell$, and $(\text{Ker}\ell)^\perp = \text{Vect}\{\vec{n}\}$ since $\dim(\text{Ker}\ell)^\perp = 1$ (in finite dimension cf. the Dimension Formula which states that the dimension of the domain of a linear map is the sum of the dimension of its range and the dimension of its kernel, and in infinite dimension see next exercise F.2). And $E = \text{Ker}\ell \oplus (\text{Ker}\ell)^\perp$ since both vector spaces are closed (an orthogonal is always closed in a Hilbert space). Thus if $\vec{x}\in E$ then $\vec{x} = \vec{x}_0 + \lambda\vec{n} \in \text{Ker}\ell \oplus (\text{Ker}\ell)^\perp$; Thus $(\vec{x},\vec{n})_g = \lambda$ and $\ell(\vec{x}) = 0 + \lambda\ell(\vec{n}) = (\vec{x},\vec{n})_g\ell(\vec{n}) = (\vec{x},\ell(\vec{n})\vec{n})_g$ (bilinearity of $(\cdot,\cdot)_g$); Thus $\vec{\ell}_g := \ell(\vec{n})\vec{n}$ satisfies (F.2). And if $\vec{\ell}_{g1}$ and $\vec{\ell}_{g2}$ satisfy (F.2) then $(\vec{\ell}_{g1}-\vec{\ell}_{g2},\vec{x})_g = 0$ for all $\vec{x}\in E$, thus $\vec{\ell}_{g1}-\vec{\ell}_{g2} = 0$. Thus $\vec{\ell}_g$ is unique.

And the Cauchy–Schwarz theorem give $||\ell||_{E^*} := \sup_{||\vec{x}||_g=1} |\ell(\vec{x})| = \sup_{||\vec{x}||_g=1} |(\vec{\ell}_g,\vec{x})_g| = ||\vec{\ell}_g||_g$.

$\vec{R}_g$ is an isomorphism between Banach spaces: linearity since $(\vec{R}_g(\ell+\lambda m),\vec{x})_g = (\ell+\lambda m).\vec{x} = \ell.\vec{x} + \lambda m.\vec{x} = (\vec{R}_g(\ell),\vec{x})_g + \lambda(\vec{R}_g(m),\vec{x})_g = (\vec{R}_g(\ell)+\lambda\vec{R}_g(m),\vec{x})_g$ for all $\vec{x}$ gives $\vec{R}_g(\ell+\lambda m) = \vec{R}_g(\ell)+\lambda\vec{R}_g(m)$, bijectivity thanks to (F.1) and (F.2), and the norm is kept since $||\vec{\ell}_g||_g = ||\ell||_{E^*}$. ∴

Exercice F.2 Prove: If $\ell\in E^*-\{0\}$ then $\dim(\text{Ker}\ell)^\perp = 1$ $(= \dim(\text{Im}(\ell)) = \dim\mathbb{R})$.

Answer. Consider the restriction $\ell_{|\text{Ker}\ell^\perp} : \left\{\begin{array}{ll} (\text{Ker}\ell)^\perp & \to \mathbb{R} \\ \vec{x} & \to \ell_{|\text{Ker}\ell^\perp}.\vec{x} = \ell.\vec{x} \end{array}\right\}$. It is linear (since ℓ is), and thus one to one: Indeed it is onto since $\ell\ne 0$, and it is one to one since if $\ell_{|\text{Ker}\ell^\perp}(\vec{x}) = 0 = \ell(\vec{x})$ then $\vec{x}\in(\text{Ker}\ell)^\perp \bigcap \text{Ker}\ell = \{\vec{0}\}$, thus $\vec{x} = 0$. Thus $\dim(\text{Ker}\ell)^\perp \le \dim(\text{Im}(\ell)) = 1$: Indeed, if $\vec{z}_1,\vec{z}_2 \in (\text{Ker}\ell)^\perp - \{\vec{0}\}$ then $\ell_{|\text{Ker}\ell^\perp}(\vec{z}_1)\in\mathbb{R}$ and $\ell_{|\text{Ker}\ell^\perp}(\vec{z}_2)\in\mathbb{R}$, thus $\exists\lambda\in\mathbb{R}$ s.t. $\ell_{|\text{Ker}\ell^\perp}(\vec{z}_2) = \lambda\ell_{|\text{Ker}\ell^\perp}(\vec{z}_1)$, thus $\ell_{|\text{Ker}\ell^\perp}(\vec{z}_2-\lambda\vec{z}_1) = 0$, thus $\vec{z}_2-\lambda\vec{z}_1 = \vec{0}$ since $\ell_{|\text{Ker}\ell^\perp}$ is one to one. And $\vec{n}\in(\text{Ker}\ell)^\perp$ gives $\dim(\text{Ker}\ell)^\perp \ge 1$ (above proof). Thus $\dim(\text{Ker}\ell)^\perp = 1 = \text{Vect}\{\vec{n}\}$. ∴

F.2 The Riesz representation operator

The Riesz representation theorem F.1 gives the $(\cdot,\cdot)_g$-Riesz representation operator

$$\vec{R}_g : \begin{cases} E^* & \to E \\ \ell & \to \vec{R}_g(\ell) := \vec{\ell}_g, \quad \text{i.e.} \quad (\vec{R}_g(\ell),\vec{v})_g = \ell.\vec{v}, \ \ \forall \vec{v}\in E. \end{cases} \tag{F.3}$$

So $\vec{R}_g$ transforms a « covariant ℓ » into a « contravariant $\vec{\ell}_g$ » thanks to the tool $(\cdot,\cdot)_g$.

NB (fundamental): $\vec{R}_g$ is a isomorphism between the Banach spaces $(E,||.||_g)$ and $(E^*,||.||_{E^*})$, but $\vec{R}_g$ is ***not*** canonical since it requires a man made tool (an inner dot product chosen by some observer) to be defined. (An isomorphism $E \leftrightarrow E^*$ can ***never*** be canonical, see § T.2.)

And with $\mathcal{G}$ the set of inner dot products in E, we have thus defined the Riesz representation mapping

$$\vec{R} : \begin{cases} \mathcal{G}\times E^* & \to E \\ (g,\ell) & \to \vec{R}(g,\ell) := \vec{\ell}_g = \vec{R}_g(\ell) = \vec{\ell}(g). \end{cases} \tag{F.4}$$

So $\vec{R}$ has two inputs: A choice $(\cdot,\cdot)_g$ by an observer for the first slot, a linear form for the second slot.

F.3 Quantification with a basis

Here E is finite dimensional, $\dim E = n$, $\ell\in E^*$ (a linear form), $(\cdot,\cdot)_g$ is an inner dot product, $(\vec{e}_i)$ is a basis, (π_{ei}) is the dual basis (classical notations). Let

$$g_{ij} = g(\vec{e}_i,\vec{e}_j), \quad \ell = \sum_{i=1}^n \ell_i \pi_{ei}, \quad \vec{\ell}_g = \sum_{i=1}^n (\vec{\ell}_g)_i \vec{e}_i, \quad \vec{R}_g.\pi_{ej} = \sum_{i=1}^n R_{ij}\vec{e}_i, \tag{F.5}$$

i.e. $[g]_{|\vec{e}} = [g_{ij}]$, $[\ell]_{|\pi_e} = (\,\ell_1 \ \ \dots \ \ \ell_n\,)$ (row matrix), $[\vec{\ell}_g]_{|\vec{e}} = \begin{pmatrix} (\vec{\ell}_g)_1 \\ \vdots \\ (\vec{\ell}_g)_n \end{pmatrix}$ (column matrix), $[\vec{R}_g]_{|\pi_e,\vec{e}} = [R_{ij}]$.

(Duality notations: $\ell = \sum_{i=1}^n \ell_i e^i$, $\vec{\ell}_g = \sum_{i=1}^n \ell_g^i \vec{e}_i$, $\vec{R}_g.e^j = \sum_{i=1}^n R^{ij}\vec{e}_i$, $[\vec{R}_g]_{e,\vec{e}} = [R^{ij}]$.)

Proposition F.3

$$\boxed{[\vec{\ell}_g] = [g]^{-1}.[\ell]^T} \quad \textit{and} \quad \boxed{[\vec{R}_g] = [g]^{-1}}, \quad \textit{i.e.} \quad (\vec{\ell}_g)_i = \sum_{j=1}^n ([g]^{-1})_{ij}(\ell)_j = \sum_{j=1}^n (\vec{R}_g)_{ij}(\ell)_j. \tag{F.6}$$

Full matrix notation: $[\vec{\ell}_g]_{|\vec{e}} = ([g]_{|\vec{e}})^{-1}.([\ell]_{|\pi_e})^T$, *and* $[\vec{R}_g]_{|\pi_e,\vec{e}} = ([g]_{|\vec{e}})^{-1}$.

Duality notation to see the change of variance induced by $(\cdot,\cdot)_g$ *(bottom index for* ℓ*, top index for* $\vec{\ell}_g$*):*

$$\ell_g^i = \sum_{j=1}^n R^{ij}\ell_j, \tag{F.7}$$

i.e. $\ell_g^i = \sum_{j=1}^n g^{ij}\ell_j$ *when* $([g]^{-1})_{ij} =^{noted} [g^{ij}]$.

(In particular, if $(\vec{e}_i)$ *is a* $(\cdot,\cdot)_g$*-orthonormal basis, then* $[\vec{R}_g] = [g]^{-1} = I$ *and* $\ell_g^i = \ell_i$*.)*

Proof. (F.2) gives $[\ell]_{|\vec{e}}.[\vec{x}]_{|\vec{e}} = [\vec{\ell}_g]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{x}]_{|\vec{e}}$ for all $\vec{x}$, thus $[\ell]_{|\vec{e}} = [\vec{\ell}_g]_{|\vec{e}}^T.[g]_{|\vec{e}}$, thus $[g]_{|\vec{e}}.[\vec{\ell}_g]_{|\vec{e}} = [\ell]_{|\vec{e}}^T$ (since $[g]_{|\vec{e}} = [g]_{|\vec{e}}^T$), thus $[\vec{\ell}_g] = [g]^{-1}.[\ell]^T$.

And $\vec{R}_g.\ell =^{(F.3)} \vec{\ell}_g$ gives $\sum_{j=1}^n (\ell)_j \vec{R}_g.\pi_j = \sum_{i=1}^n (\vec{\ell}_g)_i\vec{e}_i$, thus $\sum_{i,j=1}^n (\ell)_j R_{ij}\vec{e}_i = \sum_{i=1}^n (\vec{\ell}_g)_i\vec{e}_i$, thus $\sum_{j=1}^n R_{ij}(\ell)_j = (\vec{\ell}_g)_i$ for all i, thus $[\vec{R}_g].[\ell]^T = [\vec{\ell}_g]$. Thus $[\vec{R}_g] = [g]^{-1}$. ⁂

Remark F.4 If a chosen inner dot product $(\cdot,\cdot)_g$ is imposed (e.g. Euclidean foot based) and if duality notations are used, then a usual notation for $\vec{\ell}_g$ is $\ell^\sharp$, since $\vec{\ell}_g = \vec{R}_g(\ell) = \sum_{i=1}^n \ell^i\vec{e}_i$ with a top index for ℓ^i: the index i has been raised through $\vec{R}_g$. Then (F.2) and (F.6) read (isometric framework)

$$\ell.\vec{x} = \ell^\sharp \bullet \vec{x} \quad \text{and} \quad [\ell^\sharp]_{|\vec{e}} = [g]_{|\vec{e}}^{-1}.[\ell]_{|\vec{e}}^T. \tag{F.8}$$

We won't use this notation (we deal with objectivity). ⁂

F.4 Change of Riesz representation vector, and Euclidean case

For one linear form $\ell \in E^*$, two observers with their inner dot products $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ get two Riesz representation vectors $\vec\ell_g = \vec R_g(\ell)$ and $\vec\ell_h = \vec R_h(\ell)$ given by, cf. (F.2):

$$\forall \vec x \in E, \quad (\vec\ell_g, \vec x)_g = \ell.\vec x = (\vec\ell_h, \vec x)_h. \tag{F.9}$$

Proposition F.5 *For any basis $(\vec e_i)$ in E, we have the change of representation vector formula:*

$$[h]_{|\vec e}.[\vec\ell_h]_{|\vec e} = [g]_{|\vec e}.[\vec\ell_g]_{|\vec e}, \quad \text{i.e.} \quad [\vec\ell_h]_{|\vec e} = [h]^{-1}_{|\vec e}.[g]_{|\vec e}.[\vec\ell_g]_{|\vec e}. \tag{F.10}$$

In particular (for the Euclidean case), with $\lambda > 0$:

$$\textit{If } (\cdot,\cdot)_g = \lambda^2(\cdot,\cdot)_h \quad \textit{then} \quad \vec\ell_h = \lambda^2 \vec\ell_g. \tag{F.11}$$

Conversely, if $\vec\ell_h = \lambda^2\vec\ell_g$ for all linear forms $\ell \in E^*$, then $(\cdot,\cdot)_g = \lambda^2(\cdot,\cdot)_h$.

*So, a linear form ℓ can**not** be identified with a Riesz representation vector (which one: $\vec\ell_g$? $\vec\ell_h$?); In other words, a Riesz representation vector $\vec R_g(\ell)$ is not objective, is not intrinsic to a linear form ℓ.*

NB: (F.10)-(F.11) is a "change of vector formula" (one linear form gives two vectors relative to two inner dot products); It is not a "change of basis formula" (for one vector and its two sets of components).

Proof. (F.9) gives $[\vec x]^T_{|\vec e}.[g]_{|\vec e}.[\vec\ell_g]_{|\vec e} = [\vec x]^T_{|\vec e}.[h]_{|\vec e}.[\vec\ell_h]_{|\vec e}$ for all $\vec x$, hence $[g]_{|\vec e}.[\vec\ell_g]_{|\vec e} = [h]_{|\vec e}.[\vec\ell_h]_{|\vec e}$, i.e. (F.10).

In particular $\lambda^2(\cdot,\cdot)_h = (\cdot,\cdot)_g$ give $\lambda^2(\vec\ell_g,\vec x)_h = (\vec\ell_g,\vec x)_g =^{(F.9)} (\vec\ell_h,\vec x)_h$ for all $\vec x$, hence $\lambda^2\vec\ell_g = \vec\ell_h$.

Converse: $\lambda^2\vec\ell_g = \vec\ell_h$ for all ℓ gives $\lambda^2(\vec\ell_g,\vec x)_h = (\vec\ell_h,\vec x)_h \overset{(F.9)}{=} (\vec\ell_g,\vec x)_g$, for all $\vec x$ and for all ℓ, thus for all $\vec\ell_g$ thanks to the isomorphism $\vec R_g : E^* \to E$, cf. (F.3), thus $\lambda^2(\cdot,\cdot)_h = (\cdot,\cdot)_g$. ⁂

Example F.6 If $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ are the Euclidean dot products made with the foot and the metre then, with (F.9),

$$(\cdot,\cdot)_g = \lambda^2(\cdot,\cdot)_h \quad \Longrightarrow \quad \vec\ell_h = \lambda^2\vec\ell_g, \quad \text{with} \quad \lambda^2 > 10: \tag{F.12}$$

$\vec\ell_g$ (English) and $\vec\ell_h$ (French) are quite different! A Riesz representation vector is subjective, and certainly not "canonical" (a word that you may find in books where... nothing is defined...).

E.g., aviation: If you do want to use a Riesz representation vector to represent a $\ell \in \mathbb{R}^{n*}$, it is vital to know which Euclidean dot product is in use, see also remark A.14 (Mars Climate Orbiter Crash). Recall: The foot is the international unit of altitude for aviation. ⁂

Example F.7 If $f \in C^1(\mathbb{R}^n;\mathbb{R})$ and $p \in \mathbb{R}^n$, the differential of f at p is the linear form $df(p) \in \mathbb{R}^{n*}$ defined by, for all $\vec w \in \vec{\mathbb{R}}^n$,

$$df(p).\vec w := \lim_{h\to 0} \frac{f(p+h\vec w) - f(p)}{h} \quad \text{(definition independent of any inner dot product)}, \tag{F.13}$$

see (S.5). If you can choose an inner dot product $(\cdot,\cdot)_g$ then the gradient $\vec{\text{grad}}_g f(p)$ is the $(\cdot,\cdot)_g$-Riesz representation vector of $df(p)$:

$$\vec{\text{grad}}_g f(p) := \vec R_g(df(p)), \quad \text{i.e.} \quad df(p).\vec w = \vec{\text{grad}}_g f(p) \bullet_g \vec w, \quad \forall \vec w \in \vec{\mathbb{R}}^n. \tag{F.14}$$

And (F.12) gives

$$\vec{\text{grad}}_h f(p) = \lambda^2 \vec{\text{grad}}_g f(p) \quad \text{with} \quad \lambda^2 > 10 \ \text{(English vs French)}: \tag{F.15}$$

The gradient is very dependent on the observer (the gradient is subjective, the differential is objective). ⁂

Remark F.8 The "gradient" is observer dependent; We already had this observer dependence for the usual derivative in the 1-D case $f : x \in \mathbb{R} \to f(x) \in \mathbb{R}$; Question: What does $f'(x) = 3$ mean?

Answer. 11- For one observer, it means $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$... but... where in the departure space this observer has chosen a basis vector $\vec a$ of length 1 for him (e.g. length 1 foot) which he calls 1; So, with no abusive notations, his derivative $f'(x)$ is in fact $f'_a(x) = \lim_{h\to 0} \frac{f(x+h\vec a)-f(x)}{h}$.

12- For some other observer, it means $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$... but... where in the departure space this observer has chosen a basis vector $\vec{b}$ of length 1 for him (e.g. length 1 metre) which he calls 1; So, with no abusive notations, his derivative $f'(x)$ is in fact $f'_b(x) = \lim_{h\to 0} \frac{f(x+h\vec{b})-f(x)}{h}$.

13- Both observer use the same formula $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ but get different results! Indeed, if $\vec{b} = \lambda\vec{a}$, then $= \lim\limits_{h\to 0} \dfrac{f(x+h\vec{b})-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x+h\lambda\vec{a})-f(x)}{h} = \lambda \lim\limits_{h\to 0} \dfrac{f(x+(h\lambda)\vec{a})-f(x)}{(h\lambda)} = \lambda \lim\limits_{k\to 0} \dfrac{f(x+k\vec{a})-f(x)}{k}$ thus

$$f'_b(x) = \lambda f'_a(x), \quad \text{with} \quad \lambda \simeq 3.28 \tag{F.16}$$

with foot and metre... Quite different results! (In fact $f'(x) = \dfrac{\text{opposite side}}{\text{adjacent side}}$ depends on the length of the adjacent side...) ∴

Remark F.9 We insist on the subjectivity of the gradient:

20- The differential of f at a point x along a vector $\vec{w} \in \vec{\mathbb{R}}$ is $df(x).\vec{w} = \lim_{h\to 0} \frac{f(x+h\vec{w})-f(x)}{h}$ and is objective: The observers all use this same formula.

21- An observer chooses a Euclidean dot product $(\cdot,\cdot)_g$ (e.g. based on the foot), then represent $df(x)$ by its $(\cdot,\cdot)_g$-Riesz representation vector $\vec{R}_g(df(x)) =^{\text{noted}} \vec{\text{grad}}_g f(x)$ called the gradient of f at x relative to $(\cdot,\cdot)_g$.

22- Another observer chooses a Euclidean dot product $(\cdot,\cdot)_h$ (e.g. based on the metre), then represent $df(x)$ by its $(\cdot,\cdot)_h$-Riesz representation vector $\vec{R}_h(df(x)) =^{\text{noted}} \vec{\text{grad}}_h f(x)$ called the gradient of f at x relative to $(\cdot,\cdot)_h$.

23- Both observer use the same formula $df(x).\vec{w} = \lim_{h\to 0} \frac{f(x+h\vec{w})-f(x)}{h}$ to get a different result: $\vec{\text{grad}}_h f = \lambda^2 \vec{\text{grad}}_g f$, because they use different measuring tools (one based on the foot, the other on the metre).

24- Recall: The gradient depends on a choice of a Euclidean unit. ∴

Exercice F.10 In (F.16) we have $f'_b(x) = \lambda f'_a(x)$. And the 1-D gradient gives $\text{grad}_b f(x) = \lambda^2 \text{grad}_a f(x)$. Why?

Answer. To define a gradient $\text{grad}_a f$ we need a Euclidean dots products $(\cdot,\cdot)_a$ built from a basis $(\vec{a})$ in $\vec{\mathbb{R}}$, while to define f'_a we need a unit of length. Details: $(\vec{a})$ and $(\vec{b})$ are two bases in $\vec{\mathbb{R}}$ with $\vec{b} = \lambda\vec{a}$, thus $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$ (since $1 = (\vec{a},\vec{a})_a = (\vec{b},\vec{b})_b = (\lambda\vec{a},\lambda\vec{a})_b = \lambda^2(\vec{a},\vec{a})_b$ gives $(\vec{a},\vec{a})_a = \lambda^2(\vec{a},\vec{a})_b$ and $(\vec{a})$ is a basis). And we have $f'_b(x) =^{(F.16)} \lambda f'_a(x)$, i.e. $df(x).\vec{b} = \lambda df(x).\vec{a}$, thus $(\text{grad} f_b(x), \vec{b})_b = \lambda(\text{grad} f_a(x), \vec{a})_a$, thus $(\text{grad} f_b(x), \vec{b})_b = \lambda\lambda^2(\text{grad} f_a(x), \frac{\vec{b}}{\lambda})_b = (\lambda^2 \text{grad} f_a(x), \vec{b})_b$, thus $\text{grad} f_b(x) = \lambda^2 \text{grad} f_a(x)$. ∴

Exercice F.11 1- Prove that $(\cdot,\cdot)_g = \lambda^2(\cdot,\cdot)_h$ gives $||\vec{\ell}_h||_g = \lambda||\vec{\ell}_h||_h$. 2- Does it contradict the Riesz representation theorem which gives $||\ell||_{\mathbb{R}^{n*}} = ||\vec{\ell}_g||_{\mathbb{R}^n}$?

Answer. 1- $\vec{\ell}_h =^{(F.11)} \lambda^2\vec{\ell}_g$ gives $||\vec{\ell}_h||_h = \lambda^2||\vec{\ell}_g||_h = \lambda||\vec{\ell}_g||_g$ since $||.||_h = \lambda||.||_g$.

2- No, since $||\ell||_{\mathbb{R}^{n*}} := \sup_{||\vec{x}||_{\mathbb{R}^n}=1} |\ell.\vec{x}|$ depends on the norm $||.||_{\mathbb{R}^n}$ chosen; Here $||.||_{\mathbb{R}^n}$ is either $||.||_g$ or $||.||_h$. Thus if you write $||\ell||_{\mathbb{R}^{n*}} =^{\text{noted}} ||\ell||_{g*}$ if you use the norme $||.||_g$, then $||\ell||_{h*} = \sup_{\vec{v}\in\vec{\mathbb{R}}^n} \frac{|\ell.\vec{v}|}{||\vec{v}||_h} = \sup_{\vec{v}\in\vec{\mathbb{R}}^n} \frac{|\ell.\vec{v}|}{\frac{1}{\lambda}||\vec{v}||_g} = \lambda \sup_{\vec{v}\in\vec{\mathbb{R}}^n} \frac{|\ell.\vec{v}|}{||\vec{v}||_g} = \lambda||\ell||_{g*}$. ∴

F.5 A Riesz representation vector is contravariant

$\vec{\ell}_g$ is a vector in E, cf. (F.2), so it is contravariant. To be convinced:

Exercice F.12 Check:

$$[\vec{\ell}_g]_{|new} = P^{-1}.[\vec{\ell}_g]_{|old} \quad \text{(contravariance formula).} \tag{F.17}$$

Answer. Consider two bases $(\vec{e}_{old,i})$ and $(\vec{e}_{new,i})$ in E. With the change of basis formulas $[\vec{x}]_{|new} = P^{-1}.[\vec{x}]_{|old}$ and $[g]_{|new} = P^T.[g]_{|old}.P$, (F.2) gives (with (A.94)), for all $\vec{x}$,

$$\begin{aligned} [\vec{x}]^T_{|old}.[g]_{|old}.[\vec{\ell}_g]_{|old} = \ell.\vec{x} &= [\vec{x}]^T_{|new}.[g]_{|new}.[\vec{\ell}_g]_{|new} \\ &= ([\vec{x}]^T_{|old}.P^{-T}).(P^T.[g]_{|old}.P).[\vec{\ell}_g]_{|new} = [\vec{x}]^T_{|old}.[g]_{|old}.(P.[\vec{\ell}_g]_{|new}), \end{aligned} \tag{F.18}$$

thus $[\vec{\ell}_g]_{|old} = P.[\vec{\ell}_g]_{|new}$ since $[g]$ is invertible (an inner dot product is positive definite), thus (F.17). ∴

Remark F.13 • Dont forget: A representation vector $\vec{\ell}_g$ is not intrinsic to the linear form ℓ because it depends on a $(\cdot,\cdot)_g$ (depends on a observer: foot? metre?)). Reminder: there is no natural canonical isomorphism between E and E^*, i.e. it is impossible to identify a linear form with a vector, see § T.2.
- $\vec{\ell}_g$ is incompatible with the use of push-forwards, cf. § 7.2.
- $\vec{\ell}_g$ is incompatible with the use of Lie derivatives, cf. (9.51). ▪

F.6 What is a vector versus a $(\cdot,\cdot)_g$-vector?

1- Originally, a vector was a bipoint vector $\vec{v} = \overrightarrow{AB}$ in $\mathbb{R}^3$ used to represent of a "material object". E.g. the height of a child is represented on a wall by a vertical bipoint vector $\vec{x}$ starting from the ground up to a pencil line. The vector $\vec{x}$ is objective: Any observer uses this same vector to get the height of the child... and then use "their subjective unit" (foot, metre...) to give a value.

2- Then (mid 19th century), the concept of vector space was introduced: It is a quadruplet $(E,+,K,.)$ where $+$ is an inner law, $(E,+)$ is a group, K is a field, . is a external law on E (called a scalar multiplication) compatible with $+$ (see any math book).

And then the concept of scalar inner dot product (in a vector space) was introduced.

3- We can then get non "material" vectors ("subjectively built vectors"). E.g.: start with our usual vector space $\vec{\mathbb{R}}^n$ of bi-point vectors, then consider its dual $(\mathbb{R}^{n*},+,\mathbb{R},.) =^{\text{noted}} \mathbb{R}^{n*}$. Then, for a given $\ell \in \mathbb{R}^{n*}$ (a given measuring device), consider two observers: An English observer with his foot built Euclidean dot product $(\cdot,\cdot)_g$, and a French observer with with his metre built Euclidean dot product $(\cdot,\cdot)_h$. These observers build their own artificial Riesz representation vectors $\vec{\ell}_g = \vec{R}_g(\ell) \in \vec{\mathbb{R}}^n$ and $\vec{\ell}_h = \vec{R}_h(\ell)$, cf (F.12); They remark that $\vec{\ell}_g \neq \vec{\ell}_h$: These constructions are very subjective.

4- Then, with differential geometry, a vector $\vec{v}$ has been redefined: It is a "tangent vector", which means that there exists a C^1 curve $c : s \in [a,b] \to c(s) \in E$ such that $\vec{v}$ is defined at a $p = c(s) \in \text{Im}(c)$ by $\vec{v}(p) := \vec{c}\,'(s)$ (so a vector is part of a vector field, here defined along the range of c). (This definition of a tangent vector is applicable to "tangent vectors to a surface" i.e. tangent vectors to a manifold, see e.g. § 9.1.1,2-.) Then it is shown that $\vec{v}$ is equivalent to $\frac{\partial}{\partial \vec{v}}$ = the directional derivative in the direction $\vec{v}$ (natural canonical isomorphism between E and E^{**}). For other equivalent definitions of vectors, see Abraham–Marsden [1].

F.7 The "$(\cdot,\cdot)_g$-dual vectorial bases" of one basis (and warnings)

Framework: E is a finite dimensional vector space, $\dim E = n$ (e.g. $E = \vec{\mathbb{R}}^3$). An observer chooses an inner dot product $(\cdot,\cdot)_g$ (e.g., in $\vec{\mathbb{R}}^3$, a foot-built Euclidean dot product). Hence the results will be subjective. And $(\vec{e}_i)$ is some basis in E.

F.7.1 A basis and its many associated "dual vectorial basis"

Definition F.14 The $(\cdot,\cdot)_g$-dual vectorial basis (or $(\cdot,\cdot)_g$-vectorial dual basis, or $(\cdot,\cdot)_g$-dual basis) of the basis $(\vec{e}_i)$ is the basis $(\vec{e}_{ig})$ in E defined by

$$\forall j = 1,...,n, \quad (\vec{e}_{ig}, \vec{e}_j)_g = \delta_{ij}, \quad \text{i.e.} \quad \boxed{\vec{e}_{ig} \bullet_g \vec{e}_j = \delta_{ij}}. \tag{F.19}$$

NB: A vectorial dual basis is not unique: It depends on the chosen inner dot product, see e.g. (F.22).

NB: Pay attention to the notations: $\vec{e}_{ig}$ is a contravariant vector ($\vec{e}_{ig} \in E$), so, even if you use the Einstein convention, the index $_i$ in $\vec{e}_{ig}$ must be a bottom index.

Let (π_{ei}) be the (covariant) dual basis of the basis $(\vec{e}_i)$, i.e. the $\pi_{ei} \in E^*$ are the objective (the same for all observers) linear forms defined by $\pi_{ei}.\vec{e}_j = \delta_{ij}$ for all j, cf. (A.7).

Definition F.15 (Equivalent definition.) The $(\cdot,\cdot)_g$-dual vectorial basis of the basis $(\vec{e}_i)$ is the basis $(\vec{e}_{ig})$ in E made of the $(\cdot,\cdot)_g$-Riesz representative vectors of the π_{ei}, i.e.

$$\boxed{\vec{e}_{ig} := \vec{R}_g(\pi_{ei})}, \quad \text{i.e.} \quad \vec{e}_{ig} \bullet_g \vec{v} = \pi_{ei}.\vec{v}, \quad \forall \vec{v} \in E. \tag{F.20}$$

where $\vec{R}_g$ is the $(\cdot,\cdot)_g$-Riesz operator, see (F.3).

With duality notations, (e^i) is the dual basis and $\vec{e}_{ig} := \vec{R}_g(e^i)$, i.e. $(\vec{e}_{ig}, \vec{v})_g = e^i.\vec{v}$ for all $\vec{v} \in E$ where here the position of the index i is bottom on the left and up on the right, since $\vec{R}_g$ changes a covariant vector (a linear form) into a contravariant vector.

Exercice F.16 Prove that the vectors $\vec{e}_{ig}$ satisfy the contravariant change of basis formula

$$[\vec{e}_{ig}]_{|new} = P^{-1}.[\vec{e}_{ig}]_{|old} \tag{F.21}$$

(the $\vec{e}_{jg}$ are indeed "contravariant vectors").

Answer. • First answer: $\vec{e}_{ig}$ is a vector in E, thus it is contravariant.

• Second answer: Apply (F.17) since $\vec{e}_{ig}$ is a Riesz-representation vector.

• Third answer = direct computation: Consider two bases $(\vec{a}_i)$ and $(\vec{b}_i)$, and the transition matrix P from $(\vec{a}_i)$ to $(\vec{b}_i)$, i.e., $\vec{b}_j = \sum_{i=1}^n P_{ij}\vec{a}_i$ for all j. (F.19) and the change of basis formulas for the vectors $\vec{e}_i$ and the bilinear form $(\cdot,\cdot)_g$ give $[\vec{e}_j]_{|\vec{a}}^T.[g]_{|\vec{a}}.[\vec{e}_{ig}]_{|\vec{a}} = (\vec{e}_{ig},\vec{e}_j)_g = [\vec{e}_j]_{|\vec{b}}^T.[g]_{|\vec{b}}.[\vec{e}_{ig}]_{|\vec{b}} = (P^{-1}.[\vec{e}_j]_{|\vec{a}})^T.(P^T.[g]_{|\vec{a}}.P).[\vec{e}_{ig}]_{|\vec{a}} = [\vec{e}_j]_{|\vec{a}}^T.[g]_{|\vec{a}}.P.[\vec{e}_{ig}]_{|\vec{a}}$ for all i,j, thus $[\vec{e}_{ig}]_{|\vec{a}} = P.[\vec{e}_{ig}]_{|\vec{b}}$, thus (F.21). ∎

Exercice F.17 Consider two inner dot products $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ (e.g., a foot-built and a metre-built Euclidean dot product), and a basis $(\vec{e}_i)$ in E. Call $(\vec{e}_{ia})$ and $(\vec{e}_{ib})$ the $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$-dual vectorial bases. Prove:

$$(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b \quad\Longrightarrow\quad \vec{e}_{ib} = \lambda^2\vec{e}_{ia}, \quad \forall i. \tag{F.22}$$

E.g., $\lambda^2 > 10$ with foot and metre built Euclidean bases: $\vec{e}_{ib}$ is very different from $\vec{e}_{ia}$! A dual vectorial basis highly depends on an observer: A vectorial dual basis is ***not*** intrinsic to $(\vec{e}_i)$ (***not*** objective).

Answer. (F.19) gives $(\vec{e}_{ib},\vec{e}_j)_b = \delta_{ij} = (\vec{e}_{ia},\vec{e}_j)_a = \lambda^2(\vec{e}_{ia},\vec{e}_j)_b$, thus $(\vec{e}_{ib} - \lambda^2\vec{e}_{ia},\vec{e}_j)_b = \delta_{ij}$, for all i,j. ∎

Example F.18 If $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis we trivially get $\vec{e}_{ig} = \vec{e}_i$ for all i, i.e., $(\vec{e}_{ig}) = (\vec{e}_i)$.This particular case is not compatible with joint work by an English (foot) and French (metre) observer. ∎

F.7.2 Components of $\vec{e}_{jg}$ in the basis $(\vec{e}_i)$

Proposition F.19 *The components of $\vec{e}_{jg}$ in the basis $(\vec{e}_i)$ are given by, for any $j \in [1,n]_{\mathbb{N}}$,*

$$\boxed{[\vec{e}_{jg}]_{|\vec{e}} = [\vec{R}_g]_{|\vec{e}}.[\vec{e}_j]_{|\vec{e}}} = ([g]_{|\vec{e}})^{-1}.[\vec{e}_j]_{|\vec{e}} = \text{ the } j\text{-th column of } ([g]_{|\vec{e}})^{-1}, \tag{F.23}$$

i.e. the i-th component of $\vec{e}_{jg}$ is $([g]_{|\vec{e}}^{-1})_{ij}$.

Thus the matrix of $g(\cdot,\cdot)$ in the basis $(\vec{e}_{ig})$ is the inverse of the matrix of $g(\cdot,\cdot)$ in the basis $(\vec{e}_i)$:

$$([g(\vec{e}_{ig},\vec{e}_{jg})] =) \quad [g]_{|(\vec{e}_{ig})} = [g]_{|(\vec{e}_i)}{}^{-1} \quad (= ([g(\vec{e}_i,\vec{e}_j)])^{-1}). \tag{F.24}$$

Proof. First proof of (F.23) (straight forward calculation): (F.19) gives, for all i,j,

$$[\vec{e}_j]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{e}_{ig}]_{|\vec{e}} = \delta_{ij} = [\vec{e}_j]_{|\vec{e}}^T.[\vec{e}_i]_{|\vec{e}}, \quad \text{thus} \quad [g]_{|\vec{e}}.[\vec{e}_{ig}]_{|\vec{e}} = [\vec{e}_i]_{|\vec{e}}. \tag{F.25}$$

Second proof of (F.23): Apply (F.6) (generic Riesz representation result) to get (F.23).

Thus, $[g]_{|\vec{e}}$ being symmetric we have $[g]_{|\vec{e}}{}^{-1}$ symmetric, and $g(\vec{e}_{ig},\vec{e}_{jg}) = [\vec{e}_{ig}]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{e}_{jg}]_{|\vec{e}} = [\vec{e}_i]_{|\vec{e}}^T.[g]_{|\vec{e}}{}^{-1}.[g]_{|\vec{e}}.[g]_{|\vec{e}}{}^{-1}.[\vec{e}_j]_{|\vec{e}} = [\vec{e}_i]_{|\vec{e}}^T.[g]_{|\vec{e}}{}^{-1}.[\vec{e}_j]_{|\vec{e}} = ([g]_{|\vec{e}}{}^{-1})_{ij}$, thus (F.24). ∎

Example F.20 $\vec{\mathbb{R}^2}$, $[g]_{|\vec{e}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, thus $[g]_{|\vec{e}}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. Thus $\vec{e}_{1g} = \vec{e}_1$, $\vec{e}_{2g} = \frac{1}{2}\vec{e}_2$. ∎

Remark F.21 Warning: When $([g]_{|\vec{e}}^{-1})_{ij} \overset{\text{noted}}{=} g^{ij}$ then (F.23) reads

$$\vec{e}_{jg} = \sum_{i=1}^n g^{ij}\vec{e}_i, \tag{F.26}$$

where the Einstein convention is ***not*** satisfied: The Einstein convention is satisfied with

$$\vec{e}_{jg} = \sum_{i=1}^n (\vec{e}_{jg})^i\vec{e}_i \overset{\text{noted}}{=} \sum_{i=1}^n (P_j)^i\vec{e}_i \tag{F.27}$$

(the components of vectors have up indices), and this can be verified with $(\vec{e}_{ig},\vec{e}_j)_g =^{(F.19)} \delta_{ij}$ which gives $\sum_{k,\ell=1}^n (P_i)^k g_{kj} = \delta_{ij}$. And in (F.26) the scalars g^{ij} is just another name for $(P_j)^i$, nothing more (nothing to do with the Einstein convention).

We insist:In other words: $M = [g]_{|\vec e} = [M_{ij}]$ is a matrix, and its inverse is the matrix $M^{-1} = [M_{ij}]^{-1}$: A matrix is just a collection of scalars (has nothing to do with the Einstein convention), and its inverse is also a collection of scalars, and you do not change this fact by calling $M^{-1} =^{\text{noted}} [M^{ij}]$ (the use of up indices is irrelevant for matrices). See remark A.50.

And because $(P_j)^i$ equals $([g]^{-1}_{|\vec e})_{ij} =^{\text{noted}} g^{ij}$, some people rename $\vec e_{jg}$ as $\vec e^{\,j}$... to get $\vec e^{\,j} = \sum_{i=1}^n g^{ij}\vec e_i$... But doing so they ***despise*** Einstein's convention, despite eventual claims: They confuse covariance and contravariance... and add confusion to the confusion.

NB: Recall: If in trouble with a notation which comes as a surprise (the notation g^{ij} here), use classical notations: Then no misuse of Einstein's convention and no possible misinterpretation. In particular here $\vec e_{jg}$ is a (contravariant) vector.

⁂

F.7.3 Multiple admissible notations for the components of $\vec e_{jg}$

Let $\mathcal{P} \in \mathcal{L}(E;E)$ be the change of basis endomorphism from $(\vec e_i)$ to $(\vec e_{ig})$: defined by $\mathcal{P}.\vec e_j = \vec e_{jg}$. And let $P = [\mathcal{P}]_{|\vec e}$ (the associated transition matrix). It gives multiple admissible (non confusing) notations for the components of $\vec e_{jg}$ relative to the basis $(\vec e_i)$:

$$\vec e_{jg} = \mathcal{P}.\vec e_j = \underbrace{\sum_{j=1}^{n} P_{ij}\vec e_i = \sum_{j=1}^{n} (P_j)_i\vec e_i}_{\text{clas.}} = \underbrace{\sum_{j=1}^{n} (P_j)^i\vec e_i = \sum_{j=1}^{n} P^i{}_j\vec e_i}_{\text{dual}}, \tag{F.28}$$

i.e. the i-th component of the vector $\vec e_{jg}$ has the names $P_{ij} = (P_j)_i = (P_j)^i = P^i{}_j$, i.e. $P = [\mathcal{P}]_{|\vec e} = [P_{ij}] = [(P_j)_i] = [(P_j)^i] = [P^i{}_j]$ (four different notations for the same matrix), i.e.

$$\forall j, \quad [\vec e_{jg}]_{|\vec e} = [\mathcal{P}]_{|\vec e}.[\vec e_j]_{|\vec e} = \begin{pmatrix} P_{1j} \\ \vdots \\ P_{nj} \end{pmatrix} = \begin{pmatrix} (P_j)_1 \\ \vdots \\ (P_j)_n \end{pmatrix} = \begin{pmatrix} P^1{}_j \\ \vdots \\ P^n{}_j \end{pmatrix} = \begin{pmatrix} (P_j)^1 \\ \vdots \\ (P_j)^n \end{pmatrix} \tag{F.29}$$

= the j-th column of $[\mathcal{P}]_{|\vec e}$. You can choose any notation, depending on your current need or mood...

F.7.4 (Huge) differences between “the (covariant) dual basis” and “a dual vectorial basis”

1. A basis $(\vec e_i)$ has an infinite number of vectorial dual bases $(\vec e_{ig})$, as many as the number of inner dot products $(\cdot,\cdot)_g$ (as many as observers), see (F.23).

   While a basis $(\vec e_i)$ has a unique intrinsic (covariant) dual basis $(\pi_{ei}) \overset{\text{noted}}{=} (e^i)$, cf. (A.7): Two observers who consider the same basis $(\vec e_i)$ have the same (covariant) dual basis.

2. $\pi_{ei} = e^i$ is covariant, while $\vec e_i$ and $\vec e_{ig}$ are contravariant. And there is no transition matrix between $(\vec e_i)$ and $(\pi_{ei}) = (e^i)$, since $\vec e_i \in E$ and $\pi_{ei} = e^i \in E^*$ don't live in the same vector space.

3. If you fly, it is vital to use the dual basis $(\pi_{ei}) = (e^i)$: It is possibly fatal if you confuse foot and metre at takeoff and at landing (if you survived takeoff) because of the choice of different Euclidean dot product $(\cdot,\cdot)_g$ or $(\cdot,\cdot)_h$; See e.g. the Mars Climate Orbiter crash, remark A.14. Einstein's convention can help... only if it is really followed.

F.7.5 About the notation g^{ij} = shorthand notation for $(g^\sharp)^{ij}$

Definition F.22 The Riesz associated inner dot product $g^\sharp \in \mathcal{L}(E^*, E^*; \mathbb{R})$ is the bilinear form defined by, for all $\ell, m \in E^*$,

$$g^\sharp(\ell, m) := g(\vec\ell_g, \vec m_g), \quad \text{i.e.} \quad (\ell, m)_{g^\sharp} := (\vec\ell_g, \vec m_g)_g. \tag{F.30}$$

where $\vec\ell_g = \vec R_g(\ell)$ and $\vec m_g = \vec R_g(m)$.

Thus $g^\sharp(\cdot,\cdot) =^{\text{noted}} (\cdot,\cdot)_{g^\sharp}$ is indeed an inner dot product in E^*: trivial check.

Quantification: $(\vec e_i)$ is a basis in E and (e^i) is its dual basis (duality notations). (F.30) gives:

$$(g^\sharp)^{ij} := g^\sharp(e^i, e^j) = g(\vec e_{ig}, \vec e_{jg}), \quad \text{thus} \quad [g^\sharp]_{|e} = \boxed{[(g^\sharp)^{ij}] = [g_{ij}]^{-1}} = [g]^{-1}_{|\vec e}, \tag{F.31}$$

cf. (F.24). And

$$\boxed{[(g^\sharp)^{ij}] \underset{\text{notation}}{\overset{\text{shorthand}}{=}} [g^{ij}]}. \tag{F.32}$$

Classical notations: $[g^\sharp]_{|e} = [(g^\sharp)_{ij}] = [g^\sharp(\pi_{ei}, \pi_{ej})] = [g(\vec e_{ig}, \vec e_{jg})] = [g_{ij}]^{-1} = ([g]_{|\vec e})^{-1}$.

Exercice F.23 How do we compute $g^\sharp(\ell,m)$ with matrix computations?

Answer. $\ell = \sum_{i=1}^n \ell_i e^i$ and $m = \sum_{j=1}^n m_j e^j$ give $g^\sharp(\ell,m) = \sum_{i,j=1}^n \ell_i m_j g^\sharp(e^i,e^j) = \sum_{i,j=1}^n \ell_i (g^\sharp)^{ij} m_j = [\ell]_{|\vec{e}}.[g^\sharp]_{|\vec{e}}.[m]_{|\vec{e}}^T = [\ell]_{|\vec{e}}.[g]_{|\vec{e}}^{-1}.[m]_{|\vec{e}}^T$ (a linear form is represented by a row matrix,). ■

Exercice F.24 (F.30) tells that the $\binom{2}{0}$ tensor $g^\sharp \in \mathcal{L}(E^*,E^*;\mathbb{R})$ was created from the $\binom{0}{2}$ tensor $g = (\cdot,\cdot)_g \in \mathcal{L}(E,E;\mathbb{R})$ using twice the $(\cdot,\cdot)_g$-Riesz representation theorem.

1- Show that if you use the $(\cdot,\cdot)_g$-Riesz representation theorem just once you get the $\binom{1}{1}$ tensor $g^\natural \in \mathcal{L}(E^*,E;\mathbb{R}) \simeq \mathcal{L}(E;E)$ given by

$$g^\natural = I. \tag{F.33}$$

2- Reciprocal: What is the $\binom{0}{2}$ tensor $g^\flat \in \mathcal{L}(E,E;\mathbb{R})$ that you create from the identity $I \in \mathcal{L}(E;E)$ when using the $(\cdot,\cdot)_g$-Riesz representation theorem once?

3- Summary: $\widetilde{I} = g^\natural$ gives $(\widetilde{I})^\flat = g^\flat = g$ and $(\widetilde{I})^\sharp = g^\sharp$

Answer. 1- $g^\natural \in \mathcal{L}(E^*,E;\mathbb{R})$ is defined by $g^\natural(\ell,\vec{w}) = (\vec{\ell}_g,\vec{w})_g$ for all $(\ell,\vec{w}) \in E^* \times E$, where $\vec{\ell}_g$ is the $(\cdot,\cdot)_g$-Riesz representation vector of ℓ. Thus $g^\natural(\ell,\vec{w}) = \ell.\vec{w} = \ell.I.\vec{w}$, for all $(\ell,\vec{w}) \in E^* \times E$, hence $g^\natural \in \mathcal{L}(E^*,E;\mathbb{R})$ is naturally canonically associated with the identity $I \in \mathcal{L}(E;E)$.

2- The identity operator $I \in \mathcal{L}(E;E)$ (observer independent) is naturally canonically associated with the $\binom{1}{1}$ tensor $\widetilde{I} \in \mathcal{L}(E^*,E;\mathbb{R})$ defined by $\widetilde{I}(\ell,\vec{w}) = \ell.I.\vec{w} = \ell.\vec{w}$ for all $(\ell,\vec{w}) \in E^* \times E$, thus $\widetilde{I} = g^\natural$. ■

G Cauchy–Green deformation tensor $C = F^T.F$

Framework: $\widetilde{\Phi} : \left\{ \begin{array}{ll} \mathbb{R} \times Obj & \to \mathbb{R}^n \\ (t, P_{Obj}) & \to \widetilde{\Phi}(t,P_{Obj}) \end{array} \right\}$ is a motion of Obj, $\Omega_\tau = \widetilde{\Phi}(\tau,P_{Obj})$ is the configuration of Obj at any τ, t_0 and t are fixed, $\Phi := \Phi_t^{t_0} : \left\{ \begin{array}{ll} \Omega_{t_0} & \to \Omega_t \\ P & \to p = \Phi(P) \end{array} \right\}$ is the associated motion between t_0 and t, and $F(P) := d\Phi(P) : \left\{ \begin{array}{ll} \vec{\mathbb{R}}^n_{t_0} & \to \vec{\mathbb{R}}^n_t \\ \vec{W} & \to \vec{w} = F(P).\vec{W} := \lim\limits_{h\to 0} \dfrac{\Phi(P{+}h\vec{W}) - \Phi(P)}{h} \end{array} \right\}$ is the deformation gradient at P between t_0 and t, cf. (4.2).

G.0 Goal

Construction of C (summary of Cauchy's approach):

1- At t_0, consider two vectors $\vec{W}_1$ and $\vec{W}_2$,

2- at t, they are distorted by the motion and become the vectors $F.\vec{W}_1$ and $F.\vec{W}_2$;

3- Then choose a Euclidean dot product $(\cdot,\cdot)_g \overset{\text{noted}}{=} \cdot \bullet \cdot$, the same at all t (to simplify);

4- Then, by definition of the transposed, $(F.\vec{W}_1) \bullet (F.\vec{W}_2) = (F^T.F.\vec{W}_1) \bullet \vec{W}_2$: You have got the Cauchy strain tensor $C := F^T.F$;

5- Then $(F.\vec{W}_1) \bullet (F.\vec{W}_2) - \vec{W}_1 \bullet \vec{W}_2 = ((C{-}I).\vec{W}_1) \bullet \vec{W}_2$ gives a measure of the deformation with $\vec{W}_2$ as a reference, measure that is used to build first order constitutive laws for the stress (Cauchy).

G.1 Transposed F^T: Inner dot products required

We first give the functional definition of F^T (qualitative); Then we get the usual matrix representation of F^T relative to observers (quantification).

G.1.1 Definition of the function F^T

At t_0, a past observer chose an inner dot product $(\cdot,\cdot)_G$ in $\vec{\mathbb{R}}^n_{t_0}$, and at t the present observer chooses an inner dot product $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$.

By definition, the transposed of the linear map $F(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ relative to $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$ is the linear map $F(P)^T_{Gg} \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_{t_0})$ defined by, for all $\vec{U}_P \in \vec{\mathbb{R}}^n_{t_0}$ (vector at P) and $\vec{w}_p \in \vec{\mathbb{R}}^n_t$ (vector at p),

$$(F(P)^T_{Gg}.\vec{w}_p,\vec{U}_P)_G = (F(P).\vec{U}_P,\vec{w}_p)_g, \quad \text{in short} \quad \boxed{(F^T_{Gg}.\vec{w},\vec{U})_G = (F.\vec{U},\vec{w})_g}, \tag{G.1}$$

see (A.68). This defines $F^T_{Gg}(p) := F(P)^T_{Gg}$ when $p = \Phi(P)$:

$$F^T_{Gg} : \left\{ \begin{array}{ll} \Omega_t & \to \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0}) \\ p & \to \boxed{F^T_{Gg}(p) := F(P)^T_{Gg}} \end{array} \right\}, \quad \text{so in short} \quad \boxed{(F^T.\vec{w}) \bullet_G \vec{U} = \vec{w} \bullet_g (F.\vec{U})}, \tag{G.2}$$

without forgetting that $F^T := F^T_{Gg}$ depends on $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$.

Exercice G.1 1. In $F^T.\vec{z}.\vec{W} = \vec{z}.F.\vec{W} = F.\vec{W}.\vec{z} = \vec{W}.F^T.\vec{z}$, which dots are inner dot products?
2. What does $F.\vec{W}_1.F.\vec{W}_2 = \vec{W}_1.F^T.F.\vec{W}_2$ mean?

Answer. 1. No choice: $(\vec{W}, \vec{z}) \in \vec{\mathbb{R}}^n_{t_0} \times \vec{\mathbb{R}}^n_t$, so $(F^T.\vec{z}) \bullet_G \vec{W} = \vec{z} \bullet_g (F.\vec{W}) = (F.\vec{W}) \bullet_g \vec{z} = \vec{W} \bullet_G (F^T.\vec{z})$.
2. No choice: $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$, so $(F.\vec{W}_1) \bullet_g (F.\vec{W}_2) = \vec{W}_1 \bullet_G (F^T.(F.\vec{W}_2))$. ∴

Remark G.2 More generally, on a surface Ω (a manifold), (G.1) is defined for all $(\vec{U}_P, \vec{w}_p) \in T_P\Omega_{t_0} \times T_p\Omega_t$, where $T_{p_\tau}\Omega_\tau$ is the tangent space at Ω_τ at p_τ. ∴

G.1.2 Quantification with bases (matrix representation)

Classical notations: $(\vec{a}_i)$ is a basis in $\vec{\mathbb{R}}^n_{t_0}$, and $(\vec{b}_i)$ is a basis in $\vec{\mathbb{R}}^n_t$. Marsden–Hughes duality notations: $(\vec{E}_I)$ is a basis in $\vec{\mathbb{R}}^n_{t_0}$ and $(\vec{e}_i)$ is a basis in $\vec{\mathbb{R}}^n_t$. And the reference to the points P and p is omitted to lighten the writings (use the full notation of § G.1.1 if in doubt).

Let $[G] := [(\vec{a}_i, \vec{a}_j)_G]$, $[g] := [(\vec{b}_i, \vec{b}_j)_g]$, $[F]_{|\vec{a},\vec{b}} = [F_{ij}] =^{\text{noted}} [F]$, $[F^T]_{|\vec{b},\vec{a}} = [(F^T)_{ij}] =^{\text{noted}} [F^T]$.

(G.1) gives $[\vec{U}]^T.[G].[F^T.\vec{w}] = [F.\vec{U}]^T.[G].[\vec{w}]$, thus $[\vec{U}]^T.[G].[F^T].[\vec{w}] = [\vec{U}]^T.[F]^T.[g].[\vec{w}]$, for all $\vec{U}, \vec{w}$, thus

$$[G].[F^T] = [F]^T.[g], \quad \text{i.e.} \quad \boxed{[F^T] = [G]^{-1}.[F]^T.[g]}. \tag{G.3}$$

Remark G.3 If $(\vec{a}_i)$ and $(\vec{b}_i)$ are $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$-orthonormal bases, then $[C] = [F]^T.[F]$. But recall: If you need to work with a coordinate system, then the bases in use are the coordinate system bases which are ***not*** orthonormal in general, i.e. $[G]^{-1} \neq I$ and $[g]^{-1} \neq I$ in general. ∴

Exercice G.4 Use classical notation, then Marsden duality notations, to express (G.3) with components.

Answer. Classical notations:

$$\begin{gathered} G_{ij} = G(\vec{a}_i, \vec{a}_j), \quad g_{ij} = g(\vec{b}_i, \vec{b}_j), \quad \text{i.e.} \quad [G]_{\vec{a}} = [G_{ij}], \quad [g]_{|\vec{b}} = [g_{ij}], \\ \text{and} \quad F.\vec{a}_j = \sum_{i=1}^n F_{ij}\vec{b}_i, \quad F^T.\vec{b}_j = \sum_{i=1}^n (F^T)_{ij}\vec{a}_i, \quad \text{i.e.} \quad [F]_{|\vec{a},\vec{b}} = [F_{ij}], \quad [F^T]_{|\vec{b},\vec{a}} = [(F^T)_{ij}]. \end{gathered} \tag{G.4}$$

Then $(F^T.\vec{b}_j, \vec{a}_i)_G =^{(G.1)} (\vec{b}_j, F.\vec{a}_i)_g$ gives $(\sum_{k=1}^n (F^T)_{kj}\vec{a}_k, \vec{a}_i)_G = (\vec{b}_j, \sum_{k=1}^n F_{ki}\vec{b}_k)_g$, thus $\sum_{k=1}^n (F^T)_{kj}(\vec{a}_k, \vec{a}_i)_G = \sum_{k=1}^n F_{ki}(\vec{b}_j, \vec{b}_k)_g$ with $F_{ki} = ([F]^T)_{ik}$, thus

$$\sum_{k=1}^n G_{ik}(F^T)_{kj} = \sum_{k=1}^n ([F]^T)_{ik} g_{kj}, \quad \text{i.e.} \quad (F^T)_{ij} = \sum_{k,\ell=1}^n ([G]^{-1})_{ik} F_{\ell k} g_{\ell j}, \tag{G.5}$$

for all i, j, thus (G.3).

Marsden notations: $G_{IJ} = G(\vec{E}_I, \vec{E}_j)$, $g_{ij} = g(\vec{e}_i, \vec{e}_j)$, $F.\vec{E}_J = \sum_{i=1}^n F^i{}_J \vec{e}_i$, $F^T.\vec{e}_j = \sum_{I=1}^n (F^T)^I{}_j \vec{E}_I$, thus

$$\sum_{K=1}^n G_{IK}(F^T)^K{}_j = \sum_{k=1}^n F^k{}_I g_{kj}, \quad \text{i.e.} \quad (F^T)^I{}_j = \sum_{K,k=1}^n G^{IK} F^k{}_K g_{kj} \quad \text{where} \quad [G^{IJ}] := [G_{IJ}]^{-1}.$$ ∴

G.1.3 Remark: F^*

(For mathematicians: F^* doesn't seem to be very useful in mechanics, apart from making simple things difficult, and playing games with components and duality notations...).

Definition G.5 The adjoint of the linear map $F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ (acting on vectors) is the linear map $F^* \in \mathcal{L}(\mathbb{R}^{n*}_t; \mathbb{R}^{n*}_{t_0})$ (acting on functions) canonically defined by, for all $m \in \mathbb{R}^{n*}_t$,

$$F^*(m) := m \circ F, \quad \text{written} \quad F^*.m = m.F \ (\in \vec{\mathbb{R}}^{n*}_{t_0}). \tag{G.6}$$

So, for all $(m,\vec W)\in\vec{\mathbb{R}}^{n*}_t\times\vec{\mathbb{R}}^n_{t_0}$,

$$(F^*.m).\vec W = m.F.\vec W \quad (\in\mathbb{R}). \tag{G.7}$$

Quantification (matrix representation): (π_{ai}) and (π_{bi}) are the covariant dual bases of $(\vec a_i)$ and $(\vec b_i)$. Let $(F^*)_{ij}$ be the components of F^* relative to these dual bases:

$$F^*.\pi_{bj} = \sum_{I=1}^n (F^*)_{ij}\pi_{ai}, \quad \text{i.e.} \quad [F^*]_{|\pi_b,\pi_a} = [(F^*)_{ij}]. \tag{G.8}$$

(G.7) gives $(F^*.\pi_{bj}).\vec a_i = \pi_{bj}.F.\vec a_i$, thus

$$\forall i,j,\ \boxed{(F^*)_{ij}=F_{ji}}, \quad \text{i.e.} \quad [F^*]_{|\pi_b,\pi_a} = ([F]_{|\vec a,\vec b})^T, \quad \text{in short} \quad [F^*]=[F]^T. \tag{G.9}$$

Marsden duality notations: $F^*.e^j=\sum_{I=1}^n(F^*)_I{}^jE^I$ gives $(F^*)_I{}^j = F^j{}_I$ for all I,j.

Interpretation of F^*. As usual in classical mechanics, we use Euclidean dot products, here $(\cdot,\cdot)_G$ in $\vec{\mathbb{R}}^n_{t_0}$ and $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$. Then we use the $(\cdot,\cdot)_G$-Riesz representation vector $\vec R_G(F^*.m)\in\vec{\mathbb{R}}^n_{t_0}$ of $F^*.m\in\vec{\mathbb{R}}^{n*}_{t_0}$, and the $(\cdot,\cdot)_g$-Riesz representation vector $\vec R_g(m)\in\vec{\mathbb{R}}^n_t$ of $m\in\vec{\mathbb{R}}^{n*}_t$, so, for all $m\in\vec{\mathbb{R}}^{n*}_t$ and $\vec W\in\vec{\mathbb{R}}^n_{t_0}$,

$$(F^*.m).\vec W = \vec R_G(F^*.m)\bullet_G\vec W, \quad \text{and} \quad m.(F.\vec W) = \vec R_g(m)\bullet_g F.\vec W = (F^T.\vec R_g(m))\bullet_G\vec W. \tag{G.10}$$

Thus (G.7) gives $\vec R_G(F^*.m) = F^T.\vec R_g(m)$, thus

$$\vec R_G.F^* = F^T.\vec R_g, \quad \text{i.e.} \quad F^* = \vec R_G{}^{-1}.F^T.\vec R_g. \tag{G.11}$$

Remark G.6 The definition of F^* is intrinsic to F (objective), while the definition of F^T is ***not*** intrinsic to F (***not*** objective) since it needs inner dot products (observer choices) to be defined. ∴

G.2 Cauchy–Green deformation tensor C

G.2.1 Definition of C

Consider vectors $\vec W_i\in\vec{\mathbb{R}}^n_{t_0}$ at P, $i=1,2$, and their push forwards $\vec w_i$ toward $p=\Phi(P)$, i.e.

$$\vec w_i = F.\vec W_i, \tag{G.12}$$

short notation for $\vec w_i(p) = F(P).\vec W_i(P)$. With the chosen inner dot products $(\cdot,\cdot)_G$ in $\vec{\mathbb{R}}^n_{t_0}$ and $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$, we get $(\vec w_1(p),\vec w_2(p))_g = (F(P).\vec W_1(P),F(P).\vec W_2(P))_g =^{(G.2)} (F^T_{Gg}(p).F(P).\vec W_1(P),\vec W_2(P))_G$ when $p=\Phi(P)$, written in short:

$$(\vec w_1,\vec w_2)_g = (F.\vec W_1,F.\vec W_2)_g = (\underbrace{F^T.F}_{C}.\vec W_1,\vec W_2)_G, \tag{G.13}$$

Definition G.7 The (right) Cauchy–Green deformation tensor at $P\in\Omega_{t_0}$ relative to $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$, is the endomorphism $C_{Gg}(P)\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ defined by

$$C_{Gg}(P) := F^T_{Gg}(p)\circ F(P), \quad \text{in short} \quad \boxed{C := F^T.F}. \tag{G.14}$$

So

$$C = F^T\circ F : \vec W \xrightarrow{F} F(\vec W) \xrightarrow{F^T} F^T(F(\vec W)) = C(\vec W), \tag{G.15}$$

with F and F^T linear, thus C is linear and $C(\vec W)$ is written $C.\vec W = F^T.F.\vec W$. And (G.13) tells that C is characterized by, for all $\vec W_1,\vec W_2\in\vec{\mathbb{R}}^n_{t_0}$,

$$\vec w_1\bullet_g\vec w_2 = \boxed{(C.\vec W_1)\bullet_G\vec W_2 = (F.\vec W_1)\bullet_g(F.\vec W_2)}. \tag{G.16}$$

Moreover, $(\cdot,\cdot)_g$ being symmetric (inner dot product), C is a $(\cdot,\cdot)_G$-symmetric endomorphism in $\vec{\mathbb{R}}^n_{t_0}$, i.e., for all $\vec W_1,\vec W_2\in\vec{\mathbb{R}}^n_{t_0}$,

$$(C.\vec W_1,\vec W_2)_G = (\vec W_1,C.\vec W_2)_G, \quad \text{i.e.} \quad (C.\vec W_1)\bullet_G\vec W_2 = \vec W_1\bullet_G(C.\vec W_2), \tag{G.17}$$

since $(F^T.F.\vec W_1,\vec W_2)_G = (F.\vec W_1,F.\vec W_2)_g = (\vec W_1,F^T.F.\vec W_2)_G$.

G.2.2 Quantification

(G.14) gives $[C]=[F^T].[F]$, with $[F^T]=^{(G.3)}[G]^{-1}.[F]^T.[g]$, thus

$$\boxed{[C]=[G]^{-1}.[F]^T.[g].[F]}, \tag{G.18}$$

short notation for $[C_{Gg}]_{|\vec{a}}=[G]^{-1}_{|\vec{a}}.([F]_{|\vec{a},\vec{b}})^T.[g]_{|\vec{b}}.[F]_{|\vec{a},\vec{b}}$.

Exercice G.8 Use classical notation, then duality notations, to express (G.18) with components.

Answer. Classical notations:

$$F.\vec{a}_j=\sum_{i=1}^n F_{ij}\vec{b}_i \quad\text{and}\quad C.\vec{a}_j=\sum_{i=1}^n C_{ij}\vec{a}_i, \quad\text{i.e.}\quad [F]_{|\vec{a},\vec{b}}=[F_{ij}] \quad\text{and}\quad [C]_{|\vec{a}}=[C_{ij}]. \tag{G.19}$$

(G.16)-(G.17) give $(\vec{a}_i,C.\vec{a}_j)_G=(F.\vec{a}_i,F.\vec{a}_j)_g$, so $(\vec{a}_i,\sum_k C_{kj}\vec{a}_k)_G=(\sum_k F_{ki}\vec{b}_k,\sum_\ell F_{\ell j}\vec{b}_\ell)_g$, thus $\sum_k C_{kj}(\vec{a}_i,\vec{a}_k)_G=\sum_{k\ell}F_{ki}(\vec{b}_k,\vec{b}_\ell)_g F_{\ell j}$, i.e.

$$\sum_{k=1}^n G_{ik}C_{kj}=\sum_{k,\ell=1}^n F_{ki}\,g_{k\ell}F_{\ell j}=\sum_{k,\ell=1}^n([F]^T)_{ik}\,g_{k\ell}F_{\ell j}, \quad\text{so}\quad \boxed{[G].[C]=[F]^T.[g].[F]}, \tag{G.20}$$

so $C_{ij}=\sum_{k,\ell,m=1}^n([G]^{-1})_{im}F_{km}\,g_{k\ell}F_{\ell j}=\sum_{k,\ell,m=1}^n([G]^{-1})_{im}([F]^T)_{mk}\,g_{k\ell}F_{\ell j}$. Duality notations:

$$\begin{gathered}F.\vec{E}_J=\sum_{i=1}^n F^i{}_J\vec{e}_i \quad\text{and}\quad C.\vec{E}_J=\sum_{I=1}^n C^I{}_J\vec{E}_I, \quad\text{i.e.}\quad [F]_{|\vec{E},\vec{e}}=[F^i{}_J] \quad\text{and}\quad [C]_{|\vec{E}}=[C^I{}_J], \quad\text{and}\\ \sum_{K=1}^n G_{IK}C^K{}_J=\sum_{k,\ell=1}^n F^k{}_I\,g_{k\ell}F^\ell{}_J, \quad\text{and}\quad C^I{}_J=\sum_{k,\ell,M=1}^n G^{IM}F^k{}_M\,g_{k\ell}F^\ell{}_J \quad\text{when}\quad [G^{IJ}]:=[G_{IJ}]^{-1}.\end{gathered} \tag{G.21}$$

⁂

Exercice G.9 $(\cdot,\cdot)_G$ is a Euclidean dot product in foot, $(\cdot,\cdot)_g$ is a Euclidean dot product in metre, so $(\cdot,\cdot)_g=\mu^2(\cdot,\cdot)_G$ with $\mu\simeq 0.3048$; And $(\vec{a}_i)=(\vec{b}_i)$ is a $(\cdot,\cdot)_G$-orthonormal basis. Prove

$$[C]=\mu^2[F]^T.[F]. \tag{G.22}$$

Answer. $[C]_{|\vec{a}}=^{(G.18)}[G]^{-1}_{|\vec{a}}.[F]^T_{|\vec{a},\vec{a}}.[g]_{|\vec{a}}.[F]_{|\vec{a},\vec{a}}$ gives $[C]_{|\vec{a}}=I.[F]^T_{|\vec{a},\vec{a}}.\mu^2 I.[F]_{|\vec{a},\vec{a}}$. Shorten notation = (G.22). ⁂

G.3 Time Taylor expansion of C

Here we use a unique inner dot product $(\cdot,\cdot)_G=(\cdot,\cdot)_g$ at all time (to compare results in the vicinity of t_0). Moreover we use an orthonormal basis (to lighten the notations), thus, in short, $[C]=[F]^T.[F]$.

P is fixed, $C_t^{t_0}(P)=^{\text{noted}}C(t)$, and $[C(t)]=[F(t)]^T.[F(t)]$ (since $[G]=[g]=I$ here), and $\vec{V}_t^{t_0}(P)=^{\text{noted}}\vec{V}(t)$ and $\vec{A}_t^{t_0}(P)=^{\text{noted}}\vec{A}(t)$ are the Lagrangian velocities and accelerations, and $\vec{v}(t,p)$ and $\vec{\gamma}(t,p)$ are the Eulerian velocities and accelerations at t at $p=\Phi_t^{t_0}(t,P)$.

With Lagrangian variables (used to define C): $F(t+h)=F(t)+h\,d\vec{V}(t)+\frac{h^2}{2}\,d\vec{A}(t)+o(h^2)$ gives

$$\begin{aligned}[C(t+h)]&=[F(t+h)]^T.[F(t+h)]\\ &=[F^T+h\,d\vec{V}^T+\frac{h^2}{2}\,d\vec{A}^T+o(h^2)](t)[F+h\,d\vec{V}+\frac{h^2}{2}\,d\vec{A}+o(h^2)](t)\\ &=[C(t)+h\,\underbrace{([F^T].[d\vec{V}]+[d\vec{V}]^T.[F])(t)}_{=[(C_P^{t_0})'(t)]=^{\text{noted}}[C'(t)]}+\frac{h^2}{2}\,\underbrace{([F]^T.[d\vec{A}]+2[d\vec{V}]^T.[d\vec{V}]+[d\vec{A}]^T.[F])(t))}_{=[(C_P^{t_0})''(t)]=^{\text{noted}}[C''(t)]}(t)+o(h^2).\end{aligned} \tag{G.23}$$

(As usual with Lagrangian variables, we have three times involved: t_0, t and $t+h$).) In particular

$$[C_P^{t_0}(t_0+h)]=I+([d\vec{V}]+[d\vec{V}]^T)(t_0)+\frac{h^2}{2}\,([d\vec{A}]+2[d\vec{V}]^T.[d\vec{V}]+[d\vec{A}]^T)(t_0)+o(h^2). \tag{G.24}$$

Abusively written $C_P^{t_0}(t_0+h)=I+(d\vec{V}+d\vec{V}^T)(t_0)+\frac{h^2}{2}\,(d\vec{A}+2d\vec{V}^T.d\vec{V}+d\vec{A}^T)(t_0)+o(h^2)$, but don't forget it is a matrix meaning.

With Eulerian variables: With $p(t) = \Phi^{t_0}(t,P)$, we have $d\vec{V}^{t_0}(t,P) = d\vec{v}(t,p(t)).F(t)$ and $d\vec{A}^{t_0}(t,P) = d\vec{\gamma}(t,p(t)).F(t)$, thus writing $d\vec{v} := d\vec{v}(t,p(t))$ and $d\vec{\gamma} := d\vec{\gamma}(t,p(t))$ (for short),

$$\begin{aligned} C_P^{t_0}(t{+}h) = C_P^{t_0}(t) &+ h\,(F^T(t).(d\vec{v} + d\vec{v}^T)(t,p(t)).F(t)) \\ &+ \frac{h^2}{2}\,(F^T(t).(d\vec{\gamma} + 2d\vec{v}^T.d\vec{v} + d\vec{\gamma}^T)(t,p(t)).F(t)) + o(h^2). \end{aligned} \tag{G.25}$$

abusive notation of $[C_P^{t_0}(t{+}h)] = ...$ (matrices relative to a basis).

Remark G.10 $F'' = d\vec{A}$ is easy to interpret, but $C'' = F^T.d\vec{A} + 2d\vec{V}^T.d\vec{V} + d\vec{A}^T.F = (F^T.d\vec{A} + d\vec{V}^T.d\vec{V}) + (F^T.d\vec{A} + d\vec{V}^T.d\vec{V})^T$ is not that easy to interpret (and in not linear in $\vec{V}$).

We already had a problem with the composition of flows: The formula $F_{t_2}^{t_0} = F_{t_2}^{t_1}.F_{t_1}^{t_0}$ is simple (determinism), but the formula $C_{t_2}^{t_0} = (F_{t_2}^{t_0})^T.F_{t_2}^{t_0} = (F_{t_1}^{t_0})^T.(F_{t_2}^{t_1})^T.F_{t_2}^{t_1}.F_{t_1}^{t_0} = (F_{t_1}^{t_0})^T.C_{t_2}^{t_1}.F_{t_1}^{t_0}$ is “not that simple” ($\neq C_{t_2}^{t_1}.C_{t_1}^{t_0}$). (Indeed, to consider C instead of F amounts to consider the “motion squared”, cf. $(C.\vec{W},\vec{W})_g = ||F.\vec{W}||_g^2$.)

Since $C'(t_0) = d\vec{V}(t_0) + d\vec{V}(t_0)^T$ this may have little consequences for linear approximation near t_0, but ultimately not small consequences for second-order approximations (and large deformations) if C'' is used to make constitutive laws. The consideration of Lie derivatives may be an interesting alternative. ∴

G.4 Remark: $C^\flat$

For mathematicians: May produce errors, misuses, covariance-contravariance confusion, see next § G.4.2. For the general $^\flat$ notation see § A.10.4.

G.4.1 Definition of $C^\flat$...

Definition G.11 At $P \in \Omega_{t_0}$, the bilinear form $C^\flat_{Gg}(P) =^{\text{noted}} C^\flat \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}, \vec{\mathbb{R}}^n_{t_0}; \mathbb{R})$ associated with the linear map $C_{Gg}(P) =^{\text{noted}} C \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ is defined by, for all $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$ vectors at P,

$$C^\flat(\vec{W}_1,\vec{W}_2) := (\vec{W}_1, C.\vec{W}_2)_G \quad (= (F.\vec{W}_1, F.\vec{W}_2)_g). \tag{G.26}$$

Then $C^\flat$ is a bilinear symmetric form (trivial) and is a metric in $\vec{\mathbb{R}}^n_{t_0}$ when $F_t^{t_0} =^{\text{noted}} F$ is a diffeomorphism (usual hypothesis), but not a Euclidean one (it is iff $C = I$ i.e. for rigid body motions).

Quantification: (G.26) gives $[\vec{W}_2]^T.[C^\flat].[\vec{W}_1] = [\vec{W}_2]^T.[G].[C].[\vec{W}_1]$ for all $\vec{W}_1, \vec{W}_2$ since $C^\flat$ and $(\cdot,\cdot)_G$ are symmetric, thus

$$[C^\flat] = [G].[C] \quad (= [F]^T.[g].[F]). \tag{G.27}$$

Exercice G.12 Use duality notations to express (G.27) with components, and explain the flat $^\flat$ notation.

Answer. (G.26) gives $C^\flat(\vec{E}_J,\vec{E}_I) := (C.\vec{E}_J,\vec{E}_I)_G$. Thus with $C^\flat(\vec{E}_I,\vec{E}_J) = C_{IJ}$ and $C.\vec{E}_J = \sum_I C^I{}_J\vec{E}_I$ we get $C_{JI} = \sum_K C^K{}_J(\vec{E}_K,\vec{E}_I)_G = \sum_K C^K{}_J G_{KI}$; And $C^\flat$ and $(\cdot,\cdot)_G$ are symmetric, thus

$$C_{IJ} = \sum_K G_{IK}C^K{}_J, \quad \text{i.e.} \quad [C^\flat]_{|\vec{E}} = [G]_{|\vec{E}}.[C]_{|\vec{E}}. \tag{G.28}$$

The flat notation $C^\flat$ is due to: The top index I in $C^I{}_J$ has been transformed into a bottom index in C_{IJ} in $C^\flat$, which characterizes a change of variance because of the use of an inner dot product.

And (G.27) also gives $C_{IJ} = (F.\vec{E}_I, F.\vec{E}_J)_g = \sum_{k\ell} F^k{}_I F^\ell{}_J(\vec{b}_k,\vec{b}_\ell)_g$, thus

$$C_{IJ} = \sum_{k\ell} F^k{}_I g_{k\ell} F^\ell{}_J = \sum_{k\ell} (F^T)^I{}_k g_{k\ell} F^\ell{}_J, \quad \text{i.e.} \quad [C^\flat]_{|\vec{E}} = ([F]_{|\vec{E},\vec{e}})^T.[g]_{|\vec{e}}.[F]_{|\vec{E},\vec{e}}. \tag{G.29}$$

∴

G.4.2 ... and remarks about $C^\flat$... and Jaumann

$C^\flat$ can also be defined only with $(\cdot,\cdot)_g$ by, for all $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$,

$$C^\flat_g(\vec{W}_1,\vec{W}_2) := (F.\vec{W}_1, F.\vec{W}_2)_g, \tag{G.30}$$

i.e., $C^\flat_g := g^* =^{\text{noted}} C^\flat$. So we can also say that $C^\flat_g$ is the pull-back of the metric $(\cdot,\cdot)_g$ by Φ, see (8.9).

- However $C^\flat = C^\flat_g$ is useless in itself: $C^\flat$ is ***not*** a Euclidean dot product (it is a metric defined at each P by $C^\flat_g(P)(\vec{W}_1,\vec{W}_2) := (F(P).\vec{W}_1, F(P).\vec{W}_2)_g$ for all $\vec{W}_1,\vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$ vectors at P). In fact, $C^\flat$ is only useful to characterize a deformation if the value $C^\flat(\vec{W}_1,\vec{W}_2)$ can be compared with the initial value $(\vec{W}_1,\vec{W}_2)_G$, i.e. if a Euclidean dot product $(\cdot,\cdot)_G$ was introduced in $\vec{\mathbb{R}}^n_{t_0}$: This is why $C^\flat$ is classically defined from C, cf. (G.26).
- You may want to use the infinitesimal strain tensor $\underline{\underline{\varepsilon}} = \frac{F+F^T}{2} - I$, or the Green–Lagrange deformation tensor $E = \frac{1}{2}(C - I)$, obtained from $F^T := F^T_{Gg}$ (essential).
- There is no objective "trace" for a $\binom{0}{2}$ tensor like $C^\flat$, while $\mathrm{Tr}(C)$ is objective since C is an endomorphism ($\simeq$ a $\binom{1}{1}$ tensor).
- The Lie derivatives of a second order tensor depends on the type of the tensor, and the Lie derivative of the $\binom{1}{1}$ tensor like C gives the Jaumann derivative, which is usually preferred to the Lie derivative of the $\binom{0}{2}$ tensor like $C^\flat$ which is the lower convected Lie derivative, see remark G.13.

So the introduction and use of $C^\flat$ in mechanics mostly complicate things unnecessarily, and interferes with basic understandings like the distinction between covariance and contravariance.

Remark G.13 Interpretation issue (with Jaumann).

$2\mathcal{D} = d\vec{v} + d\vec{v}^T$ gives $2\frac{D\mathcal{D}}{Dt} = \frac{D(d\vec{v})}{Dt} + \frac{D(d\vec{v})^T}{Dt} = d\vec{\gamma} + d\vec{\gamma}^T - d\vec{v}.d\vec{v} - d\vec{v}^T.d\vec{v}^T$, thus, with (G.23) and keeping in mind the matrix meaning,

$$\begin{aligned} C''(t) &= F(t)^T.(2\frac{D\mathcal{D}}{Dt} + d\vec{v}.d\vec{v} + d\vec{v}^T.d\vec{v}^T + 2d\vec{v}^T.d\vec{v})(t,p(t)).F(t) \\ &= 2F(t)^T.(\frac{D\mathcal{D}}{Dt} + \mathcal{D}.d\vec{v} + d\vec{v}^T.\mathcal{D})(t,p(t)).F(t). \end{aligned} \tag{G.31}$$

The $\frac{D\mathcal{D}}{Dt} + \mathcal{D}.d\vec{v} + d\vec{v}^T.\mathcal{D}$ term looks like a lower-convected Lie derivative, but with $d\vec{v}^T$ instead of $d\vec{v}^*$, cf. (9.58); So you may find (G.31) written as $C'' = 2F^T.\mathcal{L}_{\vec{v}}\mathcal{D}.F$. But you get disappointing results when using the the lower convected Lie derivative (Jaumann is usually preferred). In fact, it is $\mathcal{L}_{\vec{v}}\mathcal{D}^\flat$ (lower convected Lie derivative) that should be used, where $\mathcal{D}^\flat_g := \frac{d\vec{v}^\flat_g + (d\vec{v}^\flat_g)^T}{2}$, to get $(C^\flat)'' = 2F^T.\mathcal{L}_{\vec{v}}\mathcal{D}^\flat_g.F$. ▪

G.5 Stretch ratio and deformed angle

Here $(\cdot,\cdot)_g = (\cdot,\cdot)_G$, i.e. at t_0 and t we use the same Euclidean dot product, to be able to compare the lengths relative to the same unit of measurement. (If $(\cdot,\cdot)_g \neq (\cdot,\cdot)_G$ then use $(\cdot,\cdot)_g = \mu^2(\cdot,\cdot)_G$.)

G.5.1 Stretch ratio

The stretch ratio at $P \in \vec{\mathbb{R}}^n_{t_0}$ between t_0 and t for a $\vec{W}_P \in \vec{\mathbb{R}}^n_{t_0}$ is defined by

$$\lambda(\vec{W}_P) := \frac{||\vec{w}_p||_G}{||\vec{W}_P||_G} = \frac{||F_P.\vec{W}_P||_G}{||\vec{W}_P||_G} \quad (= ||F_P.(\frac{\vec{W}_P}{||\vec{W}_P||_G})||_G) \tag{G.32}$$

where $\vec{w}_p = F_P.\vec{W}_P$ is the deformed vector by the motion at $p = \Phi(P)$. I.e., in short

$$\forall \vec{W} \in \vec{\mathbb{R}}^n_{t_0} \text{ s.t. } ||\vec{W}|| = 1, \quad \lambda(\vec{W}) := ||F.\vec{W}||. \tag{G.33}$$

(You may find: $\lambda(d\vec{X}) = ||F.d\vec{X}||$ with $d\vec{X}$ a unit vector(!); This notation should be avoided, see § 4.3.)

G.5.2 Deformed angle

Recall: The angle $\theta_{t_0} = (\widehat{\vec{W}_1,\vec{W}_2})$ formed by two vectors $\vec{W}_1$ and $\vec{W}_2$ in $\vec{\mathbb{R}}^n_{t_0} - \{\vec{0}\}$ at $P \in \Omega_{t_0}$ is given by $\cos(\theta_{t_0}) = \frac{\vec{W}_1}{||\vec{W}_1||_G} \bullet \frac{\vec{W}_2}{||\vec{W}_2||_G}$ $(= (\frac{\vec{W}_1}{||\vec{W}_1||_G}, \frac{\vec{W}_2}{||\vec{W}_2||_G})_G)$.

With the deformed vectors $\vec{w}_i = F.\vec{W}_i$ at $p = \Phi^{t_0}_t(P)$, the deformed angle is θ_t defined by

$$\cos(\theta_t) := (\widehat{\vec{w}_1,\vec{w}_2}) = \frac{\vec{w}_1}{||\vec{w}_1||} \bullet \frac{\vec{w}_2}{||\vec{w}_2||} = \frac{F.\vec{W}_1}{||F.\vec{W}_1||} \bullet \frac{F.\vec{W}_2}{||F.\vec{W}_2||} \quad (= \frac{(C.\vec{W}_1) \bullet \vec{W}_2}{||\vec{w}_1||\,||\vec{w}_2||}). \tag{G.34}$$

G.6 Decompositions of C

G.6.1 Spherical and deviatoric tensors

Definition G.14 The deformation spheric tensor is

$$C_{sph} = \frac{1}{n}\mathrm{Tr}(C)\, I, \tag{G.35}$$

with $\mathrm{Tr}(C)$ = the trace of the endomorphism C (there is no "trace" for the $\binom{0}{2}$ tensor $C^\flat$).

Definition G.15 The deviatoric tensor is

$$C_{dev} = C - C_{sph}. \tag{G.36}$$

(So $\mathrm{Tr}(C_{dev}) = 0$, and $C = C_{sph} + C_{dev}$.)

G.6.2 Rigid motion

The deformation is rigid iff, for all t_0, t,

$$(F_t^{t_0})^T.F_t^{t_0} = I, \quad \text{i.e.} \quad C_t^{t_0} = I, \quad \text{written} \quad C = I = F^T.F. \tag{G.37}$$

Thus, after a rigid body motion, lengths and angles are left unchanged.

G.6.3 Diagonalization of C

Proposition G.16 *$C = F^T.F$ being symmetric positive, C is diagonalizable, its eigenvalues are positive, and $\vec{\mathbb{R}}^n_{t_0}$ has an orthonormal basis made of eigenvectors of C.*

Proof. $(C(P).\vec W_1, \vec W_2)_G = (F(P).\vec W_1, F(P).\vec W_2)_g = (\vec W_1, C(P).\vec W_2)_G$, thus C is $(\cdot,\cdot)_G$-symmetric.

$(C.\vec W_1, \vec W_1)_G = (F.\vec W_1, F.\vec W_1)_g = ||F.\vec W_1||_g^2 > 0$ when $\vec W_1 \neq \vec 0$, since F invertible ($\Phi_t^{t_0}$ is supposed to be a diffeomorphism). Thus C est $(\cdot,\cdot)_G$-symmetric definite positive real endomorphism. ∎

Definition G.17 Let λ_i be the eigenvalues of C. Then the $\sqrt{\lambda_i}$ are called the principal stretches. And the associated eigenvectors give the principal directions.

G.6.4 Mohr circle

This § deals with general properties of $3*3$ symmetric positive endomorphism, like $C_t^{t_0}(P)$.

Consider $\vec{\mathbb{R}}^3$ with a Euclidean dot product $(\cdot,\cdot)_{\mathbb{R}^3}$ and a $(\cdot,\cdot)_{\mathbb{R}^3}$-orthonormal basis $(\vec a_i)$.

Let $\mathcal{M} : \vec{\mathbb{R}}^3 \to \vec{\mathbb{R}}^3$ be a symmetric positive endomorphism. Thus $\mathcal{M}$ is diagonalizable in a $(\cdot,\cdot)_{\mathbb{R}^3}$-orthonormal basis $(\vec e_1, \vec e_2, \vec e_3)$, that is, $\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, $\exists \vec e_1, \vec e_2, \vec e_3 \in \vec{\mathbb{R}}^3$ s.t.

$$\mathcal{M}.\vec e_i = \lambda_i \vec e_i \quad \text{and} \quad (\vec e_i, \vec e_j)_{\mathbb{R}^3} = \delta_{ij}, \quad \text{so} \quad [\mathcal{M}]_{|\vec e} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{G.38}$$

And the orthonormal basis $(\vec e_1, \vec e_2, \vec e_3)$ is ordered s.t. $\lambda_1 \geq \lambda_2 \geq \lambda_3$ (> 0).

Let S be the unit sphere in $\mathbb{R}^3$, that is the set $\{(x,y,z) : x^2 + y^2 + z^2 = 1\}$. Its image $\mathcal{M}(S)$ by $\mathcal{M}$ is the ellipsoid $\{(x,y,z) : \frac{x^2}{\lambda_1^2} + \frac{y^2}{\lambda_2^2} + \frac{z^2}{\lambda_3^2} = 1\}$. Then consider $\vec n = \sum_i n_i \vec e_i$ s.t. $||\vec n||_{\mathbb{R}^3} = 1$:

$$[\vec n]_{|\vec e} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \quad \text{with} \quad n_1^2 + n_2^2 + n_3^2 = 1. \tag{G.39}$$

Thus its image $\vec A = \mathcal{M}.\vec n \in \mathcal{M}(S)$ satisfies

$$\vec A = \mathcal{M}.\vec n, \quad [\vec A]_{|\vec e} = \begin{pmatrix} \lambda_1 n_1 \\ \lambda_2 n_2 \\ \lambda_3 n_3 \end{pmatrix}. \tag{G.40}$$

Then define

$$A_n = (\vec A, \vec n)_{\mathbb{R}^3}, \quad \vec A_\perp = \vec A - A_n \vec n, \quad A_\perp := ||\vec A_\perp||. \tag{G.41}$$

So $\vec A = A_n \vec n + \vec A_\perp \in \mathrm{Vect}\{\vec n\} \otimes \mathrm{Vect}\{\vec n\}^\perp$. (Remark: $\vec A_\perp$ is not orthonormal to the ellipsoid $\mathcal{M}(S)$, but is orthonormal to the initial sphere S.)

Mohr Circle purpose: To find a relation:

$$A_\perp = f(A_n), \tag{G.42}$$

relation between "the normal force A_n" (to the initial sphere) and the "tangent force$A_\perp$" (to the initial sphere).

(G.39), (G.40) and $A_n = (\mathcal{M}.\vec{n},\vec{n})_{\mathbb{R}^3}$ give

$$\left\{ \begin{array}{l} n_1^2+n_2^2+n_3^2 = 1, \\ \lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2 = A_n \\ \lambda_1^2 n_1^2 + \lambda_2^2 n_2^2 + \lambda_3^2 n_3^2 = ||\vec{A}||^2 = A_n^2 + A_\perp^2. \end{array} \right. \tag{G.43}$$

This is linear system with the unknowns n_1^2, n_2^2, n_3^2. The solution is

$$\left\{ \begin{array}{l} n_1^2 = \dfrac{A_\perp^2 + (A_n-\lambda_2)(A_n-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, \\ n_2^2 = \dfrac{A_\perp^2 + (A_n-\lambda_3)(A_n-\lambda_1)}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}, \\ n_3^2 = \dfrac{A_\perp^2 + (A_n-\lambda_1)(A_n-\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}. \end{array} \right. \tag{G.44}$$

The n_i^2 being non negative, and with $\lambda_1 > \lambda_2 > \lambda_3 \geq 0$, we get

$$\left\{ \begin{array}{l} A_\perp^2 + (A_n-\lambda_2)(A_n-\lambda_3) \geq 0, \\ A_\perp^2 + (A_n-\lambda_3)(A_n-\lambda_1) \leq 0, \\ A_\perp^2 + (A_n-\lambda_1)(A_n-\lambda_2) \geq 0. \end{array} \right. \tag{G.45}$$

Then let $x = A_n$ and $y = A_\perp$, and consider, for some $a, b \in \mathbb{R}$, the equation

$$y^2 + (x-a)(x-b) = 0, \quad \text{so} \quad (x - \frac{a+b}{2})^2 + y^2 = \frac{(a-b)^2}{4}.$$

This is the equation of a circle centered at $(\frac{a+b}{2}, 0)$ with radius $\frac{|a-b|}{2}$.

Thus $(\text{G.45})_2$ tells that A_n and $A_\perp$ are inside the circle centered at $(\frac{\lambda_1+\lambda_3}{2}, 0)$ with radius $\frac{\lambda_1-\lambda_3}{2}$, and $(\text{G.45})_{1,3}$ tell that A_n and $A_\perp$ are outside the other circles (adjacent and included in the first, drawing).

Exercice G.18 What happens if $\lambda_1 = \lambda_2 = \lambda_3 > 0$?

Answer. Then $\left\{ \begin{array}{l} n_1^2+n_2^2+n_3^2 = 1, \\ n_1^2+n_2^2+n_3^2 = \dfrac{A_n}{\lambda_1}, \\ n_1^2+n_2^2+n_3^2 = \dfrac{A_n^2+A_\perp^2}{\lambda_1^2}. \end{array} \right\}$ Thus $A_n = \lambda_1$ and $A_n^2 + A_\perp^2 = \lambda_1^2$, thus $A_\perp = 0$. Here $C = \lambda_1 I$, and we deal with a dilation: $A_\perp = 0$. ⁂

Exercice G.19 What happens if $\lambda_1 = \lambda_2 > \lambda_3 > 0$?

Answer. Then $\left\{ \begin{array}{l} n_1^2+n_2^2+n_3^2 = 1, \\ \lambda_1(1-n_3^2) + \lambda_3 n_3^2 = A_n, \\ \lambda_1^2(1-n_3^2) + \lambda_3^2 n_3^2 = A_n^2 + A_\perp^2. \end{array} \right\}$ Thus $A_n = \lambda_1 - (\lambda_1-\lambda_3)n_3^2 \in [\lambda_3, \lambda_1]$, and $A_\perp = \pm(\lambda_1^2 - (\lambda_1^2-\lambda_3^2)n_3^2 - A_n^2)^{\frac{1}{2}}$, with $A_n^2 + A_\perp^2$ a point on the circle with radius $\lambda_1^2(1-n_3^2)+\lambda_3^2 n_3^2$. ⁂

G.7 Green–Lagrange deformation tensor E

(G.13) gives $(\vec{w}_1, \vec{w}_2)_g = (F.\vec{W}_1, F.\vec{W}_2)_g = (C.\vec{W}, \vec{W})_G$ at $p = \Phi(P)$, thus

$$(\vec{w}_1, \vec{w}_2)_g - (\vec{W}_1, \vec{W}_2)_G = ((C-I).\vec{W}_1, \vec{W}_2)_G. \tag{G.46}$$

Definition G.20 The Green–Lagrange tensor (or Green–Saint Venant tensor) at P relative to t_0 and t is the endomorphism $E_t^{t_0}(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ defined by

$$E_t^{t_0}(P) := \frac{C_t^{t_0}(P) - I_{t_0}}{2}, \quad \text{in short} \quad \boxed{E = \frac{C-I}{2}} \quad (= \frac{F^T.F - I}{2}). \tag{G.47}$$

(In particular $E = 0$ for rigid body motions.) And $E_t^{t_0} : \Omega_{t_0} \to \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ is the Green–Lagrange tensor relative to t_0 and t.

The $\frac12$ because $(C.,.) = (F.,F.)$ corresponds to the "motion squared", see the following linearization.

And we get the time Taylor expansion of $E_P^{t_0}(t) = \frac12(C_P^{t_0}(t) - I_{t_0})$ with $p(t) = \Phi_P^{t_0}(t)$ and (G.25):

$$\begin{aligned} E_P^{t_0}(t{+}h) &= F_P^{t_0}(t)^T.\big(h\,\frac{d\vec{v} + d\vec{v}^T}{2} + \frac{h^2}{2}\,(\frac{d\vec{\gamma} + d\vec{\gamma}^T}{2} + d\vec{v}^T.d\vec{v})\big)(t,p(t)).F_P^{t_0}(t) + o(h^2) \\ &= F_P^{t_0}(t)^T.\big(h\,\mathcal{D} + h^2\,((\frac{D\mathcal{D}}{Dt} + \mathcal{D}.d\vec{v} + d\vec{v}^T.\mathcal{D})(t,p(t))).F_P^{t_0}(t) + o(h^2). \end{aligned} \tag{G.48}$$

G.8 Small deformations (linearization): The infinitesimal strain tensor $\underline{\underline{\varepsilon}}$

G.8.1 Landau notations big-O and little-o

Reminder. Let $f,g : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$.

$$f = O(g) \text{ near } x_0 \quad \Longleftrightarrow \quad \exists C>0,\ \exists \eta>0,\ \forall x \ s.t.\ |x-x_0|<\eta,\ |f(x)| < C|g(x)|. \tag{G.49}$$

and f is said to be "comparable with g" near x_0.

If $|g|>0$ then the conclusion reads $\frac{|f(x)|}{|g(x)|} < C$; And $f = O(x^n)$ near x=0 iff $\frac{|f(x)|}{|x^n|} < C$ near x=0.

And

$$f = o(g) \text{ near } x_0 \quad \Longleftrightarrow \quad \forall \varepsilon>0,\ \exists \eta>0,\ \forall x \ s.t.\ |x-x_0|<\eta,\ |f(x)| < \varepsilon|g(x). \tag{G.50}$$

and f is said to be "negligible compared with g near x_0".

If $|g|>0$ then the conclusion reads $\frac{|f(x)|}{|g(x)|} \longrightarrow_{x\to x_0} 0$. And $f = o(x^n)$ near x=0 iff $\frac{|f(x)|}{|x^n|} \longrightarrow_{x\to 0} 0$.

G.8.2 Definition of the infinitesimal strain tensor $\underline{\underline{\varepsilon}}$

The motion is supposed to be C^2. Along a trajectory, with $F_P^{t_0}(t_0) = I$ we have, near t_0,

$$F_P^{t_0}(t_0{+}h) = I + O(h), \tag{G.51}$$

thus $F_P^{t_0}(t_0{+}h).\vec{W} = \vec{W} + O(h)$ for all $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$, i.e., near t_0,

$$||\vec{w} - \vec{W}|| = O(h) \quad \text{when} \quad \vec{w} = F_P^{t_0}(t_0{+}h).\vec{W}. \tag{G.52}$$

This supposes the use of a unique inner dot product $(\cdot,\cdot)_G = (\cdot,\cdot)_g$ at all time, and (G.52) means $||\vec{w} - \vec{W}||_g = O(h)$ near t_0.

Definition G.21 If $(\cdot,\cdot)_g$ is an inner dot product, the same at all time, and if $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis, the same at all time, then the infinitesimal strain tensor at P is the matrix defined by

$$[\underline{\underline{\varepsilon}}(P)]_{|\vec{e}} = \frac{[F(P)]_{|\vec{e}} + [F(P)]^T_{|\vec{e}}}{2} - [I], \tag{G.53}$$

abusively written in short,

$$\underline{\underline{\varepsilon}} := \frac{F + F^T}{2} - I \quad \text{(matrix meaning).} \tag{G.54}$$

(And more precisely, at $P \in \Omega_{t_0}$ and between t_0 and t, $[\underline{\underline{\varepsilon}}_t^{t_0}(P)]_{|\vec{e}} = \frac{[F_t^{t_0}(P)]_{|\vec{e}} + [F_t^{t_0}(P)]^T_{|\vec{e}}}{2} - [I]$.)

So $\underline{\underline{\varepsilon}}.\vec{W} = \frac{F.\vec{W} + F^T.\vec{W}}{2} - .\vec{W}$ means $[\underline{\underline{\varepsilon}}]_{|\vec{e}}.[\vec{W}]_{|\vec{e}} = \frac{[F]_{|\vec{e}}.[\vec{W}]_{|\vec{e}} + [F]^T_{|\vec{e}}.[\vec{W}]_{|\vec{e}}}{2} - [\vec{W}]_{|\vec{e}}$.

Remark G.22 $\underline{\underline{\varepsilon}}$ in (G.54) can***not*** be a tensor (cannot be a function) since $F^{t_0}_t(P) : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t$ and $F^{t_0}_t(P)^T : \vec{\mathbb{R}}^n_t \to \vec{\mathbb{R}}^n_{t_0}$ and $I_{t_0} : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_{t_0}$ don't have the same definition domain.

So $\underline{\underline{\varepsilon}}$ is not a function, is not a tensor: It is a matrix... But is called "the infinitesimal strain tensor"... ∴

Proposition G.23 *The Green–Lagrange tensor* $E = \frac{F^T.F-I}{2} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ *satisfies near* t_0*:*

$$E = \underline{\underline{\varepsilon}} + o(t{-}t_0) \quad (= \frac{F+F^T}{2} - I + o(t{-}t_0)) \quad \textit{(matrix meaning)}, \tag{G.55}$$

which means $[E] = [\underline{\underline{\varepsilon}}] + o(t{-}t_0)$*. Thus, "for small deformations" we write* $E \simeq \underline{\underline{\varepsilon}}$*, i.e.* $E \simeq \frac{F+F^T}{2} - I$*.*

Interpretation: (G.55) is a linearization of E*, since we keep the linear part of the "quadratic"* $E = \frac{1}{2}(F^T.F - I)$ *given by* $(E.\vec{W},\vec{U})_g = \frac{1}{2}\big((F.\vec{W},F.\vec{U})_g - (\vec{W},\vec{U})_g\big)$ *for all* $\vec{U},\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$ *("motion squared" cf. the* $(F\cdot,F\cdot)_g$ *term).*

Proof. A $(\cdot,\cdot)_g$-orthonormal basis being chosen, $[F^T] =^{(G.3)} [F]^T$, thus $[C] = [F]^T.[F]$, thus

$$2[E] = [C] - [I] = [F]^T.[F] - [I] = ([F]^T - [I]).([F] - [I]) + [F]^T + [F] - 2[I]. \tag{G.56}$$

Then, near t_0 and with $h = t{-}t_0$, (G.51) gives $([F]^T - [I]).([F] - I]) = O(h)O(h) = O(h^2)$, thus $2[E] = [F]^T + [F] - 2[I] + O(h)$, thus (G.55). ∴

H Finger tensor $F.F^T$ (left Cauchy–Green tensor)

Finger's approach is consistent with the foundations of relativity (Galileo classical relativity or Einstein general relativity): We can only do measurements at the current time t, and we can refer to the past.

There is a lot of misunderstandings, as was the case for the Cauchy–Green deformation tensor C, due to the lack of precise definitions: Definition domain? Value domain? Points at stake (p or P)? Euclidean dot product (English? French?)? Covariance? Contravariance?...

H.1 Definition

Let $\widetilde{\Phi}$ be motion, $t_0 \in \mathbb{R}$, Φ^{t_0} the associated motion, $P \in \Omega_{t_0}$, $t \in \mathbb{R}$, and $F^{t_0}_t(P) := d\Phi^{t_0}_t(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$. And let $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$ be Euclidean dot products in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$.

Definition H.1 The Finger tensor $\underline{\underline{b}}^{t_0}_t(p_t)$, or left Cauchy–Green deformation tensor, at t at p_t relative to t_0 is the endomorphism $\in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$ defined by, with $P = {\Phi^{t_0}_t}^{-1}(p_t)$,

$$\underline{\underline{b}}^{t_0}_t(p_t) := F^{t_0}_t(P).(F^{t_0}_t)^T_{Gg}(p_t) \quad \text{written in short} \quad \boxed{b = F.F^T}, \tag{H.1}$$

i.e. is defined by $(\underline{\underline{b}}^{t_0}_t(p_t).\vec{w}_1,\vec{w}_2)_g = (F^{t_0}_t(P)^T.\vec{w}_1, F^{t_0}_t(P)^T.\vec{w}_2)_G = ((F^{t_0}_t)^T(p_t).\vec{w}_1, (F^{t_0}_t)^T(p_t).\vec{w}_2)_G$, for all $\vec{w}_1,\vec{w}_2$ vectors at $p_t \in \Omega_t$, written in short

$$(\underline{\underline{b}}.\vec{w}_1,\vec{w}_2)_g = (F^T.\vec{w}_1, F^T.\vec{w}_2)_G. \tag{H.2}$$

(To compare with $C = F^T.F$ and $(C.\vec{W}_1,\vec{W}_2)_G = (F.\vec{W}_1,F.\vec{W}_2)_g$.)

And the Finger tensor relative to t_0 is

$$\underline{\underline{b}}^{t_0} : \begin{cases} \mathcal{C} = \bigcup_t(\{t\} \times \Omega_t) & \to \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t) \\ \qquad\qquad (t,p_t) & \to \underline{\underline{b}}^{t_0}(t,p_t) := \underline{\underline{b}}^{t_0}_t(p_t). \end{cases} \tag{H.3}$$

NB: $\underline{\underline{b}}^{t_0}$ looks like a Eulerian function, but isn't, since it depends on a t_0.

Other definition found:

$$B^{t_0}_t := \underline{\underline{b}}^{t_0}_t \circ (\Phi^{t_0}_t)^{-1}, \quad \text{i.e.} \quad B^{t_0}_t(P) := \underline{\underline{b}}^{t_0}_t(p_t) = F^{t_0}_t(P).F^{t_0}_t(P)^T, \quad \text{written} \quad B = F.F^T. \tag{H.4}$$

Pay attention: $B^{t_0}_t(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$ is an endomorphism at t at p_t, not at t_0 at P: E.g., $B^{t_0}_t(P).\vec{w}_t(p_t) = \underline{\underline{b}}^{t_0}_t(p_t).\vec{w}_t(p_t)$ is meaningful, while $B^{t_0}_t(P).\vec{W}_{t_0}(P)$ is absurd.

Remark H.2 The push-forward by $\Phi := \Phi^{t_0}_t$ of the Cauchy–Green deformation tensor $C = F^T.F$ is $\Phi_*(C) = F.C.F^{-1} = F.F^T = \underline{\underline{b}}$, cf. (8.15): It is the Finger tensor. So the endomorphism C in $\vec{\mathbb{R}}^n_{t_0}$ is the pull-back of the endomorphism $\underline{\underline{b}}$ in $\vec{\mathbb{R}}^n_t$. (However a push-forward and a pull-back don't depend on any inner dot product while the transposed F^T does...). ∴

H.2 $\underline{\underline{b}}^{-1}$

With pull-backs (towards the virtual power principle at t). With $p_t = \Phi_t^{t_0}(P)$ and $\vec{W}_i(P) = (F_t^{t_0}(P))^{-1}.\vec{w}_i(p_t)$:

$$(\vec{W}_1, \vec{W}_2)_G = (F^{-1}.\vec{w}_1, F^{-1}.\vec{w}_2)_G = (F^{-T}.F^{-1}.\vec{w}_1, \vec{w}_2)_g = (\underline{\underline{b}}^{-1}.\vec{w}_1, \vec{w}_2)_g. \tag{H.5}$$

So $\underline{\underline{b}}^{-1} := (\underline{\underline{b}}_t^{t_0})^{-1}$ is useful:

$$(\underline{\underline{b}}_t^{t_0})^{-1} : \begin{cases} \Omega_t & \rightarrow \mathcal{L}(\vec{\mathbb{R}}_t^n; \vec{\mathbb{R}}_t^n) \\ p_t & \rightarrow (\underline{\underline{b}}_t^{t_0})^{-1}(p_t) = F_t^{t_0}(P)^{-T}.F_t^{t_0}(P)^{-1} = H_t^{t_0}(p_t)^T.H_t^{t_0}(p_t) \end{cases} \tag{H.6}$$

with $p_t = \Phi_t^{t_0}(P)$ and $H_t^{t_0}(p_t) = (F_t^{t_0}(P))^{-1}$ cf. (4.41). Thus we can define

$$(\underline{\underline{b}}^{t_0})^{-1} : \begin{cases} \displaystyle\bigcup_t (\{t\} \times \Omega_t) & \rightarrow \mathcal{L}(\vec{\mathbb{R}}_t^n; \vec{\mathbb{R}}_t^n) \\ (t, p_t) & \rightarrow (\underline{\underline{b}}^{t_0})^{-1}(t, p_t) := (\underline{\underline{b}}_t^{t_0})^{-1}(p_t). \end{cases} \tag{H.7}$$

Remark: $(\underline{\underline{b}}^{t_0})^{-1}$ looks like a Eulerian function, but isn't, since it depends on t_0.

In short:

$$\underline{\underline{b}}^{-1} = H^T.H, \quad \text{to compare with} \quad C = F^T.F, \tag{H.8}$$

and with $\vec{w} = F.\vec{W}$,

$$\underline{\underline{b}}^{-1}.\vec{w} = H^T.\vec{W}, \quad \text{to compare with} \quad C.\vec{W} = F^T.\vec{w}, \tag{H.9}$$

and with $\vec{W}_i = F^{-1}.\vec{w}_i$, i.e. $\vec{w}_i = F.\vec{W}_i$,

$$(\underline{\underline{b}}^{-1}.\vec{w}_1, \vec{w}_2)_g = (\vec{W}_1, \vec{W}_2)_G, \quad \text{to compare with} \quad (C.\vec{W}_1, \vec{W}_2)_G = (\vec{w}_1, \vec{w}_2)_g. \tag{H.10}$$

Remark H.3 $p_t = \Phi_t^{t_0}(P)$ and $b(p_t) = F(P).F(P)^T$ and $C(P) = F(P)^T.F(P)$ give

$$\underline{\underline{b}}(p_t).F(P) = F(P).C(P), \tag{H.11}$$

written $\underline{\underline{b}} = F.C.F^{-1}$. Thus $\underline{\underline{b}}^{-1} = F.C^{-1}.F^{-1}$, so

$$\Phi_t^{t_0 *}\underline{\underline{b}}^{-1} = F^{-1}.\underline{\underline{b}}^{-1}.F = F^{-1}.F^{-T} = (F^T.F)^{-1} = C^{-1}, \tag{H.12}$$

i.e. the pull-back of $\underline{\underline{b}}^{-1}$ is C^{-1}, i.e. $\underline{\underline{b}}^{-1}$ is the push-forward of C^{-1}.

⁂

H.3 Time derivatives of $\underline{\underline{b}}^{-1}$

With (H.7) let $(\underline{\underline{b}}^{t_0})^{-1} =^{\text{noted}} \underline{\underline{b}}^{-1} = H^T.H$. Thus, along a trajectory, and with (4.45), we get

$$\begin{aligned} \frac{D\underline{\underline{b}}^{-1}}{Dt} &= \frac{DH^T}{Dt}.H + H^T.\frac{DH}{Dt} = -d\vec{v}^T.H^T.H - H^T.H.d\vec{v} \\ &= -\underline{\underline{b}}^{-1}.d\vec{v} - d\vec{v}^T.\underline{\underline{b}}^{-1}. \end{aligned} \tag{H.13}$$

Exercice H.4 Prove (H.13) with (H.10).

Answer. (H.10) gives $\frac{D}{Dt}(\underline{\underline{b}}^{-1}.\vec{w}_1, \vec{w}_2)_g = 0 = (\frac{D\underline{\underline{b}}^{-1}}{Dt}.\vec{w}_1, \vec{w}_2)_g + (\underline{\underline{b}}^{-1}.\frac{D\vec{w}_1}{Dt}, \vec{w}_2)_g + (\underline{\underline{b}}^{-1}.\vec{w}_1, \frac{D\vec{w}_2}{Dt})_g$, and $\vec{w}_i(t, p(t)) = F^{t_0}(t, P).\vec{W}t_0(P)$ gives $\frac{D\vec{w}_i}{Dt} = d\vec{v}.\vec{w}_i$, thus $(\frac{D\underline{\underline{b}}^{-1}}{Dt}.\vec{w}_1, \vec{w}_2)_g + (\underline{\underline{b}}^{-1}.d\vec{v}.\vec{w}_1, \vec{w}_2)_g + (\underline{\underline{b}}^{-1}.\vec{w}_1, d\vec{v}.\vec{w}_2)_g = 0$, thus (H.13).

⁂

Exercice H.5 Prove (H.13) with $F^T.\underline{\underline{b}}^{-1}.F = I_{t_0}$.

Answer. $\underline{\underline{b}}^{-1} = (F.F^T)^{-1} = F^{-T}.F^{-1}$ gives $F^T.\underline{\underline{b}}^{-1}.F = I_{t_0}$, thus $(F^T)'.\underline{\underline{b}}^{-1}.F + F^T.\frac{D\underline{\underline{b}}^{-1}}{Dt}.F + F^T.\underline{\underline{b}}^{-1}.F' = 0$, thus $F^T.d\vec{v}^T.\underline{\underline{b}}^{-1}.F + F^T.\frac{D\underline{\underline{b}}^{-1}}{Dt}.F + F^T.\underline{\underline{b}}^{-1}.d\vec{v}.F = 0$, thus (H.13).

⁂

H.4 Euler–Almansi tensor $\underline{\underline{a}}$

Euler–Almansi approach is consistent with the foundations of relativity (Galileo relativity or Einstein general relativity): We can only do measurements at the current time t, and we can refer to the past.

At t in Ω_t, consider the Finger tensor $\underline{\underline{b}} = F.F^T$ and its inverse $\underline{\underline{b}}^{-1} = F^{-T}.F^T = H^T.H$ cf. (H.8).

Definition H.6 Euler–Almansi tenor at $p_t \in \Omega_t$ is the endomorphism $\underline{\underline{a}}_t^{t_0}(p_t) \in \mathcal{L}(\vec{\mathbb{R}}_t^n;\vec{\mathbb{R}}_t^n)$ defined by

$$\underline{\underline{a}}_t^{t_0}(p_t) = \frac{1}{2}(I_t - \underline{\underline{b}}_t^{t_0}(p_t)^{-1}) = \frac{1}{2}(I_t - H(p_t)^T.H(p_t)), \tag{H.14}$$

written

$$\underline{\underline{a}} = \frac{1}{2}(I - \underline{\underline{b}}^{-1}) = \frac{1}{2}(I - H^T.H), \tag{H.15}$$

to compare with the Green–Lagrange tensor $E = \frac{1}{2}(C-I) = \frac{1}{2}(F^T.F - I) \in \mathcal{L}(\vec{\mathbb{R}}_{t_0}^n;\vec{\mathbb{R}}_{t_0}^n)$.

Remark: $\underline{\underline{a}}^{t_0}$ looks like a Eulerian function, but isn't, since it depends on t_0.

(H.10) gives ($\vec{w}_i = F.\vec{W}_i$)

$$(\vec{w}_1,\vec{w}_2)_g - (\vec{W}_1,\vec{W}_2)_G = 2(\underline{\underline{a}}.\vec{w}_1,\vec{w}_2)_g, \tag{H.16}$$

to compare with $(\vec{w}_1,\vec{w}_2)_g - (\vec{W}_1,\vec{W}_2)_G = 2(E.\vec{W}_1,\vec{W}_2)_G$. (This also gives $(\underline{\underline{a}}.\vec{w}_1,\vec{w}_2)_g = (E.\vec{W}_1,\vec{W}_2)_G$.) And (H.15) gives

$$F^T.\underline{\underline{a}}.F = E, \quad \text{i.e.} \quad \underline{\underline{a}} = F^{-T}.E.F^{-1}, \tag{H.17}$$

standing for $F_t^{t_0}(P)^T.\underline{\underline{a}}_t^{t_0}(p).F_t^{t_0}(P) = E_t^{t_0}(P)$ when $p = \Phi_t^{t_0}(P)$.

Remark H.7 $\underline{\underline{a}}_t^{t_0}$ is not the push-forward of $E_t^{t_0}$ by $\Phi_t^{t_0}$ (the push-forward is $F.E.F^{-1}$). ∴

H.5 Time Taylor expansion for $\underline{\underline{a}}$

(H.13) gives

$$\frac{D\underline{\underline{a}}}{Dt} = \frac{\underline{\underline{b}}^{-1}.d\vec{v} + d\vec{v}^T.\underline{\underline{b}}^{-1}}{2}. \tag{H.18}$$

H.6 Almansi modified Infinitesimal strain tensor $\underline{\underline{\widetilde{\varepsilon}}}$

We are at t (present time) and remember the past: We prefer a definition of a infinitesimal strain tensor $\underline{\underline{\widetilde{\varepsilon}}}$ from the Euler–Almansi tensor $\underline{\underline{a}}$, instead of $\underline{\underline{\varepsilon}}$ from Euler–Lagrange tensor E, cf. § G.8.2.

Same Euclidean framework as in § G.8.2, and matrix meaning again.

We have $I - \underline{\underline{b}}^{-1} = I - H^T.H = -(I - H^T).(I-H) + 2I - H^T - H$ where H stands for $H_t^{t_0}(p_t)$. Thus, for small displacement we get $I - \underline{\underline{b}}^{-1} = 2I - H^T - H + O(h)$, so

$$\underline{\underline{a}}(t,p(t)) = \underline{\underline{\widetilde{\varepsilon}}}(t,p(t)) + O(h) \quad \text{where} \quad \underline{\underline{\widetilde{\varepsilon}}} := I - \frac{H+H^T}{2}. \tag{H.19}$$

And, with $t = t_0 + h$ we have $F^{t_0}(t,P) = I + (t{-}t_0)\, d\vec{v}(t,P) + o(t{-}t_0)$, cf. (4.35), thus we have $H^{t_0}(t,p(t)) = F^{t_0}(t,P)^{-1} = I - (t{-}t_0)\, d\vec{v}(t,P) + o(t{-}t_0)$ when $p(t) = \Phi^{t_0}(t,P)$. Thus

$$F^{t_0}(t,P) - I = I - H^{t_0}(t,p(t)) + O(t{-}t_0). \tag{H.20}$$

Therefore, for small displacements ($|t - t_0| << 1$):

$$\underline{\underline{a}}(t,p(t)) \simeq \underline{\underline{\widetilde{\varepsilon}}}(t,p(t)) \simeq \underline{\underline{\varepsilon}}^{t_0}(t,P) \quad \text{(matrix meaning)}. \tag{H.21}$$

I Polar decomposition, elasticity and objectivity

I.1 Polar decompositions of F ("isometric objectivity")

The motion is supposed regular, $t_0, t \in \mathbb{R}$, $p_{t_0} \in \Omega_{t_0}$, $F := F_t^{t_0}(p_{t_0})$ ($= d\Phi_t^{t_0}(p_{t_0})$), $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$ are Euclidean dot products in $\vec{\mathbb{R}}_{t_0}^n$ and $\vec{\mathbb{R}}_t^n$, and $C = F^T.F \in \mathcal{L}(\vec{\mathbb{R}}_{t_0}^n;\vec{\mathbb{R}}_{t_0}^n)$. Here the covariant objectivity is abandoned due to the need for inner dot products.

I.1.1 $F = R.U$ (right polar decomposition)

The endomorphism C being $(\cdot,\cdot)_G$-symmetric definite positive (the motion is supposed to be regular), $\exists \alpha_1, ..., \alpha_n \in \mathbb{R}^*_+$ (the eigenvalues), $\exists \vec{c}_1, ..., \vec{c}_n \in \vec{\mathbb{R}}^n_{t_0}$ (associated eigenvectors), such that, for all $i = 1, ..., n$,

$$C.\vec{c}_i = \alpha_i \vec{c}_i \quad \text{and} \quad (\vec{c}_i) \text{ is a } (\cdot,\cdot)_G\text{-orthonormal basis in } \vec{\mathbb{R}}^n_{t_0}. \tag{I.1}$$

So, if $(\vec{a}_i)$ is a $(\cdot,\cdot)_G$-Euclidean basis then $D = P^{-1}.[C]_{\vec{a}}.P$, where $D = \text{diag}(\alpha_1, ..., \alpha_n) = [C]_{\vec{c}}$ and $P^{-1} = P^T$, $P = [P_{ij}]$ being the transition matrix from $(\vec{a}_i)$ to $(\vec{c}_i)$, i.e. defined by $\vec{c}_j = \sum_i P_{ij} \vec{a}_i$ for all j.

Then, define the endomorphism $U \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$, called the right stretch tensor, by, for all $i = 1, ..., n$,

$$U.\vec{c}_i = \sqrt{\alpha_i}\, \vec{c}_i, \quad \text{and} \quad U \overset{\text{noted}}{=} \sqrt{C}, \tag{I.2}$$

the $\sqrt{\alpha_i}$ being called the principal stretches. Then, define the linear map $R \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$, called the rotation map, by

$$R := F \circ U^{-1} \overset{\text{noted}}{=} F.U^{-1}, \tag{I.3}$$

so that

$$\boxed{F = R \circ U} \overset{\text{noted}}{=} R.U, \quad \text{called the right polar decomposition of } F. \tag{I.4}$$

Proposition I.1 *We have:*

$$C = U \circ U \overset{\text{noted}}{=} U^2, \quad U \text{ is symmetric definite positive}, \quad R^{-1} = R^T. \tag{I.5}$$

And

$$\text{the right polar decomposition} \quad F = R \circ U \quad \text{is unique}: \tag{I.6}$$

If $F = \widehat{R} \circ \widehat{U}$ where $\widehat{U} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ is symmetric definite positive and $\widehat{R} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ satisfies $\widehat{R}^{-1} = \widehat{R}^T$, then $\widehat{U} = U$ and $\widehat{R} = R$.

Proof. (I.2) yields $(U \circ U).\vec{c}_j = \lambda \vec{c}_j = C.\vec{c}_j$ for all j, cf. (I.1), thus $U \circ U = C$; Then $(U^T.\vec{c}_i, \vec{c}_j)_G = (\vec{c}_i, U.\vec{c}_j)_G = (\vec{c}_i, \sqrt{\alpha_j}\vec{c}_j)_G = \sqrt{\alpha_j}(\vec{c}_i, \vec{c}_j)_G = \sqrt{\alpha_j}\delta_{ij} = \sqrt{\alpha_i}\delta_{ij} = \sqrt{\alpha_i}(\vec{c}_i, \vec{c}_j)_G = (\sqrt{\alpha_i}\vec{c}_i, \vec{c}_j)_G = (U.\vec{c}_i, \vec{c}_j)_G$ for all i, j, thus $U^T = U$ (symmetry).

Then $R^T \circ R = U^{-T} \circ F^T \circ F \circ U^{-1} = U^{-T} \circ C \circ U^{-1} = U^{-1} \circ (U \circ U) \circ U^{-1} = I_{t_0}$ identity in $\vec{\mathbb{R}}^n_{t_0}$. (Details: $(R^T.R.\vec{W}, \vec{Z})_G = (R.\vec{W}, R.\vec{Z})_g = (F.U^{-1}.\vec{W}, F.U^{-1}.\vec{Z})_g = (F^T.F.U^{-1}.\vec{W}, U^{-1}.\vec{Z})_G = (U^2.U^{-1}.\vec{W}, U^{-1}.\vec{Z})_G = (U^{-1}.U.\vec{W}, \vec{Z})_G = (\vec{W}, \vec{Z})_G$.) Thus $R^{-1} = R^T \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0})$, thus $R \circ R^T = R \circ R^{-1} = I_t$ identity in $\vec{\mathbb{R}}^n_t$.

And $F = \widehat{R} \circ \widehat{U} = R \circ U$ gives $F^T \circ F = \widehat{U}^T \circ \widehat{R}^T \circ \widehat{R} \circ \widehat{U} = U^T \circ \widehat{R}^T \circ \widehat{R} \circ U$, thus $F^T \circ F = \widehat{U}^T \circ \widehat{U}$, with $F^T \circ F = U^T \circ U$, thus $\widehat{U} \circ \widehat{U} = U \circ U = \sqrt{C}$, thus $\widehat{U} = U$ (uniqueness of the positive square root eigenvalues). Hence $\widehat{R} = R$. ∎

I.1.2 $F = S.R_0.U$ (shifted right polar decomposition for covariant objectivity)

In fact we need to be more specific if the gift of ubiquity is prohibited: Since we work with the affine space $\mathbb{R}^n$, consider the Marsden's shifter

$$S := S^{t_0}_t(p_{t_0}) : \begin{cases} T_{p_{t_0}}(\Omega_{t_0}) \overset{\text{noted}}{=} \vec{\mathbb{R}}^n_{t_0} & \to T_{p_t}(\Omega_t) \overset{\text{noted}}{=} \vec{\mathbb{R}}^n_t \\ \vec{w}_{t_0,p_{t_0}} & \to (S.\vec{w}_{t_0,p_{t_0}})(t, p_t) = \vec{w}_{t_0,p_{t_0}} \quad \text{where} \quad p_t = \Phi^{t_0}_t(p_{t_0}). \end{cases} \tag{I.7}$$

NB: 1- S looks like the algebraic identity if you have time and space ubiquity gift (otherwise it is not),

2- S is not a topological identity since it changes the norms in the general case: You consider $||\vec{w}_{t_0,p_{t_0}}||_G$ at t_0 and $||S.\vec{w}_{t_0,p_{t_0}}||_g = ||\vec{w}_{t_0,p_{t_0}}||_g$ at t.

Then, let $R_0 \in \mathcal{L}(T_{p_{t_0}}(\Omega_{t_0}); T_{p_{t_0}}(\Omega_{t_0})) =^{\text{noted}} \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ be the endomorphism defined by, in short,

$$R_0 := S^{-1} \circ R \overset{\text{noted}}{=} S^{-1}.R, \quad \text{so} \quad \boxed{R = S.R_0} \quad (= S \circ R_0). \tag{I.8}$$

Full notations: $(R_0)^{t_0}_{t,Gg}(p_{t_0}) := (S^{t_0}_t)^{-1}(R^{t_0}_{t,Gg}(p_{t_0}))$.

Proposition I.2 *The endomorphism $R_0 = S^{-1} \circ R$ is a rotation operator in $(\vec{\mathbb{R}}^n_{t_0}, (\cdot,\cdot)_G)$:*

$$R_0^{-1} = R_0^T \quad \text{in } (\vec{\mathbb{R}}^n_{t_0}, (\cdot,\cdot)_G). \tag{I.9}$$

And

$$F = S \circ R_0 \circ U. \tag{I.10}$$

Interpretation: F is composed of: The pure deformation U (endomorphism in $\vec{\mathbb{R}}^n_{t_0}$), the rotation R_0 (endomorphism in $\vec{\mathbb{R}}^n_{t_0}$), and the shift operator $S : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t$ (from past to present time and position).

Proof.

$$\begin{aligned}
(R_0^T.\vec{W}_2, \vec{W}_1)_G &= (R_0.\vec{W}_1, \vec{W}_2)_G \quad \text{(definition of the transposed)}\\
&= (S^{-1}.R.\vec{W}_1, \vec{W}_2)_G \quad \text{(definition of } R_0)\\
&= (R.\vec{W}_1, \vec{W}_2)_G \quad (S \text{ is the algebraic identity})\\
&= (R^T.\vec{W}_2, \vec{W}_1)_g \quad \text{(definition of } R^T)\\
&= (R^{-1}.\vec{W}_2, \vec{W}_1)_g \quad \text{(cf. (I.5))}\\
&= (R_0^{-1}.\vec{W}_2, \vec{W}_1)_G \quad (S \text{ is the algebraic identity}),
\end{aligned} \tag{I.11}$$

true for all $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$, thus $R_0^T = R_0^{-1}$ in $(\vec{\mathbb{R}}^n_{t_0}, (\cdot,\cdot)_G)$. And (I.8) and (I.4) give (I.10). ∴

Exercice I.3 Let $D = \text{diag}(\alpha_i)$, let $(\vec{a}_i)$ be a Euclidean basis in $\vec{\mathbb{R}}^n_{t_0}$, let P be the transition matrix from $(\vec{a}_i)$ to $(\vec{c}_i)$, so $[C]_{|\vec{a}} = P.D.P^{-1}$; Prove $[U]_{|\vec{a}} = P.\sqrt{D}.P^{-1}$. Case $(\vec{a}_i) = (\vec{E}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis?

Answer. The n equations (I.1) (for $j = 1, ..., n$), read as the matrix equation $[C]_{|\vec{a}}.P = P.D$ since $[\vec{c}_j]_{\vec{a}}$ is the j-th column of P. And he n equations (I.2) (for $j = 1, ..., n$), read as the matrix equation $[U]_{|\vec{a}}.P = P.\sqrt{D}$ since $[\vec{c}_j]_{\vec{a}}$ is the j-th column of P. ∴

Remark I.4 Instead of $R_0 \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$, cf. (I.8), you may prefer to consider $\widetilde{R}_0 \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t)$ defined by $R = \widetilde{R}_0 \circ S$, i.e., $\widetilde{R}_0 = R \circ S^{-1}$. ∴

I.1.3 $F = V.R$ (left polar decomposition)

Same steps than for the right polar decomposition, but with pull-backs (with F^{-1} instead of F).

Let $p_t = \Phi^{t_0}_t(p_{t_0}) \in \Omega_t$, let $\underline{\underline{b}}^{t_0}_t(p_t) := F^{t_0}_t(p_{t_0}) \circ (F^{t_0}_t)^T(p_t) \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t)$, written $\underline{\underline{b}} = F \circ F^T$ (the left Cauchy–Green deformation tensor also called the Finger tensor). The endomorphism b being symmetric definite positive: $\exists \beta_1, ..., \beta_n \in \mathbb{R}^*_+$ (the eigenvalues), $\exists \vec{d}_1, ..., \vec{d}_n \in \vec{\mathbb{R}}^n_t$ (associated eigenvectors), such that, for all $i = 1, ..., n$,

$$\underline{\underline{b}}.\vec{d}_i = \beta_i \vec{d}_i, \quad \text{and} \quad (\vec{d}_i) \text{ is a } (\cdot,\cdot)_g\text{-orthonormal basis in } \vec{\mathbb{R}}^n_t. \tag{I.12}$$

Then, define the unique endomorphism $V \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t)$, called the left stretch tensor, by, for all $i = 1, ..., n$,

$$V.\vec{d}_i = \sqrt{\beta_i}\vec{d}_i, \quad \text{and} \quad V \overset{\text{noted}}{=} \sqrt{\underline{\underline{b}}}. \tag{I.13}$$

(Full notation: $V^{t_0}_{t,Gg}(p_t) = \sqrt{\underline{\underline{b}}^{t_0}_t(p_t)_{Gg}}$.) Then define the linear map $R_\ell \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ by

$$R_\ell := V^{-1} \circ F \overset{\text{noted}}{=} V^{-1}.F, \tag{I.14}$$

so that

$$\boxed{F = V \circ R_\ell} \overset{\text{noted}}{=} V.R_\ell, \quad \text{called the left polar decomposition of } F. \tag{I.15}$$

Proposition I.5 *We have: 1-*

$$\underline{\underline{b}} = V \circ V \overset{\text{noted}}{=} V^2, \quad V \text{ is symmetric definite positive}, \quad R_\ell^{-1} = R_\ell^T. \tag{I.16}$$

And the left polar decomposition $F = V \circ R$ is unique.

2- $R_\ell = R$ and $V = R.U.R^{-1}$ (so U and V are similar), thus U and V have the same eigenvalues, i.e., $\alpha_i = \beta_i$ for all i, and $\vec{d}_i = R.\vec{c}_i$ for all i gives a relation between eigenvectors.

Proof. 1- Use F^{-1} and $\underline{\underline{b}}^{-1} = (F^{-1})^T.(F^{-1})$, instead of F and $C = F^T.F$, to get $F^{-1} = R_\ell^{-1}.U_\ell^{-1}$, cf. (I.3); Thus $F = U_\ell.R_\ell$; Then name $U_\ell = V$ to get (I.15) and (I.16).

2- $V.R_\ell = F = R.U = (R.U.R^{-1}).R$, thus, by uniqueness of the right polar decomposition, $V = R.U.R^{-1}$ (so U and V are similar) and $R_\ell = R$. Thus, with (I.12), $\beta_i\vec{d_i} = V.\vec{d_i} = R.U.(R^{-1}.\vec{d_i})$, thus with $\vec{c_i} = R^{-1}.\vec{d_i}$, then $(\vec{c_i})$ is an orthonormal basis in $\vec{\mathbb{R}}^n_{t_0}$ and $\beta_i R.\vec{c_i} = R.U.\vec{c_i} = \alpha_i R.\vec{c_i}$ gives $\beta_i = \alpha_i$ and the $\vec{c_i}$ are eigenvectors of U, for all i. ▪

I.2 Linear elasticity: A Classical "tensorial" approach

I.2.1 Classical approach ("isometric objectivity"), and an issue

With the infinitesimal strain "tensor" (which is not a tensor but a matrix)

$$\underline{\underline{\varepsilon}} = \frac{F + F^T}{2} - I, \tag{I.17}$$

the homogeneous isotropic elasticity constitutive law reads (matrix equation for the stress)

$$(\underline{\underline{\sigma}}(\Phi) =) \quad \underline{\underline{\sigma}} = \lambda \mathrm{Tr}(\underline{\underline{\varepsilon}}) I + 2\mu\underline{\underline{\varepsilon}}, \tag{I.18}$$

where λ, μ are the Lamé coefficients and $\underline{\underline{\sigma}}$ is the Cauchy stress "tensor".

Issue: Recall: Adding F and F^T to make $\underline{\underline{\varepsilon}}$ is functionally a mathematical nonsense since $F : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t$ and $F^T : \vec{\mathbb{R}}^n_t \to \vec{\mathbb{R}}^n_{t_0}$ and I is some identity operator: $\underline{\underline{\sigma}}$ is not a tensor. In particular the meaning of $\mathrm{Tr}(\underline{\underline{\varepsilon}})$ is questionable (since $\underline{\underline{\varepsilon}}$ is not an endomorphism and $\mathrm{Tr}(\underline{\underline{\varepsilon}})$ means $\mathrm{Tr}([\underline{\underline{\varepsilon}}]) = \frac{\mathrm{Tr}([F])+\mathrm{Tr}([F^T])}{2} - n$), as well as the meaning of $\underline{\underline{\varepsilon}}.\vec{n} = \frac{1}{2}(F.\vec{n} + F^T.\vec{n})$, or the meaning of

$$\underline{\underline{\sigma}}.\vec{n} = \lambda \mathrm{Tr}(\underline{\underline{\varepsilon}})\vec{n} + 2\mu\underline{\underline{\varepsilon}}.\vec{n} \tag{I.19}$$

since $\vec{n}$ has to be defined at (t_0, p_{t_0}) for F and at (t, p_t) for F^T. (Cauchy's approach: $\vec{n}$ is defined at (t, p_t).)

So, despite the eventual claims, neither $\underline{\underline{\varepsilon}}$ nor $\underline{\underline{\sigma}}$ are tensors (they don't have a functional meaning): They only have a questionable matrix meaning (observer dependent) $[\underline{\underline{\varepsilon}}] := \frac{[F]+[F]^T}{2} - [I]$ and

$$[\underline{\underline{\sigma}}] = \lambda \mathrm{Tr}([\underline{\underline{\varepsilon}}])[I] + 2\mu[\underline{\underline{\varepsilon}}], \quad \text{and} \quad [\underline{\underline{\sigma}}].[\vec{n}] = \lambda \mathrm{Tr}([\underline{\underline{\varepsilon}}])[\vec{n}] + 2\mu[\underline{\underline{\varepsilon}}].[\vec{n}]. \tag{I.20}$$

Remark I.6 To justify the name "tensor" applied to $\underline{\underline{\varepsilon}}$, you may read: "For small displacements the Eulerian variable p_t and the Lagrangian variable p_{t_0} can be confused": $p_t \simeq p_{t_0}$ (so Ω_{t_0} and Ω_t are "almost equal", so $F(p_{t_0}) + F^T(p_t)$ can be considered). Which means that you use the zero-th order Taylor expansions $p_t = \Phi^{t_0}_{p_{t_0}}(t) = p_{t_0} + o(1)$. But then, you can***not*** also use the first (or higher) order Taylor expansion in following calculations, e.g. you cannot use velocities... ▪

I.2.2 A functional (tensorial) formulation ("isometric objectivity")

Question: Can the constitutive law (I.18) be modified into a tensorial expression (a functional expression)? Proposal for a yes:

1. Consider the "right polar decomposition" $F = R.U$ where $U \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$, cf. (I.3). The Green Lagrange tensor $E = \frac{C-I}{2}$ (endomorphism in $\vec{\mathbb{R}}^n_{t_0}$) then reads, with (I.5),

$$E = \frac{U^2 - I_{t_0}}{2} = \frac{(U - I_{t_0})^2 + 2(U - I_{t_0})}{2} \tag{I.21}$$

(the Green–Lagrange tensor is independent of the rotation R), thus, with $U - I_{t_0} = O(h)$ (small deformation approximation), we get the modified infinitesimal strain tensor at $p_{t_0} \in \Omega_{t_0}$

$$\boxed{\underline{\underline{\widetilde{\varepsilon}}} = U - I_{t_0}} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0}), \tag{I.22}$$

endomorphism in $\vec{\mathbb{R}}^n_{t_0}$ (to compare with $\underline{\underline{\varepsilon}}$ which is not a function, cf. the previous §). (Full notation $\underline{\underline{\widetilde{\varepsilon}}}^{t_0}_{t,Gg}(p_{t_0}) = U^{t_0}_{t,Gg}(p_{t_0}) - I_{t_0}(p_{t_0})$ in $\mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$.) And, for all $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$ we get

$$\boxed{\underline{\underline{\widetilde{\varepsilon}}}.\vec{W} = U.\vec{W} - \vec{W} = R^{-1}.\vec{w} - \vec{W}} \in \vec{\mathbb{R}}^n_{t_0}, \quad \text{when} \quad \vec{w} = F.\vec{W} \text{ (push-forward).} \tag{I.23}$$

Interpretation: From $\vec{w} = F.\vec{W} = R.U.\vec{W} \in \vec{\mathbb{R}}^n_t$ (the deformed by the motion), remove the "rigid body rotation" to get $R^{-1}.\vec{w} = U.\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$, to which you remove the initial $\vec{W}$ to obtain $\underline{\underline{\widetilde{\varepsilon}}}.\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$.

In particular $||\underline{\underline{\widetilde{\varepsilon}}}.\vec{W}||_G = ||(U{-}I_{t_0}).\vec{W}||_G$ measures the relative elongation undergone by $\vec{W}$. And you can then apply R to get back into $\vec{\mathbb{R}}^n_t$ at p_t:

$$R.(\underline{\underline{\widetilde{\varepsilon}}}.\vec{W}) = F.\vec{W} - R.\vec{W} = \vec{w} - R.\vec{W} \ \in \vec{\mathbb{R}}^n_t, \quad \text{when} \quad \vec{w} = F.\vec{W} = \ \text{(push-forward)}. \tag{I.24}$$

2. Then, at $p_{t_0} \in \Omega_{t_0}$, consider the stress tensor $\widetilde{\Sigma}(\Phi) =^{\text{noted}} \widetilde{\Sigma} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ (functionally well) defined by

$$\boxed{\widetilde{\Sigma} = \lambda \mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}}) I_{t_0} + 2\mu \underline{\underline{\widetilde{\varepsilon}}}} = \lambda \mathrm{Tr}(U{-}I_{t_0}) I_{t_0} + 2\mu(U{-}I_{t_0}). \tag{I.25}$$

(The trace $\mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}})$ is well defined since $\underline{\underline{\widetilde{\varepsilon}}}$ is an endomorphism.) Then for any $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$ you get in $\vec{\mathbb{R}}^n_{t_0}$, at $p_{t_0} \in \Omega_{t_0}$,

$$\widetilde{\Sigma}.\vec{W} = \lambda \mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}}) \vec{W} + 2\mu \underline{\underline{\widetilde{\varepsilon}}}.\vec{W} = \lambda \mathrm{Tr}(U{-}I_{t_0}) \vec{W} + 2\mu(U.\vec{W}{-}\vec{W}) \tag{I.26}$$

(functionally well defined in $\vec{\mathbb{R}}^n_{t_0}$). Then rotate and shift with R to get into $\vec{\mathbb{R}}^n_t$ at p_t,

$$\begin{aligned} R.\widetilde{\Sigma}.\vec{W} &= \lambda \mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}}) R.\vec{W} + 2\mu R.\underline{\underline{\widetilde{\varepsilon}}}.\vec{W} = \lambda \mathrm{Tr}(U{-}I_{t_0}) R.\vec{W} + 2\mu R.(U{-}I_{t_0}).\vec{W} \\ &= \lambda \mathrm{Tr}(U{-}I_{t_0}) R.\vec{W} + 2\mu (F-R).\vec{W}, \\ &= \lambda \mathrm{Tr}(U{-}I_{t_0}) R.\vec{W} + 2\mu (\vec{w}-R.\vec{W}), \quad \text{where} \quad \vec{w} = F.\vec{W} = R.U.\vec{W}. \end{aligned} \tag{I.27}$$

You have defined the two point tensor (functionally well defined)

$$R.\widetilde{\Sigma} = \lambda \mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}}) R + 2\mu R.\underline{\underline{\widetilde{\varepsilon}}} \ \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t). \tag{I.28}$$

3. Then you can propose the constitutive law with the stress tensor (the symmetric endomorphism) in $\vec{\mathbb{R}}^n_t$ given by

$$(\underline{\underline{\widetilde{\sigma}}}(\Phi) =) \quad \boxed{\underline{\underline{\widetilde{\sigma}}} = R \circ \widetilde{\Sigma} \circ R^{-1}} \overset{\text{noted}}{=} R.\widetilde{\Sigma}.R^{-1} \ \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t). \tag{I.29}$$

(Functionally well defined.) And, for all vector fields $\vec{w}$ defined in Ω_t, you get the (functionally well defined) vector field

$$\underline{\underline{\widetilde{\sigma}}}.\vec{w} = R.\widetilde{\Sigma}.R^{-1}.\vec{w} \ \in \vec{\mathbb{R}}^n_t. \tag{I.30}$$

Interpretation of (I.29)-(I.30): Shift and rigid rotate backward by applying R^{-1}, apply the elastic stress law with Σ which corresponds to a rotation free motion (Noll's frame indifference principle), then shift and rigid rotate forward by applying R.

Detailed expression for (I.29)-(I.30): With $\mathrm{Tr}(R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1}) = \mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}})$ (see exercise I.8), we get, at (t,p_t),

$$\begin{aligned} \underline{\underline{\widetilde{\sigma}}} = \lambda \mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}})\, I_t + 2\mu R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1} &= \lambda \mathrm{Tr}(U{-}I_{t_0})\, I_t + 2\mu R.(U{-}I_{t_0}).R^{-1} \\ &= \lambda \mathrm{Tr}(U{-}I_{t_0})\, I_t + 2\mu (F.R^{-1}{-}I_t). \end{aligned} \tag{I.31}$$

And for any $\vec{w} \in \vec{\mathbb{R}}^n_t$, and with $\vec{w} = R.\vec{W}$, you get

$$\begin{aligned} \underline{\underline{\widetilde{\sigma}}}.\vec{w} = \lambda \mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}})\, \vec{w} + 2\mu R.\underline{\underline{\widetilde{\varepsilon}}}.\vec{W} &= \lambda \mathrm{Tr}(U{-}I_{t_0})\, \vec{w} + 2\mu R.(U{-}I_{t_0}).\vec{W} \\ &= \lambda \mathrm{Tr}(U{-}I_{t_0})\, \vec{w} + 2\mu (R.U.R^{-1}.\vec{w}{-}\vec{w}). \end{aligned} \tag{I.32}$$

To compare with the classical functionally meaningless (I.19).

Remark I.7 Doing so, you avoid the use of the Piola–Kirchhoff tensors. ⁂

Exercice I.8 Prove: $\mathrm{Tr}(R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1}) = \mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}}) = \sum_i(\alpha_i{-}1)$. (NB: $\underline{\underline{\widetilde{\varepsilon}}}$ is an endomorphism in $\vec{\mathbb{R}}^n_{t_0}$ while $R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1}$ is an endomorphism in $\vec{\mathbb{R}}^n_t$.)

Answer. $\det_{|\vec{e}}(R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1} - \lambda I_t) = \det_{|\vec{e}}(R.(\underline{\underline{\widetilde{\varepsilon}}}{-}\lambda I_{t_0}).R^{-1}) = \det_{|\vec{E},\vec{e}}(R).\det_{|\vec{E}}(\underline{\underline{\widetilde{\varepsilon}}}{-}\lambda I).\det_{|\vec{e},\vec{E}}(R^{-1}) = \det_{|\vec{E}}(\underline{\underline{\widetilde{\varepsilon}}}{-}\lambda I)$ for all Euclidean bases $(\vec{E}_i)$ and $(\vec{e}_i)$ in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$. (With $L = \underline{\underline{\widetilde{\varepsilon}}}$ and components, $\mathrm{Tr}(R.L.R^{-1}) = \sum_i (R.L.R^{-1})^i_i = \sum_{ijk} R^i_j L^j_k (R^{-1})^k_i = \sum_{jk} (R^{-1}.R)^k_j L^j_k = \sum_{jk} \delta^k_j L^j_k = \sum_j L^j_j = \mathrm{Tr}(L)$.) ⁂

Exercice I.9 Elongation in $\mathbb{R}^2$ along the first axis : origin O, same Euclidean basis $(\vec{E}_1,\vec{E}_2)$ and Euclidean dot product at all time, $\xi > 0$, $t \geq t_0$, $L, H > 0$, $P \in [0,L] \times [0,H]$, $[\overrightarrow{OP}]_{|\vec{E}} = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}$, and $[\overrightarrow{O\Phi_t^{t_0}(P)}]_{|\vec{E}} = \begin{pmatrix} X_0+\xi(t{-}t_0)X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} X_0(\kappa{+}1) \\ Y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = [\overrightarrow{Op}]_{|\vec{E}}$, where $\kappa = \xi(t{-}t_0) > 0$ for $t > t_0$.

1- Give F, C, $U = \sqrt{C}$ and $R = F.U^{-1}$. Relation with the classical expression ?

2- Spring $\overrightarrow{OP} = \overrightarrow{Oc_{t_0}}(s) = X_0\vec{E}_1 + Y_0\vec{E}_2 + s\vec{W}$, i.e. $[\overrightarrow{OP}]_{|\vec{E}} = [\overrightarrow{Oc_{t_0}}]_{|\vec{E}} = \begin{pmatrix} X_0{+}sW_1 \\ Y_0{+}sW_2 \end{pmatrix}_{|\vec{E}}$ with $s \in [0,L]$ and $\vec{W} = W_1\vec{E}_1 + W_2\vec{E}_2$. Give the deformed spring, i.e. give $p = c_t(s) = \Phi_t^{t_0}(c_{t_0}(s))$, and $\vec{c}_t{}'$, and the stretch ratio.

Answer. 1- $[F] = [d\Phi] = \begin{pmatrix} \kappa{+}1 & 0 \\ 0 & 1 \end{pmatrix}$, same Euclidean dot product and basis at all time, thus $[F^T] = [F]^T = [F]$, then $[C] = [F^T].[F] = [F]^2 = \begin{pmatrix} (\kappa{+}1)^2 & 0 \\ 0 & 1 \end{pmatrix}$, thus $[U] = [F] = \begin{pmatrix} \kappa{+}1 & 0 \\ 0 & 1 \end{pmatrix}$, thus $[R] = [I]$. All the matrices are given relative to the basis $(\vec{E}_i)$, thus F, C, U, R (e.g., $C.\vec{E}_1 = (\kappa{+}1)^2\vec{E}_1$ and $C.\vec{E}_2 = \vec{E}_2$).

Since $R = I$ and $[\underline{\underline{\varepsilon}}] = [\underline{\underline{\widetilde{\varepsilon}}}]$, (I.31) gives the usual result $[\underline{\underline{\sigma}}] = \lambda\mathrm{Tr}([\underline{\underline{\varepsilon}}])I + 2\mu[\underline{\underline{\varepsilon}}]$, cf (I.18) (matrix meaning).

2- $\overrightarrow{Oc_t(s)} = \overrightarrow{O\Phi_t^{t_0}(c_{t_0}(s))} = \begin{pmatrix} (X_0{+}sW_1)(\kappa{+}1) \\ Y_0{+}sW_2 \end{pmatrix}_{|\vec{E}}$, thus $\vec{c}_t{}'(s) = \begin{pmatrix} W_1(\kappa{+}1) \\ W_2 \end{pmatrix}_{|\vec{E}}$, stretch ration $\frac{W_1^2(\kappa+1)^2+W_2^2}{W_1^2+W_2^2}$ at (t, p_t). ⁂

Exercice I.10 Simple shear in $\mathbb{R}^2$: $[\overrightarrow{O\Phi_t^{t_0}(P)}]_{|\vec{E}} = \begin{pmatrix} X+\xi(t{-}t_0)Y \\ Y \end{pmatrix} =^{\text{noted}} \begin{pmatrix} X+\kappa Y \\ Y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = [\overrightarrow{Op}]_{|\vec{E}}$. Same questions, and moreover give the eigenvalues of C.

Answer. 1- $[F] = \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix}$, $[C] = \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix}.\begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \kappa \\ \kappa & \kappa^2{+}1 \end{pmatrix}$. Eigenvalues: $\det(C - \lambda I) = \lambda^2 - (2{+}\kappa^2)\lambda + 1$. Discriminant $\Delta = (2{+}\kappa^2)^2 - 4 = \kappa^2(\kappa^2{+}4)$. Eigenvalues $\alpha_\pm = \frac{1}{2}(2{+}\kappa^2 \pm \kappa\sqrt{\kappa^2{+}4})$. (We check that $\alpha_\pm > 0$.) Eigenvectors $\vec{v}_\pm$(main directions of deformations) given by $(1{-}\alpha_\pm)x + \kappa y = 0$, i.e., $y = \frac{1}{2}(\kappa \pm \sqrt{\kappa^2{+}4})x$, thus, e.g., $\vec{v}_\pm = \begin{pmatrix} 2 \\ \kappa \pm \sqrt{\kappa^2{+}4} \end{pmatrix}$. (We check that $\vec{v}_+ \perp \vec{v}_-$.) With P the transition matrix from $(\vec{E}_1,\vec{E}_2)$ to $(\frac{\vec{v}_+}{||\vec{v}_+||}, \frac{\vec{v}_-}{||\vec{v}_-||})$ and $D = \mathrm{diag}(\alpha_+,\alpha_-)$ we get $C = P.D.P^{-1}$ (with $P^{-1} = P^T$ since here $(\frac{\vec{v}_+}{||\vec{v}_+||}, \frac{\vec{v}_-}{||\vec{v}_-||})$ is an orthonormal basis), thus $U = P.\sqrt{D}.P^{-1}$ (we check that $U^T = U$ and $U^2 = C$). And $R = F.U^{-1}$.

2- $\overrightarrow{Oc_t(s)} = \overrightarrow{O\Phi_t^{t_0}(c_{t_0}(s))} = \begin{pmatrix} (X_0{+}sW_1) + \kappa(Y_0{+}sW_2) \\ Y_0{+}sW_2 \end{pmatrix}$, thus $[\vec{c}_t{}'(s)] = \begin{pmatrix} W_1 + \kappa W_2 \\ W_2 \end{pmatrix}$. Stretch ratio $\frac{(W_1+\kappa W_2)^2+W_2^2}{W_1^2+W_2^2}$ at (t, p_t). ⁂

I.2.3 Second functional formulation: With the Finger tensor

The above approach uses the push-forward: It uses F, i.e. you arrive with your memory. You may prefer to use the pull-back, i.e. use F^{-1} (you remember the past which is Cauchy's point of view): Then you use $F^{-1} = R^{-1}.V^{-1}$ the right polar decomposition of F^{-1}, and you consider the tensor

$$\underline{\underline{\widetilde{\widetilde{\varepsilon}}}}_t = V^{-1} - I_t \ \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t), \tag{I.33}$$

and

$$(\underline{\underline{\sigma}}_t(\Phi) =) \quad \underline{\underline{\sigma}}_t = \lambda\mathrm{Tr}(\underline{\underline{\widetilde{\widetilde{\varepsilon}}}}_t)I_t + 2\mu\underline{\underline{\widetilde{\widetilde{\varepsilon}}}}_t, \quad \text{and} \quad \underline{\underline{\sigma}}_t.\vec{n}_t = \lambda\mathrm{Tr}(\underline{\underline{\widetilde{\widetilde{\varepsilon}}}}_t)\,\vec{n}_t + 2\mu\underline{\underline{\widetilde{\widetilde{\varepsilon}}}}_t.\vec{n}_t. \tag{I.34}$$

(Quantities functionally well defined: Give a tensorial approach).

I.3 Elasticity with a covariant objective approach?

In § I.2 you need to start with Euclidean dot products, so from the start the result can't be covariant objective. Can you start without Euclidean dot products to set up general laws? Proposal:

Hypothesis: The Cauchy stress $\vec{w}$ is a Eulerian vector field.

Then we could use the (covariant objective) Lie derivative which characterizes the rate of stress, cf. § 9.3 and 9.5: With a particle $P_{Obj} \in Obj$, with $\vec{v}(\tau, p_\tau) = \frac{\partial\widetilde{\Phi}}{\partial\tau}(\tau, P_{Obj})$ its Eulerian velocity at τ at

$p_\tau = \widetilde{\Phi}(\tau, P_{Obj})$, the Lie derivative of a Eulerian vector field $\vec{w}$ along $\vec{v}$ is, at (t, p_t),

$$\mathcal{L}_{\vec{v}}\vec{w}(t,p_t) = \lim_{\tau\to t} \frac{\vec{w}(\tau,p_\tau) - \vec{w}_{t*}(\tau,p_\tau)}{\tau - t} = (\frac{\partial \vec{w}}{\partial t} + d\vec{w}.\vec{v} - d\vec{v}.\vec{w})(t,p_t). \tag{I.35}$$

Hence the proposal, with the virtual power principle to measure the rate of stress (see https://arxiv.org/abs/2208.10780v1 for a full description).

1- Hypotheses: 1.1- Suppose that n Eulerian vector fields $\vec{w}_j$ ("force fields"), $j = 1,...,n$, enable to characterize a material. (In fact, for elasticity problems it could be better to replace vector fields $\vec{w}_j$ with 1-forms α_i to characterize the work.)

1.2- With a basis $(\vec{e}_i)$ chosen in $\vec{\mathbb{R}}^n_t$, with (e^i) its (covariant) dual basis in $\vec{\mathbb{R}}^{n*}_t$, assume that the internal power density at (t, p_t) is given by (at first order):

$$p_{\rm int}(\vec{v}) = \sum_{j=1}^n e^j.\mathcal{L}_{\vec{v}}\vec{w}_j = \sum_{j=1}^n e^j.(\frac{\partial \vec{w}_j}{\partial t} + d\vec{w}_j.\vec{v} - d\vec{v}.\vec{w}_j). \tag{I.36}$$

(At second order you can add second order Lie derivatives as $\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w}_j)$, similarly for higher orders.)

2- Then, so that this $p_{\rm int}$ satisfies the frame invariance hypothesis, choose a Euclidean dot product $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$; Then, first, the internal power has to vanish if $d\vec{v} = 0$, thus we are left with

$$p_{\rm int}(\vec{v}) = -\sum_{j=1}^n e^j.d\vec{v}.\vec{w}_j = -\underline{\underline{\tau}} \mathbin{⦂} d\vec{v}, \quad \text{where} \quad \underline{\underline{\tau}} = \sum_{j=1}^n \vec{w}_j \otimes e^j, \tag{I.37}$$

defined at t. (The j-th column of $[\underline{\underline{\tau}}]_{|\vec{e}}$ is $[\vec{w}_j]_{|\vec{e}}$.) And, second, the internal power vanishes if $d\vec{v} + d\vec{v}^T = 0$ (rotation), thus we are left with

$$p_{\rm int}(\vec{v}) = -\underline{\underline{\tau}} \mathbin{⦂} \frac{d\vec{v} + d\vec{v}^T}{2} = -\underline{\underline{\sigma}} \mathbin{⦂} d\vec{v} \quad \text{where} \quad \underline{\underline{\sigma}} = \frac{\underline{\underline{\tau}} + \underline{\underline{\tau}}^T}{2}, \tag{I.38}$$

this $p_{\rm int}(\vec{v}) = -\underline{\underline{\sigma}} \mathbin{⦂} d\vec{v}$ being the usual expression of the internal power at first order.

Example I.11 (I.36) may be applied to orthotropic elasticity, e.g. for a material which fibers at some time t_0 are along $\vec{e}_1$, in a 2-D case for simplicity: 1- With an elongation type motion (Φ_e) given by $[(F_e)(p_{t_0})] = [d(\Phi_e)(p_{t_0})] = \begin{pmatrix} 1+\alpha_{11}(p_{t_0}) & 0 \\ 0 & 1-\alpha_{22}(p_{t_0}) \end{pmatrix}$ you measure the Young moduli in the directions $\vec{e}_1$ and $\vec{e}_2$; 2- And with a shear type motion given by $[(F_s)(p_{t_0})] = [d(\Phi_s)(p_{t_0})] = \begin{pmatrix} 1 & \gamma_{12} \\ 0 & 1 \end{pmatrix}$ you measure the shear modulus.

For more complex material, you may need more vectors $\vec{w}_j$ to describe the constitutive law, that is, (I.36) may be considered with $\sum_{i=1}^m e^j.\mathcal{L}_{\vec{v}}\vec{w}_j$ with $m > n$. ⁂

Remark I.12 The Lie approach is different from the usual classic approach:

1- The classic approach looks for an order two stress tensor $[\underline{\underline{\sigma}}]$ as a function of the deformation gradient $[F]$, cf. (I.18).

2- The Lie approach begins with the internal power (which measures forces), cf. (I.36), which then enable to build $\underline{\underline{\tau}}$ and the $\underline{\underline{\sigma}}$ (the stress tensor), cf. (I.37)-(I.38).

E.g., application to visco-elasticity: With the Lie approach, you automatically use Lie derivative of vector fields (and/or of differential forms), instead of Lie derivative of order 2 tensor fields (which does not seem to give good result, see e.g. the Maxwell visco-elastic type laws, as well as footnote[1] page 25). ⁂

J Displacement

J.1 The displacement vector $\vec{\mathcal{U}}$

In $\mathbb{R}^n$, let $p_t = \Phi^{t_0}_t(p_{t_0})$. Then the bi-point vector

$$\vec{\mathcal{U}}^{t_0}_t(p_{t_0}) = \Phi^{t_0}_t(p_{t_0}) - I_{t_0}(p_{t_0}) = p_t - p_{t_0} = \overrightarrow{p_{t_0}p_t} \tag{J.1}$$

is called the displacement vector at p_{t_0} relative to t_0 and t. This defines the map

$$\vec{\mathcal{U}}^{t_0}_t : \begin{cases} \Omega_{t_0} & \to \vec{\mathbb{R}}^n \\ p_{t_0} & \to \vec{\mathcal{U}}^{t_0}_t(p_{t_0}) := p_t - p_{t_0} = \overrightarrow{p_{t_0}p_t} \quad \text{when} \quad p_t = \Phi^{t_0}_t(p_{t_0}). \end{cases} \tag{J.2}$$

Remark J.1 $\vec{\mathcal{U}}_t^{t_0}(p_{t_0})$ doesn't define a vector field (it is not tensorial), because $\vec{\mathcal{U}}_t^{t_0}(p_{t_0}) = p_t - p_{t_0} = \overrightarrow{p_{t_0}p_t}$ is a bi-point vector which is neither in $\vec{\mathbb{R}}^n_{t_0}$ or in $\vec{\mathbb{R}}^n_t$ since $p_{t_0} \in \Omega_{t_0}$ and $p_t \in \Omega_t$ (it requires time and space ubiquity gift). In particular, it makes no sense on a non-plane surface (manifold). More at § J.5. ∴

Remark J.2 For elastic solids in $\mathbb{R}^n$, the function $\vec{\mathcal{U}}^{t_0}$ is often considered to be the unknown (to be computed); But the "real" unknown is the motion Φ^{t_0}. And it is sometimes confused with the extension of a spring 1-D case; But see figure 4.1 where $||\vec{w}_{t_0}(p_{t_0})||$ represents the initial length and $||\vec{w}_{t_0*}(t,p_t)||$ represents the current length of the spring, while the length of the displacement vector $\vec{\mathcal{U}}_t^{t_0} = p_t - p_{t_0}$ can be very long for a very small elongation $||\vec{w}_{t_0*}(t,p_t)|| - ||\vec{w}_{t_0}(p_{t_0})||$ of the spring. ∴

J.2 The differential of the displacement vector

The differential of $\vec{\mathcal{U}}_t^{t_0}$ at p_{t_0} is

$$d\vec{\mathcal{U}}_t^{t_0}(p_{t_0}) = d\Phi_t^{t_0}(p_{t_0}) - I_{t_0} = F_t^{t_0}(p_{t_0}) - I_{t_0}, \quad \text{written} \quad d\vec{\mathcal{U}} = F - I, \tag{J.3}$$

thus isn't defined as a function, because $F_t^{t_0}(p_{t_0}) : \vec{\mathbb{R}}^n_{t_0} \to \mathbb{R}$ while $I_{t_0} : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_{t_0}$.

So $d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})$ as to be understood as a matrix: With

$$[\vec{\mathcal{U}}_t^{t_0}(p_{t_0})] = [\overrightarrow{p_{t_0}p_t}] = [\overrightarrow{O\Phi_t^{t_0}(p_{t_0})}] - [\overrightarrow{Op_{t_0}}], \tag{J.4}$$

relative to an origin O and a unique basis at all time, compute $[d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})] = [d\Phi_t^{t_0}(p_{t_0})] - I$, abusively written $d\vec{\mathcal{U}} = d\Phi - I$. Then, with $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$,

$$d\vec{\mathcal{U}}.\vec{W} = F.\vec{W} - \vec{W}, \quad \text{which means} \quad = [F_t^{t_0}(p_{t_0})].[\vec{W}] - [\vec{W}]. \tag{J.5}$$

Thus we have defined (matrix meaning)

$$\vec{\mathcal{U}}^{t_0} : \begin{cases} [t_0,T] \times \Omega_{t_0} & \to \vec{\mathbb{R}}^n \\ (t,p_{t_0}) & \to \vec{\mathcal{U}}^{t_0}(t,p_{t_0}) := \vec{\mathcal{U}}_t^{t_0}(p_{t_0}), \end{cases} \quad \text{and} \quad \vec{\mathcal{U}}_{p_{t_0}}^{t_0} : \begin{cases} [t_0,T] & \to \vec{\mathbb{R}}^n \\ t & \to \vec{\mathcal{U}}_{p_{t_0}}^{t_0}(t) := \vec{\mathcal{U}}_t^{t_0}(p_{t_0}). \end{cases} \tag{J.6}$$

J.3 Deformation "tensor" $\underline{\underline{\varepsilon}}$ (matrix), bis

(J.3) gives (matrix meaning)

$$F_t^{t_0}(p_{t_0}) = I_{t_0} + d\vec{\mathcal{U}}_t^{t_0}(p_{t_0}), \quad \text{written} \quad F = I + d\vec{\mathcal{U}}. \tag{J.7}$$

Therefore, Cauchy–Green deformation tensor $C = F^T.F$ reads, in short, (matrix meaning)

$$C = I + d\vec{\mathcal{U}} + d\vec{\mathcal{U}}^T + d\vec{\mathcal{U}}^T.d\vec{\mathcal{U}} \quad \text{(matrix meaning)}, \tag{J.8}$$

i.e. $[C_t^{t_0}(p_{t_0})] = [I_{t_0}] + [d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})] + [d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})]^T + [d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})]^T.[d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})]$.

Thus the Green–Lagrange deformation tensor $E = \frac{C-I}{2}$, cf. (G.47), reads, in short, (matrix meaning)

$$E = \frac{d\vec{\mathcal{U}} + d\vec{\mathcal{U}}^T}{2} + \frac{1}{2} d\vec{\mathcal{U}}^T.d\vec{\mathcal{U}} \quad \text{(matrix meaning)}. \tag{J.9}$$

Thus the deformation tensor $\underline{\underline{\varepsilon}}$, cf. (G.54), reads (matrix meaning)

$$\underline{\underline{\varepsilon}} = E - \frac{1}{2}(d\vec{\mathcal{U}})^T.d\vec{\mathcal{U}}, \tag{J.10}$$

with $\underline{\underline{\varepsilon}}$ the "linear part" of E (small displacements: we only used the first order derivative $d\Phi_t^{t_0}$).

J.4 Small displacement hypothesis, bis

(Usual introduction.) Let $p_t = \Phi_t^{t_0}(p_{t_0})$, $\vec W_i \in \vec{\mathbb{R}}^n_{t_0}$, $\vec w_i(p_t) = F_t^{t_0}(p_{t_0}).\vec W_i(p_{t_0}) \in \vec{\mathbb{R}}^n_t$ (the push-forwards), written $\vec w_i = F.\vec W_i$. Then define (matrix meaning)

$$\vec\Delta_i := \vec w_i - \vec W_i = d\mathcal{U}.\vec W_i, \quad \text{and} \quad ||\vec\Delta||_\infty = \max(||\vec\Delta_1||_{\mathbb{R}^n}, ||\vec\Delta_2||_{\mathbb{R}^n}). \tag{J.11}$$

Then the small displacement hypothesis reads (matrix meaning):

$$||\vec\Delta||_\infty = o(||\vec W||_\infty). \tag{J.12}$$

Thus $\vec w_i = \vec W_i + \vec\Delta_i$ (with $\vec\Delta_i$ "small") and the hypothesis $(\cdot,\cdot)_g = (\cdot,\cdot)_G$ (same inner dot product at t_0 and t) give

$$(\vec w_1, \vec w_2)_G - (\vec W_1, \vec W_2)_G = (\vec\Delta_1, \vec W_2)_G + (\vec\Delta_2, \vec W_1)_G + (\vec\Delta_1, \vec\Delta_2)_G.$$

So (J.10) gives $2(E.\vec W_1, \vec W_2)_G = 2(\underline{\underline{\varepsilon}}.\vec W_1, \vec W_2)_G + (d\vec{\mathcal{U}}^T.d\vec{\mathcal{U}}.\vec W_1, \vec W_2)_G$, And (J.12) gives

$$(E.\vec W_1, \vec W_2)_G = (\underline{\underline{\varepsilon}}.\vec W_1, \vec W_2)_G + O(||\vec\Delta||_\infty^2), \tag{J.13}$$

so $E_t^{t_0}$ is approximated by $\underline{\underline{\varepsilon}}_t^{t_0}$, that is, $E_t^{t_0} \simeq \underline{\underline{\varepsilon}}_t^{t_0}$ (matrix meaning).

J.5 Displacement vector with differential geometry

J.5.1 The shifter

We give the steps, see Marsden–Hughes [12]. The complexity introduced is due to the small displacement hypothesis applied to the Green–Lagrange tensor $E = \frac{F^T.F-I}{2}$ which linearization gives $\underline{\underline{\varepsilon}} = \frac{F+F^T}{2} - I$ (the classical approach "squares the motion" to get E, then "linearizes" E ... to get back to F... with a spurious F^T).

Let $P \in \Omega_{t_0}$, $\vec W_P \in \vec{\mathbb{R}}^n_{t_0}$, $p_t = \Phi_t^{t_0}(P) \in \Omega_t$, and $\vec w_{p_t} = F_t^{t_0}(P).\vec W_P \in \vec{\mathbb{R}}^n_t$ (push-forward).

- **Affine case $\mathbb{R}^n$ (continuum mechanics):** With $p_t = \Phi_t^{t_0}(P)$, the shifter is:

$$\widetilde{S_t^{t_0}} : \begin{cases} \Omega_{t_0} \times \vec{\mathbb{R}}^n_{t_0} & \to \Omega_t \times \vec{\mathbb{R}}^n_t \\ (P, \vec Z_P) & \to \widetilde{S_t^{t_0}}(P, \vec Z_P) = (p_t, S_t^{t_0}(\vec Z_P)) \quad \text{with} \quad S_t^{t_0}(\vec Z_P) = \vec Z_P. \end{cases} \tag{J.14}$$

(The vector is unchanged but the time and the application point have changed: A real observer has no ubiquity gift). So:

$$S_t^{t_0} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t) \quad \text{and} \quad [S_t^{t_0}]_{|\vec e} = I \text{ identity matrix}, \tag{J.15}$$

the matrix equality being possible after the choice of a unique basis at t_0 and at t. And (simplified notation) $\widetilde{S_t^{t_0}}(P, \vec Z_P) =^{\text{noted}} S_t^{t_0}(\vec Z_P)$. Then the deformation tensor $\underline{\underline{\varepsilon}}$ at P can be defined by

$$\underline{\underline{\varepsilon}}_t^{t_0}(P).\vec Z(P) = \frac{(S_t^{t_0})^{-1}(F_t^{t_0}(P).\vec Z(P)) + F_t^{t_0}(P)^T.(S_t^{t_0}(P).\vec Z(P))}{2} - \vec Z(P), \tag{J.16}$$

in short: $\underline{\underline{\varepsilon}}.\vec Z = \frac{(S_t^{t_0})^{-1}(F.\vec Z) + F^T.(S_t^{t_0}.\vec Z)}{2} - \vec Z$).

- **In a manifold:** Ω is a manifold (like a surface in $\mathbb{R}^3$ from which we cannot take off). Let $T_P\Omega_{t_0}$ be the tangent space à P (the fiber at P), and $T_{p_t}\Omega_t$ be the tangent space à p_t (the fiber at p_t). In general $T_P\Omega_{t_0} \neq T_{p_t}\Omega_t$ (e.g. on a sphere "the Earth"). The bundle (the union of fibers) at t_0 is $T\Omega_{t_0} = \bigcup_{P \in \Omega_{t_0}} (\{P\} \times T_P\Omega_{t_0})$, and the bundle at t is $T\Omega_t = \bigcup_{p_t \in \Omega_t} (\{p_t\} \times T_{p_t}\Omega_t)$. Then the shifter

$$\widetilde{S_t^{t_0}} : \begin{cases} T\Omega_{t_0} & \to T\Omega_t \\ (P, \vec Z_P) & \to \widetilde{S_t^{t_0}}(P, \vec Z_P) = (p_t, S_t^{t_0}(\vec Z_P)), \end{cases} \tag{J.17}$$

is defined such that $\vec Z_P \in T_P\Omega_{t_0}$ "as little distorted as possible" along a path. E.g., on a sphere, if the path is a geodesic, if θ_{t_0} is the angle between $\vec Z_P$ and the tangent vector to the geodesic at P, then θ_{t_0} is also the angle between $S_t^{t_0}(\vec Z_P)$ and the tangent vector to the geodesic at p_t, and $S_t^{t_0}(\vec Z_P)$ has the same length than $\vec Z_P$ (at constant speed in a car you think the geodesic is a straight line, although $S_t^{t_0}(\vec Z_P) \neq \vec Z_P$: the Earth is not flat).

J.5.2 The displacement vector

(Affine space framework, Ω_{t_0} open set in $\mathbb{R}^n$.) Let $P \in \Omega_{t_0}$, $\vec{W}_P \in \vec{\mathbb{R}}^n_{t_0}$, $p_t = \Phi_t^{t_0}(P) \in \Omega_t$, and $d\Phi_t^{t_0} = F_t^{t_0} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$. Define

$$\widetilde{\delta\vec{\mathcal{U}}_t^{t_0}} : \begin{cases} \Omega_{t_0} \times \vec{\mathbb{R}}^n_{t_0} & \to \Omega_t \times \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t) \\ (P, \vec{Z}_P) & \to \widetilde{\delta\vec{\mathcal{U}}_t^{t_0}}(P, \vec{Z}_P) = (p_t, \delta\vec{\mathcal{U}}_t^{t_0}(\vec{Z}_P)) \quad \text{with} \quad \delta\vec{\mathcal{U}}_t^{t_0}(\vec{Z}_P) = (F_t^{t_0} - S_t^{t_0}).\vec{Z}_P. \end{cases} \tag{J.18}$$

Then $\widetilde{\delta\vec{\mathcal{U}}_t^{t_0}} = F_t^{t_0} - S_t^{t_0}$ is a two-point tensor. And

$$\begin{aligned} C_t^{t_0} &= (F_t^{t_0})^T.F_t^{t_0} = (\delta\mathcal{U}_t^{t_0} + S_t^{t_0})^T.(\delta\mathcal{U}_t^{t_0} + S_t^{t_0}) \\ &= I + (S_t^{t_0})^T.\delta\mathcal{U}_t^{t_0} + (\delta\mathcal{U}_t^{t_0})^T.S_t^{t_0} + (\delta\mathcal{U}_t^{t_0})^T.\delta\mathcal{U}_t^{t_0}, \end{aligned} \tag{J.19}$$

since $(S_t^{t_0})^T.S_t^{t_0} = I$ identity in $T\Omega_{t_0}$: Indeed, $((S_t^{t_0})^T.S_t^{t_0}.\vec{A}, \vec{B})_{\mathbb{R}^n} = (S_t^{t_0}.\vec{A}, S_t^{t_0}.\vec{B})_{\mathbb{R}^n} = (\vec{A}, \vec{B})_{\mathbb{R}^n}$, cf. (J.14), for all $\vec{A}, \vec{B}$. Then the Green–Lagrange tensor is defined on Ω_{t_0} by

$$E_t^{t_0} = \frac{1}{2}(C_t^{t_0} - I_{t_0}) = \frac{(S_t^{t_0})^T.\delta\mathcal{U}_t^{t_0} + S_t^{t_0}.(\delta\mathcal{U}_t^{t_0})^T}{2} + \frac{1}{2}(\delta\mathcal{U}_t^{t_0})^T.\delta\mathcal{U}_t^{t_0}, \tag{J.20}$$

to compare with (G.47).

K Determinants

K.1 Alternating multilinear form

Let E be a vector space, and let $\mathcal{L}(E, ..., E; \mathbb{R}) =^{\text{noted}} \mathcal{L}(E^n; \mathbb{R})$ be the set of multilinear forms, i.e. $m \in \mathcal{L}(E^n; \mathbb{R})$ iff

$$m(..., \vec{x} + \lambda\vec{y}, ...) = m(..., \vec{x}, ...) + \lambda m(..., \vec{y}, ...) \tag{K.1}$$

for all $\vec{x}, \vec{y} \in E$ and all $\lambda \in \mathbb{R}$ and for all “slot”. In particular, $m(\lambda_1\vec{x}_1, ..., \lambda_n\vec{x}_n) = (\prod_{i=1,...,n} \lambda_i)\, m(\vec{x}_1, ..., \vec{x}_n)$, for all $\lambda_1, ..., \lambda_n \in \mathbb{R}$ and all $\vec{x}_1, ..., \vec{x}_n \in E$.

Definition K.1 If $n = 1$ then a 1-alternating multilinear function is a linear form, also called a 1-form. If $n \geq 2$ then $\mathcal{A}\ell : \left\{ \begin{array}{cl} E^n & \to \mathbb{R} \\ (\vec{v}_1, ..., \vec{v}_n) & \to \mathcal{A}\ell(\vec{v}_1, ..., \vec{v}_n) \end{array} \right\} \in \mathcal{L}(E^n; \mathbb{R})$ is a n-alternating multilinear form iff, for all $\vec{u}, \vec{v} \in E$,

$$\mathcal{A}\ell(..., \vec{u}, ..., \vec{v}, ...) = -\mathcal{A}\ell(..., \vec{v}, ..., \vec{u}, ...), \tag{K.2}$$

the other elements being unchanged. If $n = 1$, the set of 1-forms is $\Omega^1(E) = E^*$. If $n \geq 2$, the set of n-alternating multilinear forms is

$$\Omega^n(E) = \{m \in \mathcal{L}(E^n; \mathbb{R}) : m = \mathcal{A}\ell \text{ is alternating}\}. \tag{K.3}$$

If $\mathcal{A}\ell, \mathcal{B}\ell \in \Omega^n(E)$ and $\lambda \in \mathbb{R}$ then $\mathcal{A}\ell + \lambda\mathcal{B}\ell \in \Omega^n(E)$ thanks to the linearity for each variable. Thus $\Omega^n(E)$ is a vector space, sub-space in $(\mathcal{F}(E^n; \mathbb{R}), +, .)$.

K.2 Leibniz formula

Particular case dim E=n. Let $\mathcal{A}\ell \in \Omega^n(E)$ (a n-alternating multilinear form). Recall (see e.g. Cartan [5]):

1- A permutation $\sigma : [1, n]_{\mathbb{N}} \to [1, n]_{\mathbb{N}}$ is a bijective map (i.e. one-to-one and onto); Let S_n be the set of permutations of $[1, n]_{\mathbb{N}}$.

2- A transposition $\tau : [1, n]_{\mathbb{N}} \to [1, n]_{\mathbb{N}}$ is a permutation that exchanges two elements, that is, $\exists i, j$ s.t. $\tau(..., i, ..., j, ...) = (..., j, ..., i, ...)$, the other elements being unchanged.

3- A permutation is a composition of transpositions (theorem left as an exercise, of see Cartan). And a permutation is even iff the number of transpositions is even, and a permutation is odd iff the number of transpositions is odd. Based on: The parity (even or odd) of a permutation is an invariant.

4- The signature $\varepsilon(\sigma) = \pm 1$ of a permutation σ is $+1$ if σ is even, and is -1 if σ is odd.

Proposition K.2 (Leibniz formula) *Let* $\mathcal{A}\ell \in \Omega^n(E)$*. Let* $(\vec{e}_i)_{i=1,...,n} =^{noted} (\vec{e}_i)$ *be a basis in* E*. For all vectors* $\vec{v}_1, ..., \vec{v}_n \in E$*, with* $\vec{v}_j = \sum_{i=1}^n v_j^i \vec{e}_i$ *for all* j*,*

$$\mathcal{A}\ell(\vec{v}_1, ..., \vec{v}_n) = c \sum_{\sigma\in S_n} \varepsilon(\sigma) \prod_{i=1}^n v_i^{\sigma(i)} = c \sum_{\tau\in S_n} \varepsilon(\tau) \prod_{i=1}^n v_{\tau(i)}^i \quad \textit{(with } c := \mathcal{A}\ell(\vec{e}_1, ..., \vec{e}_n)\textit{)}. \tag{K.4}$$

Thus if $c = \mathcal{A}\ell(\vec{e}_1, ..., \vec{e}_n)$ *is known, then* $\mathcal{A}\ell$ *is known. Thus* $\dim(\Omega^n(E)) = 1$*.*
(Classic not.: $\vec{v}_j = \sum_{i=1}^n v_{ij} \vec{e}_i$*,* $\mathcal{A}\ell(\vec{v}_1, ..., \vec{v}_n) = c \sum_{\sigma\in S_n} \varepsilon(\sigma) \prod_{i=1}^n v_{\sigma(i),i} = c \sum_{\tau\in S_n} \varepsilon(\tau) \prod_{i=1}^n v_{i,\tau(i)}$*.)*

Proof. Let $F := \mathcal{F}([1,n]_{\mathbb{N}}; [1,n]_{\mathbb{N}}) =^{\text{noted}} [1,n]_{\mathbb{N}}^{[1,n]_{\mathbb{N}}}$ be the set of functions $i : \left\{ \begin{array}{rl} [1,n]_{\mathbb{N}} & \to [1,n]_{\mathbb{N}} \\ k & \to i_k = i(k) \end{array} \right\}$. $\mathcal{A}\ell$ being multilinear, $\mathcal{A}\ell(\vec{v}_1, ..., \vec{v}_n) = \sum_{j_1=1}^n v_1^{j_1} \mathcal{A}\ell(\vec{e}_{j_1}, \vec{v}_2, ..., \vec{v}_n)$ ("the first column" development). By recurrence we get $\mathcal{A}\ell(\vec{v}_1, ..., \vec{v}_n) = \sum_{j_1,...,j_n=1}^n v_1^{j_1} ... v_n^{j_n} \mathcal{A}\ell(\vec{e}_{j_1}, ..., \vec{e}_{j_n}) = \sum_{j\in F} \prod_{k=1}^n v_k^{j(k)} \mathcal{A}\ell(\vec{e}_{j(1)}, ..., \vec{e}_{j(n)})$.

And $\mathcal{A}\ell(\vec{e}_{i_1}, ..., \vec{e}_{i_n}) \neq 0$ iff $i : k \in \{1, ..., n\} \to i(k) = i_k \in \{1, ..., n\}$ is one-to-one (thus bijective). Thus $\mathcal{A}\ell(\vec{v}_1, ..., \vec{v}_n) = \sum_{\sigma\in S_n} \prod_{i=1}^n v_i^{\sigma(i)} \mathcal{A}\ell(\vec{e}_{\sigma(1)}, ..., \vec{e}_{\sigma(n)}) = \sum_{\sigma\in S_n} \varepsilon(\sigma) \prod_{i=1}^n v_i^{\sigma(i)} \mathcal{A}\ell(\vec{e}_1, ..., \vec{e}_n)$, which is the first equality in (K.4). Then $\sum_{\sigma\in S_n} \varepsilon(\sigma) \prod_{i=1}^n v_i^{\sigma(i)} = \sum_{\sigma\in S_n} \varepsilon(\sigma) \prod_{i=1}^n v_{\sigma^{-1}(i)}^{\sigma(\sigma^{-1}(i))}$ since σ is bijectif, thus $\sum_{\sigma\in S_n} \varepsilon(\sigma) \prod_{i=1}^n v_i^{\sigma(i)} = \sum_{\tau\in S_n} \varepsilon(\tau^{-1}) \prod_{i=1}^n v_{\tau(i)}^i$, thus the second equality in (K.4) since $\varepsilon(\tau)^{-1} = \varepsilon(\tau)$. (See Cartan [5].) ∴

K.3 Determinant of vectors

Definition K.3 $(\vec{e}_i)_{i=1,...,n}$ being a basis in E, the alternating multilinear form $\det_{|\vec{e}} \in \Omega^n(E)$ defined by

$$\det_{|\vec{e}}(\vec{e}_1, ..., \vec{e}_n) = 1 \tag{K.5}$$

is called the determinant relative to $(\vec{e}_i)$. And, with prop. K.2 (here $c = 1$),

$$\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = \sum_{\sigma\in S_n} \varepsilon(\sigma) \prod_{i=1}^n v_i^{\sigma(i)} = \sum_{\tau\in S_n} \varepsilon(\tau) \prod_{i=1}^n v_{\tau(i)}^i \tag{K.6}$$

is called the determinant of the vectors $\vec{v}_i$ relative to $(\vec{e}_i)$. And we write

$$\Omega^n(E) = \text{Vect}\{\det_{|\vec{e}}\} \quad \text{(the 1-D vector space spanned by } \det_{|\vec{e}}\text{)}. \tag{K.7}$$

Thus, if $\mathcal{A}\ell \in \Omega^n(E)$ then

$$\mathcal{A}\ell = \mathcal{A}\ell(\vec{e}_1, ..., \vec{e}_n) \det_{|\vec{e}}, \tag{K.8}$$

thus if $(\vec{b}_i)$ is another basis then

$$\exists c \in \mathbb{R}, \quad \det_{|\vec{b}} = c \det_{|\vec{e}}, \quad \text{with} \quad c = \det_{|\vec{b}}(\vec{e}_1, ..., \vec{e}_n). \tag{K.9}$$

Exercice K.4 Change of measuring unit: If $(\vec{a}_i)$ is a basis and $\vec{b}_j = \lambda \vec{a}_j$ for all j, prove

$$\forall j = 1, ..., n, \quad \vec{b}_j = \lambda \vec{a}_j \quad \Longrightarrow \quad \det_{|\vec{a}} = \lambda^n \det_{|\vec{b}} \tag{K.10}$$

(relation between volumes relative to a change of measuring unit in the Euclidean case).

Answer. $\det_{|\vec{a}}(\vec{b}_1, ..., \vec{b}_n) = \det_{|\vec{a}}(\lambda\vec{a}_1, ..., \lambda\vec{a}_n) \overset{multi}{\underset{linear}{=}} \lambda^n \det_{|\vec{a}}(\vec{a}_1, ..., \vec{a}_n) \overset{(K.5)}{=} \lambda^n \overset{(K.5)}{=} \lambda^n \det_{|\vec{b}}(\vec{b}_1, ..., \vec{b}_n)$. ∴

Proposition K.5 $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) \neq 0$ *iff* $(\vec{v}_1, ..., \vec{v}_n)$ *is a basis, or equivalently,* $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = 0$ *iff* $\vec{v}_1, ..., \vec{v}_n$ *are linearly dependent.*

Proof. If one of the $\vec{v}_i$ is $= \vec{0}$ then $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = 0$ (multilinearity), and if $\sum_{i=1}^n c_i \vec{v}_i = 0$ and one of the $c_i \neq 0$ and then a $\vec{v}_i$ is a linear combination of the others thus $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = 0$ (since $\det_{|\vec{e}}$ is alternate). Thus $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) \neq 0 \Rightarrow$ the $\vec{v}_i$ are independent. And if the $\vec{v}_i$ are independent then $(\vec{v}_1, ..., \vec{v}_n)$ is a basis, thus $\det_{|\vec{v}}(\vec{v}_1, ..., \vec{v}_n) = 1 \neq 0$, with $\det_{|\vec{v}} = c \det_{|\vec{e}}$, thus $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) \neq 0$. ∴

Exercice K.6 In $\mathbb{R}^2$. Let $\vec{v}_1 = \sum_{i=1}^2 v_1^i \vec{e}_i$ and $\vec{v}_2 = \sum_{j=1}^2 v_2^j \vec{e}_j$ (duality notations). Prove:

$$\det_{|\vec{e}}(\vec{v}_1, \vec{v}_2) = v_1^1 v_2^2 - v_1^2 v_2^1. \tag{K.11}$$

Answer. Development relative to the first column (linearity used for the first vector $\vec{v}_1 = v_1^1 \vec{e}_1 + v_1^2 \vec{e}_2$): $\det_{|\vec{e}}(\vec{v}_1, \vec{v}_2) = \det_{|\vec{e}}(v_1^1 \vec{e}_1 + v_1^2 \vec{e}_2, \vec{v}_2) = v_1^1 \det_{|\vec{e}}(\vec{e}_1, \vec{v}_2) + v_1^2 \det_{|\vec{e}}(\vec{e}_2, \vec{v}_2)$. Thus (linearity used for the second vector $\vec{v}_2 = v_2^1 \vec{e}_1 + v_2^2 \vec{e}_2$): $\det_{|\vec{e}}(\vec{v}_1, \vec{v}_2) = 0 + v_1^1 v_2^2 \det(\vec{e}_1, \vec{e}_2) + v_1^2 v_2^1 \det(\vec{e}_2, \vec{e}_1) + 0 = v_1^1 v_2^2 - v_1^2 v_2^1$. ⁂

Exercice K.7 In $\mathbb{R}^3$, with $\vec{v}_j = \sum_{i=1}^3 v_j^i \vec{e}_i$, prove:

$$\det(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} v_1^i v_2^j v_3^k, \tag{K.12}$$

where $\varepsilon_{ijk} = \frac{1}{2}(j-i)(k-j)(k-i)$, i.e. $\varepsilon_{ijk} = 1$ if $(i,j,k) = (1,2,3)$, $(3,1,2)$ or $(2,3,1)$ (even signature), $\varepsilon_{ijk} = -1$ if $(i,j,k) = (3,2,1)$, $(1,3,2)$ and $(2,1,3)$ (odd signature), and $\varepsilon_{ijk} = 0$ otherwise.

Answer. Development relative to the first column (as in exercise K.6).
Result $= v_1^1 v_2^2 v_3^3 + v_2^1 v_3^2 v_1^3 + v_3^1 v_1^2 v_2^3 - v_1^3 v_2^2 v_3^1 - v_2^3 v_3^2 v_1^1 - v_3^3 v_1^2 v_2^1$. ⁂

K.4 Determinant of a matrix

Let $M = [M_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$ be a n^2 real matrix. Let $\vec{\mathbb{R}}^n = \mathbb{R} \times \dots \times \mathbb{R}$ (Cartesian product n-times) with its canonical basis $(\vec{E}_i)$. Let $\vec{v}_j \in \vec{\mathbb{R}}^n$, $\vec{v}_j = \sum_{i=1}^n M_{ij} \vec{E}_i$; So $M = \big([\vec{v}_1]_{|\vec{E}}, \dots, [\vec{v}_n]_{|\vec{E}} \big)$.

Definition K.8 The determinant of the matrix $M = \big([\vec{v}_1]_{|\vec{E}}, \dots, [\vec{v}_n]_{|\vec{E}} \big)$ is

$$\det(M) := \det_{|\vec{E}}(\vec{v}_1, \dots, \vec{v}_n). \tag{K.13}$$

Proposition K.9 *Let M^T be the transposed matrix, i.e., $(M^T)_{ij} = M_{ji}$ for all i,j. Then*

$$\det(M^T) = \det(M). \tag{K.14}$$

Proof. $\det[M_{ij}] = \det_{\vec{E}}(\vec{v}_1, \dots, \vec{v}_n) \overset{(K.6)}{=} \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^n v_i^{\sigma(i)} = \sum_{\tau \in S_n} \varepsilon(\tau) \prod_{i=1}^n v_{\tau(i)}^i = \det[M_{ji}]$. ⁂

K.5 Volume

Definition K.10 *Let $(\vec{e}_i)$ be a Euclidean basis. Consider a parallelepiped in $\mathbb{R}^n$ which sides are vectors $\vec{v}_1, \dots, \vec{v}_n$; Its algebraic volume relative to $(\vec{e}_i)$ is*

$$\textit{algebraic volume} = \det_{|\vec{e}}(\vec{v}_1, \dots, \vec{v}_n). \tag{K.15}$$

And its volume relative to $(\vec{e}_i)$ is (non negative)

$$\textit{volume} = \big|\det_{|\vec{e}}(\vec{v}_1, \dots, \vec{v}_n)\big|. \tag{K.16}$$

E.g., if $n = 1$ and $\vec{v} = v^1 \vec{e}_1$, then $\det_{|\vec{e}}(\vec{v}) = v^1$ is the algebraic length of $\vec{v}$ (relative to the unit of measurement given by $\vec{e}_1$). And $|\det_{|\vec{e}}(\vec{v})| = |v^1|$ is the length of $\vec{v}$ (the norm of $\vec{v}$). (The volume function $(\vec{v}_1, \dots, \vec{v}_n) \to \big|\det_{|\vec{e}}(\vec{v}_1, \dots, \vec{v}_n)\big|$ is ***not*** a multilinear form, because the absolute value function is not linear.) E.g., if $n = 2$ or 3, see exercises K.6-K.7.

Notation. Let $(\vec{e}_i)$ be a Cartesian basis and $(e^i) = (dx^i)$ be the dual basis. Then, cf. Cartan [6],

$$\det_{|\vec{e}} \overset{\text{noted}}{=} e^1 \wedge \dots \wedge e^n = dx^1 \wedge \dots \wedge dx^n. \tag{K.17}$$

And, for integration, the volume element (non negative) uses a Euclidean basis $(\vec{e}_i)$ and is

$$d\Omega(\vec{x}) = |\det_{|\vec{e}}| = |dx^1 \wedge \dots \wedge dx^n| \overset{\text{noted}}{=} dx^1 \dots dx^n. \tag{K.18}$$

Thus the volume of a parallelepiped at $\vec{x}$ which sides are given by $\delta x_1 \vec{u}_1, \dots, \delta x_n \vec{u}_n$ is $d\Omega(\vec{x})(\delta x_1 \vec{u}_1, \dots, \delta x_n \vec{u}_n) = |\delta x_1 \dots \delta x_n| \, |\det_{|\vec{e}}(\vec{u}_1, \dots, \vec{u}_n)|$; Thus the volume of a polygonal domain $\Omega =$

$\bigcup_{i=1}^N P_i$ where P_i is a parallelepiped which sides are given by $\delta x_{i,1}\vec u_{i,1}, ..., \delta x_{i,n}\vec u_{i,n}$ is

$$|\Omega| = \sum_{i=1}^N |\underset{|\vec e}{\det}(\vec u_{i,1}, ..., \vec u_{i,n})| \delta x_{i,1} ... \delta x_{i,n}. \tag{K.19}$$

And thus (Riemann approach), the volume of a regular domain Ω is written

$$|\Omega| = \int_\Omega d\Omega = \int_{\vec x \in \Omega} |\underset{|\vec e}{\det}(\vec u_{i,1}, ..., \vec u_{i,n})| \, dx^1 ... dx^n. \tag{K.20}$$

In particular, since any regular volume Ω can be approximated with cubes as small as wished, $|\Omega| = \sum_{i=1}^N |\delta x_{i,1} ... \delta x_{i,n} \det_{|\vec e}(\vec e_1, ..., \vec e_n)| = \sum_{i=1}^N |\delta x_{i,1} ... \delta x_{i,n}|$ gives

$$|\Omega| = \int_\Omega d\Omega = \int_{\vec x \in \Omega} dx^1 ... dx^n. \tag{K.21}$$

Exercice K.11 Let $\Psi : \vec q = (q_1, ..., q_n) \in [a_1, b_1] \times ... \times [a_n, b_n] \to \vec x = (x_1 = \Psi_1(\vec q), ..., x_n = \Psi_n(\vec q)) \in \Omega$ be a parametric description of a domain Ω; Prove

$$d\Omega(\vec x) = |J_\Psi(\vec q)| \, dq^1 ... dq^n \quad (= |\underset{|\vec e}{\det}(\vec p_1(\vec x), ..., \vec p_n(\vec x))| \, dq^1 ... dq^n), \tag{K.22}$$

where $(\vec p_i(x)) = (\frac{\partial \Psi}{\partial q_i}(\vec q))$ is the parametric basis at $\vec x = \Psi(\vec q)$ and $J_\Psi(\vec q) = \det_{|\vec e}[d\Psi(\vec q)]_{|\vec e}$ is the Jacobian matrix of Ψ at $\vec q$. And thus $|\Omega| = \int_{\vec q} |J_\Psi(\vec q)| \, dq^1 ... dq^n$.

Answer. Polar coordinates for illustration purpose (immediate generalization): Consider the disk Ω parametrized with the polar coordinate system $\Psi : \vec q = (\rho, \theta) \in [0, R] \times [0, 2\pi] \to \vec x = (x = \rho\cos\theta, y = \rho\sin\theta) \in \mathbb{R}^2$ where a Euclidean basis $(\vec e_1, \vec e_2)$ has been used in $\mathbb{R}^2$ (so $\vec x = \rho\cos\theta\vec e_1 + \rho\sin\theta\vec e_2$). The associated polar basis at $\vec x = \Psi(\vec q)$ is $(\vec p_1(\vec x) = \frac{\partial\Psi}{\partial\rho}(\rho, \theta), \vec p_2(\vec x) = \frac{\partial\Psi}{\partial\theta}(\rho, \theta))$, so $[\vec p_1(\vec x)]_{|\vec e} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $[\vec p_2(\vec x)]_{|\vec e} = \begin{pmatrix} -\rho\sin\theta \\ \rho\cos\theta \end{pmatrix}$. Thus $\det_{|\vec e}(\vec p_1(\vec x), \vec p_2(\vec x)) = \rho$ (> 0 here), thus $d\Omega = |\rho| \, d\rho d\theta = \rho \, d\rho d\theta$. Thus the volume is $|\Omega| = \int_{\vec x \in \Omega} d\Omega = \int_{\rho=0}^R \int_{\theta=0}^{2\pi} \rho \, d\rho d\theta$ $(= \pi R^2)$. ∴

Exercice K.12 What is the "volume element" on a regular surface Σ in $\mathbb{R}^3$, called the "surface element"?

Answer. Let $(\vec e_1, \vec e_2, \vec e_3)$ be a Euclidean basis in $\mathbb{R}^3$. We need a regular parametric description $\Psi : (u, v) \in [a_1, b_2] \times [a_2, b_2] \to \vec x = \Psi(u, v) = x_1(u, v)\vec e_1 + ... + x_3(u, v)\vec e_3$ of the geometric surface $\Sigma = \operatorname{Im}(\Psi)$. Thus $\vec t_1(\vec x) = \frac{\partial\Psi}{\partial u}(u, v)$ and $\vec t_2(\vec x) = \frac{\partial\Psi}{\partial v}(u, v)$ are tangent vectors at Σ at $\vec x = \Psi(u, v)$. Hence a normal unit vector is $\vec n(\vec x) = \frac{\vec t_1(\vec x) \wedge \vec t_2(\vec x)}{||\vec t_1(\vec x) \wedge \vec t_2(\vec x)||}$, and thus $\det_{|\vec e}(\vec t_1, \vec t_2, \vec n) = ||\vec t_1(\vec x) \wedge \vec t_2(\vec x)||$ is the area of the parallelogram which sides are given by $\vec t_1$ and $\vec t_2$ (volume with height 1). Thus the surface element at $\vec x = \Psi(u, v)$ is $d\Sigma(\vec x) = ||\frac{\partial\Psi}{\partial u}(u, v) \wedge \frac{\partial\Psi}{\partial v}(u, v)|| \, dudv$. Thus $|\Sigma| = \int_{\vec x \in \Sigma} d\Sigma(\vec x) = \int_{u=a_1}^{b_1} \int_{v=a_2}^{b_2} ||\frac{\partial\Psi}{\partial u}(u, v) \wedge \frac{\partial\Psi}{\partial v}(u, v)|| \, dudv$. ∴

K.6 Determinant of an endomorphism

K.6.1 Definition and basic properties

Definition K.13 The determinant of an endomorphism $L \in \mathcal{L}(E; E)$ relative to a basis $(\vec e_i)$ is

$$\underset{|\vec e}{\widetilde{\det}}(L) := \underset{|\vec e}{\det}(L.\vec e_1, ..., L.\vec e_n). \tag{K.23}$$

This define $\widetilde{\det}_{|\vec e} : \mathcal{L}(E; E) \to \mathbb{R}$. (If the context is not ambiguous, then $\widetilde{\det}_{|\vec e} =^{\text{noted}} \det_{|\vec e}$.)

Proposition K.14 *Let $L \in \mathcal{L}(E; E)$.*

1- If $L = I$ the identity, then $\widetilde{\det}_{|\vec e}(I) = 1$ for all basis $(\vec e_i)$.

2- For all $\vec v_1, ..., \vec v_n \in E$,

$$\underset{|\vec e}{\det}(L.\vec v_1, ..., L.\vec v_n) = \underset{|\vec e}{\widetilde{\det}}(L) \underset{|\vec e}{\det}(\vec v_1, ..., \vec v_n). \tag{K.24}$$

3- If $L.\vec e_j = \sum_{i=1}^n L_{ij}\vec e_i$, i.e. $[L]_{|\vec e} = [L_{ij}]$, then

$$\underset{|\vec e}{\widetilde{\det}}(L) = \det([L]_{|\vec e}) = \det([L_{ij}]). \tag{K.25}$$

4- For all $M \in \mathcal{L}(E; E)$, and with $M \circ L =^{noted} M.L$ (thanks to linearity),

$$\underset{|\vec e}{\widetilde{\det}}(M.L) = \underset{|\vec e}{\widetilde{\det}}(M) \underset{|\vec e}{\widetilde{\det}}(L) = \underset{|\vec e}{\widetilde{\det}}(L.M). \tag{K.26}$$

5- L is invertible iff $\widetilde{\det}_{|\vec e}(L) \neq 0$.

6- If L is invertible then

$$\widetilde{\det}_{|\vec e}(L^{-1}) = \frac{1}{\widetilde{\det}_{|\vec e}(L)}. \tag{K.27}$$

7- If $(\cdot,\cdot)_g$ is an inner dot product in E and L^T_g is the $(\cdot,\cdot)_g$ transposed of L (i.e., $(L^T_g\vec w,\vec u)_g = (\vec w, L.\vec u)_g$ for all $\vec u,\vec w \in E$) then

$$\widetilde{\det}_{|\vec e}(L^T_g) = \widetilde{\det}_{|\vec e}(L). \tag{K.28}$$

8- If $(\vec e_i)$ and $(\vec b_i)$ are two $(\cdot,\cdot)_g$-orthonormal bases in $\vec{\mathbb{R}}^n_t$ (e.g. two Euclidean basis for the same measuring unit), then $\det_{|\vec b} = \pm\det_{|\vec e}$.

Proof. 1- $\widetilde{\det}_{|\vec e}(I) =^{(K.23)} \det_{|\vec e}(I.\vec e_1,...,I.\vec e_n) =^{(K.5)} \det_{|\vec e}(\vec e_1,...,\vec e_n) = 1$, true for all basis.

2- Let $m : (\vec v_1,...,\vec v_n) \to m(\vec v_1,...,\vec v_n) := \det_{|\vec e}(L.\vec v_1,...,L.\vec v_n)$: It is a multilinear alternated form, since L is linear; Thus $m =^{(K.8)} m(\vec e_1,...,\vec e_n)\det_{|\vec e}$; With $m(\vec e_1,...,\vec e_n) =^{(K.23)} \widetilde{\det}_{|\vec e}(L)$, thus (K.24).

3- Apply (K.13) with $M = [L]_{|\vec e}$ to get (K.25).

4- $\det_{|\vec e}((M.L).\vec e_1,...,(M.L).\vec e_n) = \det_{|\vec e}(M.(L.\vec e_1),...,M.(L.\vec e_n)) =^{(K.24)} \widetilde{\det}_{|\vec e}(M)\det_{|\vec e}(L.\vec e_1,...,L.\vec e_n)$.

5- If L is invertible, then $1 = \widetilde{\det}_{|\vec e}(I) = \widetilde{\det}_{|\vec e}(L.L^{-1}) = \widetilde{\det}_{|\vec e}(L)\,\widetilde{\det}_{|\vec e}(L^{-1})$, thus $\widetilde{\det}_{|\vec e}(L) \neq 0$.

If $\widetilde{\det}_{|\vec e}(L) \neq 0$ then $\det_{|\vec e}(L.\vec e_1,...,L.\vec e_n) \neq 0$, thus $(L.\vec e_1,...,L.\vec e_n)$ is a basis, thus L is invertible.

6- (K.26) gives $1 = \widetilde{\det}_{|\vec e}(I) = \widetilde{\det}_{|\vec e}(L^{-1}.L) = \widetilde{\det}_{|\vec e}(L).\widetilde{\det}_{|\vec e}(L^{-1})$, thus (K.27).

7- $(L^T_g\vec w,\vec u)_g = (\vec w, L.\vec u)_g$ gives $[g]_{|\vec e}.[L^T_g]_{|\vec e} = ([L]_{|\vec e})^T.[g]_{|\vec e}$, thus $\det([g]_{|\vec e})\det([L^T_g]_{|\vec e}) = \det(([L]_{|\vec e})^T)\det([g]_{|\vec e})$, and $\det([g]_{|\vec e}) \neq 0$ (exercise), thus (K.28).

8- Let $\mathcal{P}$ be the change of basis endomorphism from $(\vec e_i)$ to $(\vec b_i)$, and P be the transition matrix from $(\vec e_i)$ to $(\vec b_i)$. Both basis being $(\cdot,\cdot)_g$-orthonormal, $P^T.P = I$, thus $\det(P) = \pm 1 = \widetilde{\det}_{|\vec e}(\mathcal{P})$. And $\det_{|\vec e}(\vec b_1,...,\vec b_n) = \det_{|\vec e}(\mathcal{P}.\vec e_1,...,\mathcal{P}.\vec e_n) = \widetilde{\det}_{|\vec e}(\mathcal{P})\det_{|\vec e}(\vec e_1,...,\vec e_n) = \widetilde{\det}_{|\vec e}(\mathcal{P})\det_{|\vec b}(\vec b_1,...,\vec b_n)$, thus $\det_{|\vec e} = \widetilde{\det}_{|\vec e}(\mathcal{P})\det_{|\vec b} = \pm\det_{|\vec b}$. ∎

Definition K.15 Two $(\cdot,\cdot)_g$-orthonormal bases $(\vec e_i)$ and $(\vec b_i)$ have the same orientation iff $\det_{|\vec b} = +\det_{|\vec e}$.

Exercice K.16 Prove $\widetilde{\det}_{|\vec e}(\lambda L) = \lambda^n\,\widetilde{\det}_{|\vec e}(L)$.

Answer. $\widetilde{\det}_{|\vec e}(\lambda L) = \det_{|\vec e}(\lambda L.\vec e_1,...,\lambda L.\vec e_n) = \lambda^n\det_{|\vec e}(L.\vec e_1,...,L.\vec e_n) = \lambda^n\,\widetilde{\det}_{|\vec e}(L)$. ∎

K.6.2 The determinant of an endomorphism is objective

Proposition K.17 *Let $(\vec a_i)$ and $(\vec b_i)$ be bases in E. The determinant of an endomorphism $L \in \mathcal{L}(E;E)$ is objective (observer independent, here basis independent):*

$$(\det([L]_{|\vec a}) =)\quad \widetilde{\det}_{|\vec a}(L) = \widetilde{\det}_{|\vec b}(L)\quad (= \det([L]_{|\vec b})). \tag{K.29}$$

NB: *But the determinant of n vectors is* **not** *objective, cf. (K.9) (compare the change of basis formula for vectors $[\vec w]_{|\vec b} = P^{-1}.[\vec w]_{|\vec a}$ with the change of basis formula for endomorphisms $[L]_{|\vec b} = P^{-1}.[L]_{|\vec a}.P$).*

Proof. Let $(\vec a_i)$ and $(\vec b_i)$ be bases in E, and P be the transition matrix from $(\vec a_i)$ to $(\vec b_i)$. The change of basis formula $[L]_{|\vec b} = P^{-1}.[L]_{|\vec a}.P$ and (K.26) give $\det([L]_{|\vec b}) = \det(P^{-1})\det([L]_{|\vec a})\det(P) = \det([L]_{|\vec a})$, thus (K.25) gives (K.29). ∎

Exercice K.18 Let $(\vec a_i)$ and $(\vec b_i)$ be bases in E, and $\mathcal{P} \in \mathcal{L}(E;E)$ be the change of basis endomorphism from $(\vec a_i)$ to $(\vec b_i)$ (i.e., $\mathcal{P}.\vec a_j = \vec b_j$ for all j). Prove

$$\det_{|\vec a}(\vec b_1,...,\vec b_n) = \widetilde{\det}_{|\vec a}(\mathcal{P}),\quad \text{thus}\quad \det_{|\vec a} = \widetilde{\det}_{|\vec a}(\mathcal{P})\det_{|\vec b},\quad \text{i.e.}\quad \det_{|\vec b} = \frac{\det_{|\vec a}}{\widetilde{\det}_{|\vec a}(\mathcal{P})}, \tag{K.30}$$

Answer. $\det_{|\vec a}(\vec b_1,...,\vec b_n) = \det_{|\vec a}(\mathcal{P}.\vec a_1,...,\mathcal{P}.\vec a_n) \overset{(K.24)}{=} \widetilde{\det}_{|\vec a}(\mathcal{P})\det_{|\vec a}(\vec a_1,...,\vec a_n) = \widetilde{\det}_{|\vec a}(\mathcal{P})\,1 = \widetilde{\det}_{|\vec a}(\mathcal{P})\det_{|\vec b}(\vec b_1,...,\vec b_n)$, thus (K.30) and $\det_{|\vec a} = \widetilde{\det}_{|\vec a}(\mathcal{P})\det_{|\vec b}$ and $\det_{|\vec a}(\vec v_1,...,\vec v_n) = \widetilde{\det}_{|\vec a}(\mathcal{P})\det_{|\vec b}(\vec v_1,...,\vec v_n)$. ∎

K.7 Determinant of a linear map

(Needed for the deformation gradient $F_t^{t_0}(P) = d\Phi_t^{t_0}(P) : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t$.)

Let A and B be vector spaces, $\dim A = \dim B = n$, and $(\vec{a}_i)$ and $(\vec{b}_i)$ be bases in A and B.

K.7.1 Definition and first properties

Definition K.19 The determinant of a linear map $L \in \mathcal{L}(A;B)$ relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$ is

$$\underset{|\vec{a},\vec{b}}{\widetilde{\det}}(L) := \underset{|\vec{b}}{\det}(L.\vec{a}_1, ..., L.\vec{a}_n). \tag{K.31}$$

(And $\widetilde{\det}_{|\vec{a},\vec{b}}(L) =^{\text{noted}} \det(L)$ if the bases are implicit.)

Thus, (K.13) gives, with $L.\vec{a}_j = \sum_{i=1}^n L_{ij}\vec{b}_i$, i.e. $[L]_{|\vec{a},\vec{b}} = [L_{ij}]$:

$$\underset{|\vec{a},\vec{b}}{\widetilde{\det}}(L) = \det([L]_{|\vec{a},\vec{b}}) = \det([L_{ij}]). \tag{K.32}$$

Proposition K.20 *Let $\vec{u}_1, ..., \vec{u}_n \in A$. Then*

$$\underset{|\vec{b}}{\det}(L.\vec{u}_1, ..., L.\vec{u}_n) = \underset{|\vec{a},\vec{b}}{\widetilde{\det}}(L)\ \underset{|\vec{a}}{\det}(\vec{u}_1, ..., \vec{u}_n). \tag{K.33}$$

Proof. $m : (\vec{u}_1, ..., \vec{u}_n) \in A^n \to m(\vec{u}_1, ..., \vec{u}_n) := \det_{|\vec{b}}(L.\vec{u}_1, ..., L.\vec{u}_n) \in \mathbb{R}$ is a multilinear alternated form, since L is linear; And $m(\vec{a}_1, ..., \vec{a}_n) = \det_{|\vec{b}}(L.\vec{a}_1, ..., L.\vec{a}_n) =^{(K.31)} \widetilde{\det}_{|\vec{a},\vec{b}}(L) = \widetilde{\det}_{|\vec{a},\vec{b}}(L) \det_{|\vec{a}}(\vec{a}_1, ..., \vec{a}_n)$. Thus $m = \widetilde{\det}_{|\vec{a},\vec{b}}(L) \det_{|\vec{a}}$, cf. (K.9), thus (K.33). ∎

Corollary K.21 *Let A, B, C be vector spaces such that $\dim A = \dim B = \dim C = n$. Let $(\vec{a}_i)$, $(\vec{b}_i)$, $(\vec{c}_i)$ be bases in A, B, C. Let $L : A \to B$ and $M : B \to C$ be linear. Then, with $M \circ L =^{noted} M.L$ (thanks to linearity),*

$$\underset{|\vec{a},\vec{c}}{\widetilde{\det}}(M.L) = \underset{|\vec{a},\vec{b}}{\widetilde{\det}}(L)\, \underset{|\vec{b},\vec{c}}{\widetilde{\det}}(M). \tag{K.34}$$

Proof. $\underset{|\vec{a},\vec{c}}{\widetilde{\det}}(M.L) = \underset{|\vec{c}}{\det}(M.L.\vec{a}_1), ..., M.L.\vec{a}_n)) = \underset{|\vec{b},\vec{c}}{\widetilde{\det}}(M) \underset{|\vec{b}}{\det}(L.\vec{a}_1, ..., L.\vec{a}_n) = \underset{|\vec{b},\vec{c}}{\widetilde{\det}}(M)\, \underset{|\vec{a},\vec{b}}{\widetilde{\det}}(L)$. ∎

K.7.2 Jacobian of a motion, and dilatation

Let $\widetilde{\Phi}$ be a motion, let $t_0, t \in \mathbb{R}$, let $\Phi_t^{t_0}$ be the associated motion, let $F_t^{t_0}(p_{t_0}) := d\Phi_t^{t_0}(p_{t_0}) : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t$ the deformation gradient at $p_{t_0} \in \Omega_{t_0}$ relative to t_0 and t, cf. (4.1). Let $(\vec{E}_i)$ be a Euclidean basis in $\vec{\mathbb{R}}^n_{t_0}$ and $(\vec{e}_i)$ be a Euclidean basis in $\vec{\mathbb{R}}^n_t$ for all $t \geq t_0$, and $[F_t^{t_0}(p_{t_0})]_{|\vec{E},\vec{e}} = [F_{ij}(p_{t_0})]$, i.e., $F_t^{t_0}(p_{t_0}).\vec{E}_j = \sum_{i,j=1}^n F_{ij}(p_{t_0})\vec{e}_i$ for all j.

Definition K.22 The "volume dilatation" at p_{t_0}, relative to the Euclidean bases $(\vec{E}_i)$ in $\vec{\mathbb{R}}^n_{t_0}$ and $(\vec{e}_i)$ in $\vec{\mathbb{R}}^n_t$, is

$$J_{|\vec{E},\vec{e}}(\Phi_t^{t_0})(p_{t_0}) := \underset{|\vec{E},\vec{e}}{\widetilde{\det}}(F_t^{t_0}(p_{t_0})) \quad (= \underset{|\vec{e}}{\det}(F_t^{t_0}(p_{t_0}).\vec{E}_1, ..., F_t^{t_0}(p_{t_0}).\vec{E}_n) = \det([F_{ij}(p_{t_0})])), \tag{K.35}$$

usually written $J_{|\vec{E},\vec{e}} := \det([F]_{|\vec{E},\vec{e}})$ (or simply $J = \det(F)$ when everything is implicit).

So, at t_0 at p_{t_0}, $(p_{t_0}, \vec{E}_1, ..., \vec{E}_n)$ is a unit parallelepiped which volume is 1 relative to the unit of measurement chosen in $\vec{\mathbb{R}}^n_{t_0}$, and, at t at $p_t = \Phi_t^{t_0}(p_{t_0})$, $J_{|\vec{E},\vec{e}}(\Phi_t^{t_0})(p_{t_0}) = \det_{|\vec{e}}(F_t^{t_0}(p_{t_0}).\vec{E}_1, ..., F_t^{t_0}(p_{t_0}).\vec{E}_n)$ is the volume of the parallelepiped $(p_t, F_t^{t_0}(p_{t_0}).\vec{E}_1, ..., F_t^{t_0}(p_{t_0}).\vec{E}_n)$ relative to the unit of measurement chosen in $\vec{\mathbb{R}}^n_t$.

Interpretation: With $t_2 > t_1 \geq t_0$, and $[\vec{e}_i)$ is the basis at t_1 and t_2:

- Dilatation if $J_{|\vec{E},\vec{e}}(\Phi_{t_2}^{t_0})(p_{t_0}) > J_{|\vec{E},\vec{e}}(\Phi_{t_1}^{t_0})(p_{t_0})$ (volume increase),

- contraction if $J_{|\vec{E},\vec{e}}(\Phi^{t_0}_{t_2})(p_{t_0}) < J_{|\vec{E},\vec{e}}(\Phi^{t_0}_{t_1})(p_{t_0})$ (volume decrease), and
- incompressibility if $J_{|\vec{E},\vec{e}}(\Phi^{t_0}_{t_2})(p_{t_0}) = J_{|\vec{E},\vec{e}}(\Phi^{t_0}_{t_1})(p_{t_0})$ for all t (volume conservation).

In particular, if $(\vec{e}_i) = (\vec{E}_i)$ then $J_{|\vec{e},\vec{e}}(\Phi^{t_0}_{t_0})(p_{t_0}) = 1$, and if $t > t_0$, then

- Dilatation if $J_{|\vec{e},\vec{e}}(\Phi^{t_0}_{t})(p_{t_0}) > 1$ (volume increase),
- contraction if $J_{|\vec{e},\vec{e}}(\Phi^{t_0}_{t})(p_{t_0}) < 1$ (volume decrease), and
- incompressibility if $J_{|\vec{e},\vec{e}}(\Phi^{t_0}_{t})(p_{t_0}) = 1$ for all t (volume conservation).

Exercice K.23 Let $(\vec{E}_i)$ be a Euclidean basis in $\vec{\mathbb{R}}^n_{t_0}$, and let $(\vec{a}_i)$ and $(\vec{b}_i)$ be two Euclidean bases in $\vec{\mathbb{R}}^n_t$ for the same Euclidean dot product $(\cdot,\cdot)_g$. Prove:

$$J_{|\vec{E},\vec{a}}(\Phi^{t_0}_t(P)) = \pm J_{|\vec{E},\vec{b}}(\Phi^{t_0}_t(P)). \tag{K.36}$$

Answer. P being the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$, $\det(P) = \pm 1$ here. And (4.30) gives $[F]_{|\vec{E},\vec{a}} = P.[F]_{|\vec{E},\vec{b}}$, thus $\det([F]_{|\vec{E},\vec{a}}) = \pm\det([F]_{|\vec{E},\vec{b}})$, thus $\det_{|\vec{a}}(F.\vec{E}_1,...,F.\vec{E}_n) = \pm\det_{|\vec{b}}(F.\vec{E}_1,...,F.\vec{E}_n)$. ▪

K.7.3 Determinant of the transposed

Let $(A,(\cdot,\cdot)_g)$ and $(B,(\cdot,\cdot)_h)$ be finite dimensional Hilbert spaces. Let $L \in \mathcal{L}(A;B)$ (a linear map). Recall: The transposed $L^T_{gh} \in \mathcal{L}(B;A)$ is defined by, for all $\vec{u} \in A$ and all $\vec{w} \in B$, cf. (A.68)

$$(L^T_{gh}.\vec{w},\vec{u})_g := (\vec{w},L.\vec{u})_h. \tag{K.37}$$

Let $(\vec{a}_i)$ be a basis in A and $(\vec{b}_i)$ be a basis in B. Then

$$\widetilde{\det}([L^T_{gh}]_{|\vec{b},\vec{a}}) = \det([L]_{|\vec{a},\vec{b}})\frac{\det([(\cdot,\cdot)_g]_{|\vec{a}})}{\det([(\cdot,\cdot)_h]_{|\vec{b}})}. \tag{K.38}$$

Indeed, (K.37) gives $[(\cdot,\cdot)_g]_{|\vec{a}}.[L^T_{gh}]_{|\vec{b},\vec{a}} = ([L]_{|\vec{a},\vec{b}})^T.[(\cdot,\cdot)_h]_{|\vec{b}}$.

K.8 Dilatation rate

A unique Euclidean basis $(\vec{e}_i)$ at all time is chosen, and $(\cdot,\cdot)_g$ is the associated inner dot product.

K.8.1 $\frac{\partial J^{t_0}}{\partial t}(t,p_{t_0}) = J^{t_0}(t,p_{t_0})\,\mathrm{div}\vec{v}(t,p_t)$

A regular motion $\widetilde{\Phi}$ is considered, cf. (1.5), and the Eulerian velocity is $\vec{v}(t,p_t) = \frac{\partial\widetilde{\Phi}}{\partial t}(t,P_{Obj})$ at $p_t = \widetilde{\Phi}(t,P_{Obj})$. Let t_0 be given; The associated motion Φ^{t_0} is given by $\Phi^{t_0}(t,p_{t_0}) = \widetilde{\Phi}(t,P_{Obj}) =^{\text{noted}} p_t$ when $p_{t_0} = \widetilde{\Phi}(t_0,P_{Obj})$, (3.1), and is supposed to be at least C^2; The Lagrangian velocity is $\vec{V}(t,p_{t_0}) = \frac{\partial\Phi^{t_0}}{\partial t}(t,p_{t_0})$, and the Eulerian velocity satisfies $\vec{v}(t,p_t) = \frac{\partial\Phi^{t_0}}{\partial t}(t,p_{t_0})$ when $p_t = \Phi^{t_0}(t,p_{t_0})$, cf. (3.25). Let $F^{t_0}(t,p_{t_0}) = d\Phi^{t_0}(t,p_{t_0}) = F^{t_0}_t(p_{t_0}) = d\Phi^{t_0}_t(p_{t_0})$, and consider the Jacobian

$$J^{t_0}_t(p_{t_0}) = \det_{|\vec{e}}(F^{t_0}_t(p_{t_0})) = J^{t_0}(t,p_{t_0}), \tag{K.39}$$

Lemma K.24 *$\frac{\partial J^{t_0}}{\partial t}(t,p_{t_0})$ satisfies, with $p_t = \Phi^{t_0}_t(p_{t_0})$,*

$$\frac{\partial J^{t_0}}{\partial t}(t,p_{t_0}) = J^{t_0}(t,p_{t_0})\,\mathrm{div}\vec{v}(t,p_t) \tag{K.40}$$

(value to be considered at t at p_t). In particular, $\widetilde{\Phi}$ is incompressible iff $\mathrm{div}\vec{v}(t,p_t) = 0$.

Proof. Let $\mathcal{O}$ be a origin in $\mathbb{R}^n$. Let $\overrightarrow{\mathcal{O}\Phi^{t_0}} = \sum_{i=1}^n \Phi^i\vec{e}_i$, $\vec{V}^{t_0} = \sum_{i=1}^n V^i\vec{e}_i$, $\vec{v} = \sum_{i=1}^n v^i\vec{e}_i$, $F^{t_0}.\vec{E}_j = d\Phi^{t_0}.\vec{E}_j = \sum_{i=1}^n \frac{\partial\Phi^i}{\partial X^j}\vec{e}_i$. Let $[F^{t_0}]_{|\vec{E},\vec{e}} =^{\text{noted}} F$, $J^{t_0} =^{\text{noted}} J$ and $[d\Phi^i]_{|\vec{E}} = \left(\frac{\partial\Phi^i}{\partial X^1} \;\; \dots \;\; \frac{\partial\Phi^i}{\partial X^n}\right) =^{\text{noted}} d\Phi^i$

(row matrix). Thus $J = \det F = \det\begin{pmatrix} d\Phi^1 \\ \vdots \\ d\Phi^n \end{pmatrix}$, thus (a determinant is multilinear)

$$\frac{\partial J}{\partial t} = \det\begin{pmatrix} \dfrac{\partial(d\Phi^1)}{\partial t} \\ d\Phi^2 \\ \vdots \\ d\Phi^n \end{pmatrix} + \ldots + \det\begin{pmatrix} d\Phi^1 \\ \vdots \\ d\Phi^{n-1} \\ \dfrac{\partial(d\Phi^n)}{\partial t}) \end{pmatrix}.$$

With Φ^{t_0} C^2, thus $\dfrac{\partial(d\Phi^i)}{\partial t}(t,p_{t_0}) \overset{\text{Swhartz}}{=} d(\dfrac{\partial\Phi^i}{\partial t})(t,p_{t_0}) = dV^i(t,p_{t_0}) = dv^i(t,p_t).F(t,p_{t_0})$, cf. (3.27). Thus

$$\det\begin{pmatrix} \dfrac{\partial(d\Phi^1)}{\partial t} \\ d\Phi^2 \\ \vdots \\ d\Phi^n \end{pmatrix} = \det\begin{pmatrix} \displaystyle\sum_{i=1}^n \frac{\partial v^1}{\partial x^i} d\Phi^i \\ d\Phi^2 \\ \vdots \\ d\Phi^n \end{pmatrix} \overset{\text{det is}}{\underset{\text{alternating}}{=}} \det\begin{pmatrix} \dfrac{\partial v^1}{\partial x^1} d\Phi^1 \\ d\Phi^2 \\ \vdots \\ d\Phi^n \end{pmatrix} = \frac{\partial v^1}{\partial x^1} \det\begin{pmatrix} d\Phi^1 \\ d\Phi^2 \\ \vdots \\ d\Phi^n \end{pmatrix} = \frac{\partial v^1}{\partial x^1} J$$

Idem for the other terms, thus

$$\frac{\partial J}{\partial t}(t,p_{t_0}) = \frac{\partial v^1}{\partial x^1}(t,p_t)\ J(t,p_{t_0}) + \ldots + \frac{\partial v^n}{\partial x^n}(t,p_t)\ J(t,p_{t_0}) = \mathrm{div}\vec{v}(t,p_t)\ J(t,p_{t_0}),$$

i.e. (K.40). ∴

Definition K.25 $\mathrm{div}\vec{v}(t,p_t)$ is the dilatation rate.

K.8.2 Leibniz formula

Proposition K.26 (Leibniz formula) *Under regularity assumptions (e.g. hypotheses of the Lebesgue theorem to be able to derive under $\int$) we have*

$$\begin{aligned} \frac{d}{dt}\Big(\int_{p_t\in\Omega_t} f(t,p_t)\,d\Omega_t\Big) &= \int_{p_t\in\Omega_t} \big(\frac{Df}{Dt} + f\,\mathrm{div}\vec{v}\big)(t,p_t)\,d\Omega_t \\ &= \int_{p_t\in\Omega_t} \big(\frac{\partial f}{\partial t} + df.\vec{v} + f\,\mathrm{div}(\vec{v})\big)(t,p_t)\,d\Omega_t \\ &= \int_{p_t\in\Omega_t} \big(\frac{\partial f}{\partial t} + \mathrm{div}(f\vec{v})\big)(t,p_t)\,d\Omega_t. \end{aligned} \tag{K.41}$$

Proof. Let

$$Z(t) := \int_{p\in\Omega_t} f(t,p)\,d\Omega_t = \int_{P\in\Omega_{t_0}} f(t,\Phi^{t_0}(t,P))\,J^{t_0}(t,P)\,d\Omega_{t_0}.$$

(The Jacobian is positive for a regular motion.) Then (derivation under $\int$)

$$\begin{aligned} Z'(t) &= \int_{P\in\Omega_{t_0}} \frac{Df}{Dt}(t,p_t)\,J^{t_0}(t,P) + f(t,p_t)\frac{\partial J^{t_0}}{\partial t}(t,P)\,d\Omega_{t_0} \\ &= \int_{P\in\Omega_{t_0}} \big(\frac{Df}{Dt}(t,p_t)\ + f(t,p_t)\,\mathrm{div}\vec{v}(t,p_t)\big)J^{t_0}(t,P)\,d\Omega_{t_0}, \end{aligned}$$

thanks to (K.40). And $\mathrm{div}(f\vec{v}) = df.\vec{v} + f\,\mathrm{div}\vec{v}$ gives (K.41). ∴

Corollary K.27 *With $(\vec{u},\vec{w})_g =^{noted} \vec{u} \bullet \vec{w}$ (in the given Euclidean framework),*

$$\frac{d}{dt}\int_{\Omega_t} f(t,p_t)\,d\Omega_t = \int_{\Omega_t} \frac{\partial f}{\partial t}(t,p_t)\,d\Omega_t + \int_{\partial\Omega_t} (f\vec{v}\bullet\vec{n})(t,p_t)\,d\Gamma_t, \tag{K.42}$$

sum of the temporal variation within Ω_t and the flux through the surface $\partial\Omega_t$.

Proof. Apply $(\text{K.41})_3$. ∴

K.9 $\partial J/\partial F = J\,F^{-T}$

K.9.1 Meaning of $\frac{\partial \det}{\partial M_{ij}}$?

Let $\mathcal{M}_{nn} = \{M = [M_{ij}] \in \mathbb{R}^{n^2}\}$ be the set of $n * n$ matrices, and consider the function

$$Z := \det : \left\{ \begin{array}{rl} \mathcal{M}_{nn} & \to \mathbb{R} \\ M = [M_{ij}] & \to Z(M) := \det(M) = \det([M_{ij}]). \end{array} \right. \tag{K.43}$$

Question: What does $\frac{\partial Z}{\partial M_{ij}}(M)$ mean?

Answer: It is the "standard meaning" of a directional derivative $\frac{\partial f}{\partial x_i}(\vec{x}) = df(\vec{x}).\vec{e}_i$... where here $f = Z$, thus $\vec{x} =^{\text{noted}} M$ is a matrix (a vector in $\mathcal{M}_{nn}$), and $(\vec{e}_i)$ is the canonical basis (m_{ij}) in $\mathcal{M}_{nn}$ (all the elements of the matrix m_{ij} vanish but the element at intersection of line i and column j which equals 1). So:

$$\frac{\partial Z}{\partial M_{ij}}(M) := dZ(M).m_{ij} = \lim_{h\to 0} \frac{Z(M + h m_{ij}) - Z(M)}{h} \quad (\in \mathbb{R}). \tag{K.44}$$

K.9.2 Calculation of $\frac{\partial \det}{\partial M_{ij}}$

Proposition K.28

$$\forall i, j, \quad \frac{\partial Z}{\partial M_{ij}}(M) = Z(M)\,(M^{-T})_{ij}, \quad \textit{written} \quad \frac{\partial Z}{\partial M} = Z\,M^{-T}. \tag{K.45}$$

Proof. $\frac{\partial Z}{\partial M_{ij}}(M) := \lim_{h\to 0} \frac{\det(M+hm_{ij})-\det(M)}{h}$; The development of the determinant $\det(M + hm_{ij})$ relative to the column j gives

$$\det(M + h[m_{ij}]) = \det(M) + h\,c_{ij} \tag{K.46}$$

where c_{ij} is the (i, j)-th cofactor of M; Thus $\frac{\partial Z}{\partial M_{ij}}(M) = \lim_{h\to 0} \frac{Z(M+hm_{ij})-Z(M)}{h} = c_{ij}$; And since $M^{-1} = \frac{1}{\det(M)}[c_{ij}]^T$, i.e. $[c_{ij}] = \det(M)M^{-T}$, we get $\frac{\partial Z}{\partial M_{ij}}(M) = \det(M)(M^{-T})_{ij}$, i.e. (K.45). ∴

K.9.3 $\partial J/\partial F = J\,F^{-T}$ usually written $[\frac{\partial J}{\partial F_{ij}}] = J\,F^{-T}$

Setting of § K.8: With $F := d\Phi(p_{t_0})$ we have $F.\vec{E}_j = \sum_{i=1}^n F_{ij}\vec{e}_i$ where $F_{ij} = \frac{\partial \Phi_i}{\partial X_j}(p_{t_0})$, and

$$J_{\Phi,p_{t_0},\vec{E},\vec{e}} \overset{\text{noted}}{=} J : \left\{ \begin{array}{rl} \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t) & \to \mathbb{R} \\ F & \to J(F) := \det([F_{ij}]) \quad (= \det([\dfrac{\partial \Phi_i}{\partial X_j}(p_{t_0})]), \end{array} \right. \tag{K.47}$$

so, $J(F)$ is the Jacobian $\widetilde{\det}_{|\vec{E},\vec{e}}(d\Phi(p_{t_0}))$ of Φ at p_{t_0} relative to $(\vec{E}_i)$ and $(\vec{e}_i)$. Thus (K.45) gives:

Corollary K.29

$$\forall i, j, \quad \frac{\partial J}{\partial F_{ij}}(F) = J(F)\,([F]^{-T})_{ij}, \quad \textit{written} \quad \frac{\partial J}{\partial F} = J\,F^{-T}. \tag{K.48}$$

K.9.4 Interpretation of $\frac{\partial J}{\partial F_{ij}}$?

The first derivations into play are along the directions $\vec{E}_j$ at t_0: The $F_{ij} = \frac{\partial \Phi_i}{\partial X_j}(p_{t_0}) := d\Phi_i(p_{t_0}).\vec{E}_j$.

Question: $\frac{\partial J}{\partial F_{ij}}$ is the usual notation for a directional derivative, cf. § K.9.1. So $\frac{\partial J}{\partial F_{ij}}$ is the derivative in which direction?

Answer: 1- "Identify" $F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ with the tensor $\widetilde{F} \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t, \vec{\mathbb{R}}^n_{t_0};\mathbb{R})$ given by $\widetilde{F}(\ell, \vec{U}) = \ell.(F.\vec{U})$; So, if $F.\vec{E}_j = \sum_{i=1}^n F_{ij}\vec{e}_i$ then $\widetilde{F} = \sum_{i,j=1}^n F_{ij}\vec{e}_i \otimes \pi_{Ej}$, relative to a basis $(\vec{E}_i)$ and its covariant dual basis (π_{Ei}) in $\vec{\mathbb{R}}^n_{t_0}$ and a basis $(\vec{e}_i)$ and $\vec{\mathbb{R}}^n_t$.

2- Define the function $\widetilde{\det}_{\vec{E},\vec{e}} = \widetilde{J} : \left\{ \begin{array}{rl} \mathcal{L}(\vec{\mathbb{R}}^{n*}_t, \vec{\mathbb{R}}^n_{t_0};\mathbb{R}) & \to \mathbb{R} \\ \widetilde{F} & \to \widetilde{J}(\widetilde{F}) := J(F) = \det\limits_{\vec{E},\vec{e}}(F) = \det([F_{ij}]) \end{array} \right\}$;

3- Then it is meaningful to differentiate $\widetilde{J}$ along the direction $\vec{e}_i \otimes \pi_{Ej} \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t, \vec{\mathbb{R}}^n_{t_0}; \mathbb{R})$ to get

$$\frac{\partial \widetilde{J}}{\partial F_{ij}}(\widetilde{F}) := \lim_{h\to 0} \frac{\widetilde{J}_{|\vec{e},\vec{E}}(\widetilde{F} + h\vec{e}_i \otimes \pi_{Ej}) - \widetilde{J}_{|\vec{e},\vec{E}}(\widetilde{F})}{h} \quad (\overset{\text{noted}}{=} \frac{\partial J}{\partial F_{ij}}(F)); \tag{K.49}$$

This is a derivation in both directions π_{Ej} in $\vec{\mathbb{R}}^n_{t_0}$ (past at p_{t_0}) ***and*** $\vec{e}_i$ in $\vec{\mathbb{R}}^n_t$ (present at p_t). What does this derivative mean? (The answer is unknown to the author.)

L Transport of volumes and areas

Here $\mathbb{R}^n = \mathbb{R}^3$ the usual affine space. Let $t_0, t \in \mathbb{R}$, and $\Phi^{t_0}_t : \mathbb{R} \times \Omega_{t_0} \to \Omega_t$, see (3.1). Let $F_P = d\Phi^{t_0}_t(P)$. Let $(\cdot,\cdot)_g$ be a Euclidean dot product in $\vec{\mathbb{R}}^n$ (English, French...), with $||.||_g$ the associated norm.

L.1 Transformed parallelepiped

The Jacobian of $\Phi^{t_0}_t$ at P relative to a $(\cdot,\cdot)_g$-Euclidean bases is defined in (K.35): With $F_P = F^{t_0}_t(P)$,

$$J_P = J(P) := \det_{|\vec{e}}(F^{t_0}_t(P).\vec{E}_1, ..., F^{t_0}_t(P).\vec{E}_n)), \quad \text{and} \quad J_P > 0 \tag{L.1}$$

the motion being supposed regular. Thus, if $(\vec{U}_{1P}, ..., \vec{U}_{nP})$ is a parallelepiped at P at t_0, if $\vec{u}_{ip} = F_P.\vec{U}_{iP}$, then $(\vec{u}_{1p}, ..., \vec{u}_{np})$ is a parallelepiped at p at t which volume is

$$\det_{|\vec{e}}(\vec{u}_{1p}, ..., \vec{u}_{np}) = J_P \det_{|\vec{e}}(\vec{U}_{1P}, ..., \vec{U}_{nP}). \tag{L.2}$$

L.2 Transformed volumes

Riemann integrals and (L.2) give the change of variable formula: For any regular function $f : \Omega_t \to \mathbb{R}$,

$$\int_{p_t\in\Omega_t} f(p_t)\, d\Omega_t = \int_{P\in\Omega_{t_0}} f(\Phi^{t_0}_t(P))\, |J(P)|\, d\Omega_{t_0}. \tag{L.3}$$

(See (K.18): $d\Omega_t$ is a positive measure: It is not a multilinear form.) In particular,

$$|\Omega_t| = \int_{p_t\in\Omega_t} d\Omega_t(p_t) = \int_{P\in\Omega_{t_0}} |J(P)|\, d\Omega_{t_0}(P). \tag{L.4}$$

(With $J(P) > 0$ for regular motions.)

L.3 Transformed parallelogram

Consider two independent vectors $\vec{U}_{1P}, \vec{U}_{2P} \in \vec{\mathbb{R}}^n_{t_0}$ at t_0 at P, and the vectors $\vec{u}_{1p} = F_P.\vec{U}_{1P}$ and $\vec{u}_{2p} = F_P.\vec{U}_{2P}$ at t at $p = \Phi^{t_0}_t(P)$. Since $\Phi^{t_0}_t$ is a diffeomorphism, $\vec{u}_{1p}$ and $\vec{u}_{2p}$ are independent.

Then choose a Euclidean dot product $(\cdot,\cdot)_g$ (English, French...) to be able to use the vectorial product, cf. (E.15), the same at all time t. Then the areas of the parallelograms are

$$||\vec{U}_{1P} \wedge \vec{U}_{2P}||_g \quad \text{and} \quad ||\vec{u}_{1p} \wedge \vec{u}_{2p})||_g, \tag{L.5}$$

and unit normal vectors to the quadrilaterals are

$$\vec{N}_P = \frac{\vec{U}_{1P} \wedge \vec{U}_{2P}}{||\vec{U}_{1P} \wedge \vec{U}_{2P}||_g} \in \vec{\mathbb{R}}^n_{t_0}, \quad \text{and} \quad \vec{n}_p = \frac{\vec{u}_{1p} \wedge \vec{u}_{2p}}{||\vec{u}_{1p} \wedge \vec{u}_{2p}||_g} \in \vec{\mathbb{R}}^n_t. \tag{L.6}$$

Proposition L.1 *If $\vec{u}_{1p} = F_P.\vec{U}_{1P}$ and $\vec{u}_{2p} = F_P.\vec{U}_{2P}$, then*

$$\vec{u}_{1p} \wedge \vec{u}_{2p} = J_P\, F_P^{-T}.(\vec{U}_{1P} \wedge \vec{U}_{2P}), \quad \textit{and} \quad ||\vec{u}_{1p} \wedge \vec{u}_{2p}||_g = J_P\, ||F_P^{-T}.(\vec{U}_{1P} \wedge \vec{U}_{2P})||_g, \tag{L.7}$$

since $J_P > 0$ (for regular motions), and

$$\vec{n}_p = \frac{F_P^{-T}.\vec{N}_P}{||F_P^{-T}.\vec{N}_P||_g} \quad (\neq F_P.\vec{N}_P \textit{ in general}). \tag{L.8}$$

Proof. Let $\vec{W}_P \in \vec{\mathbb{R}}^n_{t_0}$ and $\vec{w}_p = F_P.\vec{W}_P$. Then the volume of the parallelepiped $(\vec{u}_{1p}, \vec{u}_{2p}, \vec{w}_p)$ is

$$\begin{aligned}(\vec{u}_{1p} \wedge \vec{u}_{2p}, \vec{w}_p)_g &= \det(\vec{u}_{1p}, \vec{u}_{2p}, \vec{w}_p) = \det(F_P.\vec{U}_{1P}, F_P.\vec{U}_{2P}, F_P.\vec{W}_P) = \det(F_P)\det(\vec{U}_{1P}, \vec{U}_{2P}, \vec{W}_P) \\ &= J_P\,(\vec{U}_{1P} \wedge \vec{U}_{2P}, \vec{W}_P)_g = J_P\,(\vec{U}_{1P} \wedge \vec{U}_{2P}, F_P^{-1}.\vec{w}_p)_g = J_P\,(F_P^{-T}.(\vec{U}_{1P} \wedge \vec{U}_{2P}), \vec{w}_p)_g,\end{aligned} \tag{L.9}$$

for all $\vec{w}_p$, thus (L.7), thus $\frac{\vec{u}_{1p} \wedge \vec{u}_{2p}}{||\vec{u}_{1p} \wedge \vec{u}_{2p}||_g} = \frac{J_P\,F_P^{-T}.(\vec{U}_{1P} \wedge \vec{U}_{2P})}{J_P||F_P^{-T}.(\vec{U}_{1P} \wedge \vec{U}_{2P})||_g}$, thus (L.8). ∎

L.4 Transformed surface

L.4.1 Deformation of a surface

A parametrized surface Ψ_{t_0} in Ω_{t_0} and the associated geometric surface S_{t_0} are defined by

$$\Psi_{t_0} : \left\{ \begin{array}{rl} [a,b] \times [c,d] & \to \Omega_{t_0} \\ (u,v) & \to P = \Psi_{t_0}(u,v) \end{array} \right\} \quad \text{and} \quad S_{t_0} = \operatorname{Im}(\Psi_{t_0}) \subset \Omega_{t_0}. \tag{L.10}$$

(It is also represented after a choice of an origin O by the vector valued parametrized surface $\vec{r}_{t_0} = \overrightarrow{O\Psi_{t_0}}$.)

The transformed parametric surface is $\Psi_t := \Phi_t^{t_0} \circ \Psi_{t_0}$ and the associated geometric surface is S_t:

$$\Psi_t := \Phi_t^{t_0} \circ \Psi_{t_0} : \left\{ \begin{array}{rl} [a,b] \times [c,d] & \to \Omega_{t_0} \\ (u,v) & \to p = \Psi_t(u,v) = \Phi_t^{t_0}(\Psi_{t_0}(u,v)) = \Phi_t^{t_0}(P) \end{array} \right\} \quad \text{and} \quad S_t = \Phi_t^{t_0}(S_{t_0}). \tag{L.11}$$

(After a choice of an origin O, the associated vector valued parametrized surface is $\vec{r}_t = \overrightarrow{O\Psi_t}$.)

Let $(\vec{E}_1, \vec{E}_2)$ be the canonical basis in the space $\mathbb{R} \times \mathbb{R} \supset [a,b] \times [c,d] = \{(u,v)\}$ of parameters.

The surface Ψ_{t_0} is supposed to be regular, that is, Ψ_{t_0} is C^1 and, for all $P = \Psi_{t_0}(u,v) \in S_{t_0}$, the tangents vectors $\vec{T}_{1P}$ and $\vec{T}_{2P}$ at P are independent, that is,

$$\left. \begin{array}{l} \vec{T}_{1P} := d\Psi_{t_0}(u,v).\vec{E}_1 \overset{\text{noted}}{=} \dfrac{\partial \Psi_{t_0}}{\partial u}(u,v), \\ \vec{T}_{2P} := d\Psi_{t_0}(u,v).\vec{E}_2 \overset{\text{noted}}{=} \dfrac{\partial \Psi_{t_0}}{\partial v}(u,v), \end{array} \right\} \quad \text{and} \quad \vec{T}_{1P} \wedge \vec{T}_{2P} \neq \vec{0}. \tag{L.12}$$

And the tangent vectors at S_t at $p = \Phi_t^{t_0}(P)$ at t are

$$\left\{ \begin{array}{l} \vec{t}_{1p} := d\Psi_t(u,v).\vec{E}_1 = \dfrac{\partial \Psi_t}{\partial u}(u,v), \quad \text{so} \quad \vec{t}_{1p} = F_P.\vec{T}_{1P} \quad (= d\Phi_t^{t_0}(P).\dfrac{\partial \Psi_{t_0}}{\partial u}(u,v)), \\ \vec{t}_{2p} := d\Psi_t(u,v).\vec{E}_2 = \dfrac{\partial \Psi_t}{\partial v}(u,v), \quad \text{so} \quad \vec{t}_{2p} = F_P.\vec{T}_{2P} \quad (= d\Phi_t^{t_0}(P).\dfrac{\partial \Psi_{t_0}}{\partial v}(u,v)). \end{array} \right. \tag{L.13}$$

These vectors are independent since $\Phi_t^{t_0}$ is a diffeomorphism and Ψ_{t_0} is regular. In facts, we used tangent vectors to curves and their push-forwards, cf. figure 4.1 and § 6.5.1.

L.4.2 Euclidean dot product and unit normal vectors

Then choose a Euclidean dot product $(\cdot,\cdot)_g$ (English, French...), to be able to use the vectorial product, cf. (E.15), the same at all time t. Then the scalar area elements $d\Sigma_P$ at P at S_{t_0} relative to Ψ_{t_0}, and $d\sigma_p$ at p at S_t relative to Ψ_t, are

$$\left\{ \begin{array}{l} d\Sigma_P := ||\dfrac{\partial \Psi_{t_0}}{\partial u}(u,v) \wedge \dfrac{\partial \Psi_{t_0}}{\partial v}(u,v)||_g \, du\, dv \quad (= ||\vec{T}_{1P} \wedge \vec{T}_{2P}||_g \, du\, dv), \\ d\sigma_p := ||\dfrac{\partial \Psi_t}{\partial u}(u,v) \wedge \dfrac{\partial \Psi_t}{\partial v}(u,v)||_g \, du\, dv \quad (= ||\vec{t}_{1p} \wedge \vec{t}_{2p}||_g \, du\, dv). \end{array} \right. \tag{L.14}$$

And the areas of S_{t_0} and S_t are

$$\left\{ \begin{array}{l} |S_{t_0}| = \displaystyle\int_{P \in S_{t_0}} d\Sigma_P := \int_{u=a}^{b} \int_{v=c}^{d} ||\dfrac{\partial \Psi_{t_0}}{\partial u}(u,v) \wedge \dfrac{\partial \Psi_{t_0}}{\partial v}(u,v)||_g \, du\, dv, \\ |S_t| = \displaystyle\int_{p \in S_t} d\sigma_p := \int_{u=a}^{b} \int_{v=c}^{d} ||\dfrac{\partial \Psi_t}{\partial u}(u,v) \wedge \dfrac{\partial \Psi_t}{\partial v}(u,v)||_g \, du\, dv. \end{array} \right. \tag{L.15}$$

(See (K.18): $d\Sigma_P$ and $d\sigma_p$ are positive measures: They are not multilinear forms.)

And the unit normal vectors $\vec{N}_P$ at S_{t_0} at P at t_0 and $\vec{n}_p$ at S_t at p at t are

$$\begin{cases} \vec{N}_P = \dfrac{\frac{\partial\Psi_{t_0}}{\partial u}(u,v) \wedge \frac{\partial\Psi_{t_0}}{\partial v}(u,v)}{||\frac{\partial\Psi_{t_0}}{\partial u}(u,v) \wedge \frac{\partial\Psi_{t_0}}{\partial v}(u,v)||_g} \quad (= \dfrac{\vec{T}_{1P} \wedge \vec{T}_{2P}}{||\vec{T}_{1P} \wedge \vec{T}_{2P}||_g}) \\ \vec{n}_p = \dfrac{\frac{\partial\Psi_t}{\partial u}(u,v) \wedge \frac{\partial\Psi_t}{\partial v}(u,v)}{||\frac{\partial\Psi_t}{\partial u}(u,v) \wedge \frac{\partial\Psi_t}{\partial v}(u,v)||_g} \quad (= \dfrac{\vec{t}_{1p} \wedge \vec{t}_{2p}}{||\vec{t}_{1p} \wedge \vec{t}_{2p}||_g} = . \end{cases} \tag{L.16}$$

Then the vectorial area elements $d\vec{\Sigma}_P$ at P at $S_{t_0} = \mathrm{Im}(\Psi_{t_0})$ relative to $\vec{r}_{t_0}$ and $d\vec{\sigma}_p$ at p at $S_t = \mathrm{Im}(\Psi_t)$ relative to Ψ_t are

$$\begin{cases} d\vec{\Sigma}_P := \vec{N}_P\, d\Sigma_P = \dfrac{\partial\Psi_{t_0}}{\partial u}(u,v) \wedge \dfrac{\partial\Psi_{t_0}}{\partial v}(u,v)\, du\, dv \quad (= \vec{T}_{1P} \wedge \vec{T}_{2P}\, du\, dv) \\ d\vec{\sigma}_p := \vec{n}_p\, d\sigma_p = \dfrac{\partial\Psi_t}{\partial u}(u,v) \wedge \dfrac{\partial\Psi_t}{\partial v}(u,v)\, du\, dv \quad (= \vec{t}_{1p} \wedge \vec{t}_{2p}\, du\, dv). \end{cases} \tag{L.17}$$

(Useful to get the flux through a surface: $\int_\Gamma \vec{f} \bullet \vec{n}\, d\sigma = \int_\Gamma \vec{f} \bullet d\vec{\sigma}$.)

(NB: $d\vec{\Sigma}_P$ and $d\vec{\sigma}_p$ are not multilinear since $d\Sigma_P$ and $d\sigma_p$ are not.)

L.4.3 Relations between surfaces

$\vec{t}_{1p} \wedge \vec{t}_{2p} = J_P\, F_P^{-T}.(\vec{T}_{1P} \wedge \vec{T}_{2P})$, cf. (L.7), gives

$$\frac{\partial\Psi_t}{\partial u}(u,v) \wedge \frac{\partial\Psi_t}{\partial v}(u,v) = J_P\, F_P^{-T}.(\frac{\partial\Psi_{t_0}}{\partial u}(u,v) \wedge \frac{\partial\Psi_{t_0}}{\partial v}(u,v)). \tag{L.18}$$

This gives the relation between vectorial and scalar area elements,

$$\vec{n}\, d\sigma_p = d\vec{\sigma}_p = J_P\, F_P^{-T}.d\vec{\Sigma}_P = J_P\, F_P^{-T}.\vec{N}_P\, d\Sigma_P, \quad \text{and} \quad d\sigma_p = J_P\, ||F_P^{-T}.\vec{N}_P||_g\, d\Sigma_P. \tag{L.19}$$

(Check with (L.8).)

L.5 Piola identity

Reminder: Let $M = [M^i_j]$ be a $3*3$ matrix function. We use the usual divergence in continuum mechanics (non objective) given by $\mathrm{div} M := \begin{pmatrix} \frac{\partial M^1_1}{\partial X^1} + \frac{\partial M^1_2}{\partial X^2} + \frac{\partial M^1_3}{\partial X^3} \\ \frac{\partial M^2_1}{\partial X^1} + \frac{\partial M^2_2}{\partial X^2} + \frac{\partial M^2_3}{\partial X^3} \\ \frac{\partial M^3_1}{\partial X^1} + \frac{\partial M^3_2}{\partial X^2} + \frac{\partial M^3_3}{\partial X^3} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n \frac{\partial M^1_j}{\partial X^j} \\ \sum_{j=1}^n \frac{\partial M^2_j}{\partial X^j} \\ \sum_{j=1}^n \frac{\partial M^3_j}{\partial X^j} \end{pmatrix}$, cf. (S.65). And if $\mathrm{Cof}(M)$ is the matrix of cofactors (in $\mathbb{R}^3$: $\mathrm{Cof}(M)^i_j = M^{i+1}_{j+1} M^{i+2}_{j+2} - M^{i+1}_{j+2} M^{i+2}_{j+1}$), then $M^{-1} = \frac{1}{\det M}\mathrm{Cof}(M)^T$, i.e.,

$$(\det M) M^{-1} = \mathrm{Cof}(M)^T. \tag{L.20}$$

The framework being Euclidean, we use a Euclidean basis and the associated matrix, and thus (matrix meaning)

$$J(P)F(P)^{-T} = \mathrm{Cof}(F(P)) \overset{\text{noted}}{=} \mathrm{Cof}(F)(P), \quad \text{written} \quad JF^{-T} = \mathrm{Cof}(F). \tag{L.21}$$

Proposition L.2 (Piola identity) *In $\mathbb{R}^3$, we have*

$$\mathrm{div}(JF^{-T})(P) = 0, \quad \textit{i.e.} \quad \forall i, \ \sum_{j=1}^n \frac{\partial \mathrm{Cof}(F)^i_j}{\partial X^j}(P) = 0. \tag{L.22}$$

Also written $\sum_{j=1}^n \frac{\partial}{\partial X^j}(J \frac{\partial X^i}{\partial x^j}) = 0$... NB: (L.22) is a just a matrix computation since we used the divergence of a matrix (we used components relative to a given basis).

Proof. We are in $\mathbb{R}^3$, thus $\mathrm{Cof}(F)^i_j = F^{i+1}_{j+1} F^{i+2}_{j+2} - F^{i+1}_{j+2} F^{i+2}_{j+1}$, and $F = [d\Phi_t] = [\frac{\partial \varphi^i}{\partial X^j}]$, that is, $F^i_j = \frac{\partial \varphi^i}{\partial X^j}$. Thus

$$\frac{\partial \mathrm{Cof}(F)^i_j}{\partial X^j} = \frac{\partial^2 \varphi^{i+1}}{\partial X^j \partial X^{j+1}} \frac{\partial \varphi^{i+2}}{\partial X^{j+2}} + \frac{\partial \varphi^{i+1}}{\partial X^{j+1}} \frac{\partial^2 \varphi^{i+2}}{\partial X^j \partial X^{j+2}} - \frac{\partial^2 \varphi^{i+1}}{\partial X^j \partial X^{j+2}} \frac{\partial \varphi^{i+2}}{\partial X^{j+1}} - \frac{\partial \varphi^{i+1}}{\partial X^{j+2}} \frac{\partial^2 \varphi^{i+2}}{\partial X^j \partial X^{j+1}}.$$

Thus, for all $i = 1,2,3$, we get $\sum_{j=1}^n \frac{\partial \mathrm{Cof}(F)^i_j}{\partial X^j} = 0$ (the terms cancel each other out two by two). ∎

L.6 Piola transformation

Let $\vec{u}$ be a vector field in Ω_t. The goal is to find a vector field $\vec{U}_{\rm Piola}$ in Ω_{t_0} s.t. for all open subset $\omega_t \subset \Omega_t$ with $\omega_{t_0} = {\Phi_t^{t_0}}^{-1}(\omega_t) \subset \Omega_{t_0}$,

$$\int_{\partial\omega_{t_0}} \vec{U}_{\rm Piola} \bullet \vec{N}\, d\Sigma = \int_{\partial\omega_t} \vec{u} \bullet \vec{n}\, d\sigma, \tag{L.23}$$

or

$$\int_{\omega_{t_0}} {\rm div}(\vec{U}_{\rm Piola})\, d\Omega_{t_0} = \int_{\omega_t} {\rm div}(\vec{u})\, d\Omega_t, \tag{L.24}$$

i.e.

$$\int_{P\in\omega_{t_0}} {\rm div}(\vec{U}_{\rm Piola})(P)\, d\Omega_{t_0} = \int_{P\in\omega_{t_0}} {\rm div}(\vec{u})(\Phi_t^{t_0}(P))\, J(P)\, d\Omega_{t_0}. \tag{L.25}$$

(The motion is supposed to be regular, so $J(P) > 0$). Thus we want

$$ {\rm div}\vec{U}_{\rm Piola}(P) = J(P)\, {\rm div}\vec{u}(p) \quad \text{when} \quad p = \Phi_t^{t_0}(P). \tag{L.26}$$

Definition L.3 The Piola transform is the map

$$\begin{cases} C^\infty(\Omega_t;\mathbb{R}^n) & \to C^\infty(\Omega_{t_0};\mathbb{R}^n) \\ \qquad\qquad \vec{u} & \to \vec{U}_{\rm Piola}, \quad \boxed{\vec{U}_{\rm Piola}(P) := J(P)F(P)^{-1}.\vec{u}(p)} \quad \text{when} \quad p = \Phi_t^{t_0}(P). \end{cases} \tag{L.27}$$

(So $\vec{U}_{\rm Piola}(P) = J(P)\Phi^*(\vec{u})(P)$ where $\Phi^*(\vec{u})(P) = F(P)^{-1}.\vec{u}(p) =$ the pull-back with $\Phi = \Phi_t^{t_0}$.)

Proposition L.4 *With* $p = \Phi_t^{t_0}(P)$, $\vec{U}_{\rm Piola} = \sum_{i=1}^n U_{\rm Piola}^i \vec{e}_i$ *and* $\vec{u} = \sum_{i=1}^n u^i \vec{e}_i$ *we get*

$$ {\rm div}\vec{U}_{\rm Piola}(P) = J(P)\, {\rm div}\vec{u}(p), \quad \textit{i.e.} \quad \sum_{i=1}^n \frac{\partial U^i_{\rm Piola}}{\partial X^i}(P) = J(P) \sum_{i=1}^n \frac{\partial u^i}{\partial x^i}(p). \tag{L.28}$$

Proof. ${\rm div}(\underline{\underline{\tau}}.\vec{w}) =^{(S.61)} \widetilde{\rm div}(\underline{\underline{\tau}}).\vec{w} + \underline{\underline{\tau}} \,\textcircled{:}\, d\vec{w}$ gives ${\rm div}((JF^{-1}).(\vec{u} \circ \Phi_t^{t_0}))(P) = ({\rm div}(JF^{-T})(P), \vec{u}(p))_g + (J(P)F(P)^{-1}) \,\textcircled{:}\, (d\vec{u}(p).F(P)) =^{(L.22)} 0 + J(P)(F(P).F(P)^{-1}) \,\textcircled{:}\, d\vec{u}(p) = J(P)\, I \,\textcircled{:}\, d\vec{u}(p) = J(P){\rm div}\vec{u}(p)$, which gives (L.28). ∴

M Work and power

M.1 Definitions

M.1.1 Work

(Thermodynamic like approach.) The elementary work is a differential form α, e.g. $\alpha = dU$ (internal energy density), $\alpha = \delta W =$ (elementary work). Consider a regular curve $c : t \in [t_0, T] \to c(t) \in \mathbb{R}^n$. And let $\vec{v}(t, c(t)) := \vec{c}\,'(t)$. The work of α along the curve is

$$\begin{aligned} \int_c \alpha := \int_{t=t_0}^T \alpha(t, c(t)).\vec{c}\,'(t)\, dt &\overset{\text{noted}}{=} \int_{t=t_0}^T \alpha.d\vec{c} \\ = \int_{t=t_0}^T \alpha(t, c(t)).\vec{v}(t, c(t))\, dt &\overset{\text{noted}}{=} \int_{t=t_0}^T \alpha.\vec{v}\, dt. \end{aligned} \tag{M.1}$$

E.g., $W_T^{t_0}(\alpha, c) = \int_c \delta W =$ work along c of the differential form $\alpha = \delta W$.

Then consider an object *Obj* and its motion $\widetilde{\Phi} : (t, P_{Obj}) \to p(t) = \widetilde{\Phi}(t, P_{Obj}) = \widetilde{\Phi}_{P_{Obj}}(t) \in \mathbb{R}^n$, the curves $c_{P_{Obj}} = \widetilde{\Phi}_{P_{Obj}} : t \in [t_0, T] \to p(t) = \widetilde{\Phi}_{P_{Obj}}(t) \in \mathbb{R}^n$, and the Eulerian velocities $\vec{v}(t, p(t)) = {\widetilde{\Phi}_{P_{Obj}}}'(t)$. The work for *Obj* and a Eulerian differential form α along $\widetilde{\Phi}$ is the sum of work of α of all particles, formally $\sum_{P_{Obj} \in Obj} (\int_{c_{P_{Obj}}} \alpha_{P_{Obj}})$. So with the associated motion $\Phi^{t_0}(t, p_{t_0}) = \widetilde{\Phi}(t, P_{Obj}) = p(t) = \Phi_{p_{t_0}}^{t_0}(t)$ when $p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj})$, and with $\Omega_t = \widetilde{\Phi}(t, Obj)$,

$$W_T^{t_0}(\widetilde{\Phi}) := \int_{p_{t_0} \in \Omega_{t_0}} \int_{t=t_0}^T \alpha(t, \Phi_{p_{t_0}}^{t_0}(t)).\vec{v}(t, \Phi_{p_{t_0}}^{t_0}(t))\, dt\, d\Omega_{t_0} = \int_{t=t_0}^T \int_{p_t \in \Omega_t} \alpha(t, p_t)).\vec{v}(t, p_t)\, d\Omega_t\, dt. \tag{M.2}$$

(The last equality if Fubini theorem can be applied, e.g. if α is C^0 and Φ^{t_0} is C^1, *Obj* being bounded.)

Exercice M.1 If α is a stationary and exact differential form, $\alpha = dU$, then prove that

$$\int_c dU = U(c(T)) - U(c(t_0)) \overset{\text{noted}}{=} \Delta U \tag{M.3}$$

only depends on the extremities $c(t_0)$ and $c(T)$ of the curve c.

Answer. $\int_c dU = \int_{t=t_0}^T dU(c(t)).\vec{c}\,'(t)\,dt = \int_{t=t_0}^T \frac{d(U\circ c)}{dt}(t)\,dt = [U\circ c]_{t_0}^T = U(c(T)) - U(c(t_0))$. ∴

Remark (continuum mechanics): An observer chooses a Euclidean dot product $(\cdot,\cdot)_g = .\bullet_g. = .\bullet.$ (if $(\cdot,\cdot)_g$ is imposed and implicit). And if he chooses to represent a linear form $\alpha_t(p_t)$ with its $(\cdot,\cdot)_g$-Riesz representation vector $\vec{f}_t(p_t)$ (observer dependent), cf. (B.8), then

$$\int_c \alpha = \int_{t=t_0}^T \alpha(t,c(t)).\vec{c}\,'(t)\,dt = \int_{t=t_0}^T \vec{f}\bullet d\vec{c} = \int_{t=t_0}^T \vec{f}\bullet\vec{v}\,dt. \tag{M.4}$$

M.1.2 And its associated power density

Definition: The power density of a differential form α along $\widetilde{\Phi}$ is the Eulerian function

$$\psi := \alpha.\vec{v} : \begin{cases} \mathcal{C} = \bigcup\limits_{t\in[t_0,T]} (\{t\}\times\Omega_t) & \to \mathbb{R} \\ \qquad\qquad (t,p) & \to \psi(t,p) = \alpha(t,p).\vec{v}(t,p). \end{cases} \tag{M.5}$$

And the power at t is

$$\mathcal{P}_t(\widetilde{\Phi}) = \mathcal{P}(t,\widetilde{\Phi}) := \int_{p\in\Omega_t} \psi(t,p)\,d\Omega_t = \int_{p\in\Omega_t} \alpha_t(p).\vec{v}_t(p)\,d\Omega_t \overset{\text{noted}}{=} \mathcal{P}(t,\vec{v}_t) = \mathcal{P}_t(\vec{v}_t). \tag{M.6}$$

Remark: With a Euclidean dot product $(\cdot,\cdot)_g$, then with the $(\cdot,\cdot)_g$-Riesz representation vector $\vec{f}$ of α (observer dependent) we get

$$\psi = \vec{f}\bullet\vec{v}, \quad \text{i.e.} \quad \psi(t,p) = \vec{f}(t,p)\bullet_g\vec{v}(t,p) \quad (= (\vec{f}(t,p),\vec{v}(t,p))_g), \tag{M.7}$$

which gives $\mathcal{P}(t,\widetilde{\Phi}) := \int_{p\in\Omega_t} \vec{f}(t,p)\bullet\vec{v}(t,p)\,d\Omega_t$.

M.2 Piola–Kirchhoff tensors

Consider a regular Eulerian velocity field $\vec{v}$, so $d\vec{v}$ is an endomorphism (identified with a $\binom{1}{1}$ tensor). Then we need another endomorphism $\underline{\underline{\tau}}$ (identified with a $\binom{1}{1}$ to get the objective double contraction

$$\underline{\underline{\tau}} \mathbin{0\!\!\!\!:} d\vec{v} := \mathrm{Tr}(\underline{\underline{\tau}}.d\vec{v}), \tag{M.8}$$

which means $\underline{\underline{\tau}}(t,p) \mathbin{0\!\!\!\!:} d\vec{v}(t,p) := \mathrm{Tr}(\underline{\underline{\tau}}(t,p).d\vec{v}(t,p))$.

Quantification: With a basis $(\vec{e}_i)$ at t and $\underline{\underline{\tau}} = \sum_{ij}\tau^i_j\vec{e}_i\otimes e^j$ and $d\vec{v} = \sum_{jk} v^j_{|k}\vec{e}_j\otimes e^k$ (the endomorphisms have been written like $\binom{1}{1}$ tensors for calculation purpose), $\underline{\underline{\tau}}.d\vec{v} = \sum_{ijk}\tau^i_j v^j_{|k}\vec{e}_i\otimes e^k$ and

$$\underline{\underline{\tau}} \mathbin{0\!\!\!\!:} d\vec{v} = \sum_{i,j=1}^n \tau^i_j v^j_{|i} \quad \text{(objective value)}, \tag{M.9}$$

see (Q.32). Then choose a Euclidean dot product $(\cdot,\cdot)_g$, to be able to use the double matrix product

$$\underline{\underline{\tau}} : d\vec{v} := \sum_{i,j=1}^n \tau^i_j v^i_{|j} = [\underline{\underline{\tau}}]_{|\vec{e}}{}^T : [d\vec{v}]_{|\vec{e}} \quad \text{(subjective value)}. \tag{M.10}$$

M.2.1 Objective internal power for the stress: function of $d\vec{v}$

Usual hypothesis for the internal stress in a material: At first order, the power density is of the type

$$\psi = \underline{\underline{\tau}} \mathbin{0\!\!\!\!:} d\vec{v}, \quad \text{i.e.} \quad \psi(t,p) = \underline{\underline{\tau}}(t,p) \mathbin{0\!\!\!\!:} d\vec{v}(t,p), \quad \forall (t,p)\in\mathcal{C}, \tag{M.11}$$

thus the power at t is

$$\mathcal{P}_t(\vec{v}_t) = \int_{p\in\Omega_t} \psi(t,p)\,d\Omega_t = \int_{p\in\Omega_t} \underline{\underline{\tau}}_t(p) \mathbin{0\!\!\!\!:} d\vec{v}_t(p)\,d\Omega_t. \tag{M.12}$$

M.2.2 The first Piola–Kirchhoff tensor

The Piola–Kirchhoff approach consists in transforming Eulerian quantities into Lagrangian quantities to refer to the initial configuration. (M.12) gives

$$\mathcal{P}_t(\vec{v}_t) = \int_{P\in\Omega_{t_0}} \psi_t(\Phi_t^{t_0}(P))\,|J_t^{t_0}(P)|\,d\Omega_{t_0} = \int_{P\in\Omega_{t_0}} \underline{\underline{\tau}}_t(\Phi_t^{t_0}(P)) \odot d\vec{v}_t(\Phi_t^{t_0}(P))\; J_t^{t_0}(P)\,d\Omega_{t_0} \tag{M.13}$$

(the Jacobian $J_t^{t_0}(P)=\det(F_t^{t_0}(P))$ of $\Phi_t^{t_0}$ at P is positive for a regular motion). The Lagrangian velocity $\vec{V}^{t_0}(t,P)=\vec{v}_t(\Phi_t^{t_0}(P))$ satisfies $d\vec{V}_t^{t_0}(P)=d\vec{v}_t(p_t).F_t^{t_0}(P)$ where $p_t=\Phi^{t_0}(t,P)$. Thus

$$\underline{\underline{\tau}}_t(p_t) \odot d\vec{v}_t(p_t) = \underline{\underline{\tau}}(p_t) \odot (d\vec{V}_t^{t_0}(P).F_t^{t_0}(P)^{-1}) = (F_t^{t_0}(P)^{-1}.\underline{\underline{\tau}}(p_t)) \odot d\vec{V}_t^{t_0}(P). \tag{M.14}$$

Quantification: Choose a basis and a Euclidean dot product $(\cdot,\cdot)_g$, thus

$$\mathcal{P}_t(\vec{v}_t) = \int_{P\in\Omega_{t_0}} \underbrace{(J_t^{t_0}(P)\underline{\underline{\tau}}(p_t)^T.F_t^{t_0}(P)^{-T})}_{\mathbb{K}_t^{t_0}(P)} : d\vec{V}_t^{t_0}(P)\,d\Omega_{t_0}. \tag{M.15}$$

Definition M.2 The first Piola–Kirchhoff (two point) tensor at $P\in\Omega_{t_0}$, relative to t_0, t and a basis $(\vec{e}_i)$, is the linear map $\mathbb{K}_t^{t_0}(P)\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ defined by

$$\mathbb{K}_t^{t_0}(P) = J_t^{t_0}(P)\,\underline{\underline{\sigma}}_t(\Phi_t^{t_0}(P)).F_t^{t_0}(P)^{-T}, \quad \text{where} \quad \underline{\underline{\sigma}} = \underline{\underline{\tau}}^T, \tag{M.16}$$

abusively written

$$\boxed{\mathbb{K} = J\,\underline{\underline{\sigma}}.F^{-T}}. \tag{M.17}$$

Hence

$$\mathcal{P}_t(\vec{v}_t) = \int_{\Omega_{t_0}} \mathbb{K}_t^{t_0}(P) : d\vec{V}_t^{t_0}(P)\,d\Omega_{t_0}. \tag{M.18}$$

Remark M.3 The Piola–Kirchhoff tensor is not that easy to master: Everything is quite simple in a Eulerian framework (the configuration at t where the laws are expressed to begin with), but then everything is made more complicated when expressed in an initial configuration (at t_0)... So, when the Piola–Kirchhoff tensor is used to introduce the Lie derivatives (Eulerian type), it makes the Lie derivative quite a mysterious mathematical object, see footnote page 25. ⁂

Remark M.4 Continuation of the remark: With the pull-backs, (M.13) reads ($J_t^{t_0}(P)$ being positive)

$$\mathcal{P}(t,\widetilde{\Phi}) = \int_{p_t\in\Omega_t} \psi_t(p_t)\,d\Omega_t = \int_{P\in\Omega_{t_0}} ((\Phi_t^{t_0})^*\psi_t)(P)\,\big((\Phi_t^{t_0})^*d\Omega_t\big), \tag{M.19}$$

since $((\Phi_t^{t_0})^*d\Omega_t) = J_t^{t_0}(P)\,d\Omega_{t_0}$ and $((\Phi_t^{t_0})^*\psi_t)(P) = \psi_t(p_t)$ (scalar valued functions). It gives the Piola–Kirchhoff tensor (pull-back to the initial configuration) since $(\Phi_t^{t_0})^*(\alpha_t.\vec{v}_t)(p_t) = (\alpha_t.\vec{v}_t)(\Phi_t^{t_0}(P)) = \alpha_t(\Phi_t^{t_0}(P)).\vec{v}_t(\Phi_t^{t_0}(P))$. ⁂

M.2.3 The second Piola–Kirchhoff tensor

The first Piola–Kirchhoff tensor $\mathbb{K}$ may confuse Eulerian and Lagrangian variables, linear maps and endomorphisms... And $\mathbb{K}(p_{t_0})$ is not symmetric: It can't be since $\mathbb{K}(p_{t_0})\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ is not an endomorphism. To get a symmetric tensor, the second Piola–Kirchhoff tensor is defined:

Definition M.5 The second Piola–Kirchhoff tensor is the endomorphism $\mathbb{SK}_t^{t_0}(P)\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ defined by, in short,

$$\mathbb{SK} = F^{-1}.\mathbb{K} = JF^{-1}.\underline{\underline{\sigma}}.F^{-T}. \tag{M.20}$$

Full notation: $\mathbb{SK}_t^{t_0}(P) = (F_t^{t_0}(P))^{-1}.\mathbb{K}_t^{t_0}(P) = J_t^{t_0}(P)(F_t^{t_0}(P))^{-1}.\underline{\underline{\sigma}}_t(p).(F_t^{t_0}(P))^{-T}$.

So if $\underline{\underline{\sigma}}_t(p)\in\mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$ is symmetric then $\mathbb{SK}_t^{t_0}(P)\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ is symmetric.

Thus, with the pull-back of the endomorphism $d\vec{v}_t \in \mathcal{L}(\vec{\mathbb{R}}^n_t, \vec{\mathbb{R}}^n_t)$:

$$((\Phi^{t_0}_t)^* d\vec{v}_t)(P) = F^{t_0}_t(P)^{-1}.d\vec{v}_t(p_t).F^{t_0}_t(P), \tag{M.21}$$

and with $d\vec{v}_t(p_t) = d\vec{V}^{t_0}_t(P).F^{t_0}_t(P)^{-1}$ and $\underline{\underline{\sigma}}_t(p)$ symmetric (so ${S\!\!K}^{t_0}_t$ is symmetric),

$$\begin{aligned}\mathcal{P}_t(\vec{v}_t) &= \int_{\Omega_{t_0}} {P\!\!K}^{t_0}_t : d\vec{V}^{t_0}_t \, d\Omega_{t_0} = \int_{\Omega_{t_0}} (F^{t_0}_t.{S\!\!K}^{t_0}_t) : d\vec{V}^{t_0}_t \, d\Omega_{t_0} = \int_{\Omega_{t_0}} ([F^{t_0}_t].[{S\!\!K}^{t_0}_t]) : [d\vec{V}^{t_0}_t]^T \, d\Omega_{t_0} \\ &= \int_{\Omega_{t_0}} {S\!\!K}^{t_0}_t : ((d\vec{V}^{t_0}_t)^T.F^{t_0}_t) \, d\Omega_{t_0} = \int_{\Omega_{t_0}} {S\!\!K}^{t_0}_t : (\frac{{F^{t_0}_t}^T.d\vec{V}^{t_0}_t + d(\vec{V}^{t_0}_t)^T.F^{t_0}_t}{2}) \, d\Omega_{t_0}.\end{aligned} \tag{M.22}$$

Remark M.6 It is a "chosen time derivative" of ${S\!\!K}(t) = J(t)F(t)^{-1}.\underline{\underline{\sigma}}(t).F(t)^{-T}$ that leads to some kind of Lie derivative as explain in books in continuum mechanics, as in footnote page 25. ⁂

M.3 Classical hyper-elasticity: $\partial W/\partial F$

E and F are finite dimensional spaces, $\dim E = n$, $\dim F = m$.

M.3.1 Definition

Reminder: Consider a function

$$\widehat{W} : \begin{cases} \mathcal{L}(E;F) & \to \mathbb{R} \\ \qquad L & \to \widehat{W}(L) \end{cases} \tag{M.23}$$

Its differential $d\widehat{W} : \left\{ \begin{array}{ll} \mathcal{L}(E;F) & \to \mathcal{L}(\mathcal{L}(E;F);\mathbb{R}) \\ \qquad L & \to d\widehat{W}(L) \end{array} \right\}$ is defined at L by, in a direction M, cf. (S.3),

$$d\widehat{W}(L)(M) = \lim_{h\to 0} \frac{\widehat{W}(L+hM) - \widehat{W}(L)}{h} \overset{\text{noted}}{=} \frac{\partial \widehat{W}}{\partial L}(L)(M). \tag{M.24}$$

Also written $d\widehat{W}(L)(M) = d\widehat{W}(L).M$ since $d\widehat{W}(L)$ is linear.

Example M.7 $\widehat{W} : F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t) \to \widehat{W}(F) \in \mathbb{R}$ (real valued function), with $F := F^{t_0}_t(p_{t_0})$ the deformation gradient at $\in \Omega_{t_0}$ at t at p_{t_0}. Thus $d\widehat{W}(F).M = \lim_{h\to 0} \frac{\widehat{W}(F+hM) - \widehat{W}(F)}{h} =^{\text{noted}} \frac{\partial \widehat{W}}{\partial F}(F).M \in \mathbb{R}$ is the derivative of $\widehat{W}$ at F in a direction $M \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$. ⁂

Example M.8 $m = n$, endomorphisms $L \in \mathcal{L}(\vec{\mathbb{R}}^n; \vec{\mathbb{R}}^n)$, and $\widehat{W}(L) := \mathrm{Tr}(L)$ (the trace). Here $d\mathrm{Tr}(L)(M) = \lim_{h\to 0} \frac{\mathrm{Tr}(L+hM) - \mathrm{Tr}(L)}{h} = \mathrm{Tr}(M)$ (the trace is linear), thus $d\mathrm{Tr}(L) = \mathrm{Tr}$ for all L. ⁂

M.3.2 Expression with bases (quantification): The $\partial W/\partial L_{ij}$

Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be bases in E and F, with (π_{ai}) the (covariant) dual basis of $(\vec{a}_i)$. Let $(\mathcal{L}_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = (\vec{b}_i \otimes \pi_{aj})$ be the associated basis in $\mathcal{L}(E;F)$, i.e. the $\mathcal{L}_{ij}$ are defined by $\mathcal{L}_{ij}.\vec{a}_\ell = \delta_{j\ell}\vec{b}_i$ for all $i = 1, ..., m$ and $j, \ell = 1, ..., n$, cf. prop. A.41 (all the elements of the matrix $[\mathcal{L}_{ij}]_{|\vec{a},\vec{b}}$ vanish except the element at the intersection of row i and column j which equals 1). Let $L \in \mathcal{L}(E;F)$. The derivation of $\widehat{W}$ at L in a direction $\mathcal{L}_{ij}$ is

$$d\widehat{W}(L).\mathcal{L}_{ij} = \lim_{h\to 0} \frac{\widehat{W}(L+h\mathcal{L}_{ij}) - \widehat{W}(L)}{h} \overset{\text{noted}}{=} \frac{\partial \widehat{W}}{\partial L_{ij}}(L) \tag{M.25}$$

(usual notation). Associated matrix relative to the chosen bases:

$$[d\widehat{W}(L)]_{|\mathcal{L}_{ij}} := [\frac{\partial \widehat{W}}{\partial L_{ij}}]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \overset{\text{noted}}{=} [d\widehat{W}(L)]_{|\vec{a},\vec{b}} \quad (\overset{\text{noted}}{=} [d\widehat{W}(L)_{ij}]). \tag{M.26}$$

So if $M = \sum_{i=1}^m \sum_{j=1}^n M_{ij}\mathcal{L}_{ij}$ then $d\widehat{W}(L)(M) = \sum_{ij} M_{ij}\, d\widehat{W}(L)(\mathcal{L}_{ij})$ since $d\widehat{W}(L)$ is linear, so

$$d\widehat{W}(L)(M) = \sum_{i,j=1}^n \frac{\partial \widehat{W}}{\partial L_{ij}}(L) M_{ij} = [d\widehat{W}(L)]_{|\vec{a},\vec{b}} : [M]_{|\vec{a},\vec{b}}, \tag{M.27}$$

double matrix contraction. Duality notations: $d\widehat{W}(L)(M) = \sum_{ij} \frac{\partial \widehat{W}}{\partial L^i_j}(L) M^i_j$.

Remark M.9 The notation $[M]_{|\vec{E},\vec{e}} : [d\widehat{W}(L)]_{|\vec{E},\vec{e}}$ is just a matrix product, since $M = \mathcal{L}(\vec{\mathbb{R}^n};\vec{\mathbb{R}^m})$ and $d\widehat{W}(L) \in \mathcal{L}(\mathcal{L}(\vec{\mathbb{R}^n};\vec{\mathbb{R}^m});\mathbb{R})$ are different kinds of mathematical objects. ∴

Example M.10 Continuing example M.8 with $(\vec{e}_i) = (\vec{E}_i)$: Then $\widehat{W}(L) = \mathrm{Tr}(L)$ gives $d\widehat{W}(L).M = \mathrm{Tr}(M) = \sum_i M_{ii}$, thus $\frac{\partial\widehat{W}}{\partial L_{ij}}(L) = \delta_{ij}$ for all i,j, thus $[d\widehat{W}(L)]_{|\vec{e}} = [I] = [\frac{\partial \mathrm{Tr}}{\partial L_{ij}}(L)]$ (identity matrix), and we recover $d\mathrm{Tr}(L)(M) = [\frac{\partial \mathrm{Tr}}{\partial L_{ij}}(L)] : [M] = [I] : [M] = \sum_{i=1}^n M_{ii} = \mathrm{Tr}(M)$. ∴

Example M.11 Continuing example M.7: The meaning of the derivation $\frac{\partial\widehat{W}}{\partial F_{ij}}$ is intriguing: It is a derivation in the direction $\mathcal{L}_{ij} =^{\text{noted}} \vec{e}_i \otimes \pi_{Ej}$, where $(\vec{e}_i)$ is a basis at $p = \Phi_t^{t_0}(P)$ in $\vec{\mathbb{R}}_t^n$ and (π_{Ej}) is the dual basis of a basis $(\vec{E}_j)$ at P in $\vec{\mathbb{R}}_{t_0}^n$, i.e. $\frac{\partial\widehat{W}}{\partial F_{ij}}(F) = d\widehat{W}(F).\mathcal{L}_{ij} =^{\text{noted}} d\widehat{W}(F).(\vec{e}_i \otimes \pi_{Ej})$ is a derivation "at the same time" in the directions $\vec{e}_i$ (at (t,p)) and π_{Ej} (at (t_0,P)), where F stands for $F_t^{t_0}(P)$. ∴

M.3.3 Motions and ω-lemma

Generalization of (M.23) to C^1 functions, with $\mathcal{U}_{\mathcal{E}}$ open subset in a affine space which associated vector space is E,

$$\widehat{W} : \begin{cases} \mathcal{U}_{\mathcal{E}} \times \mathcal{L}(E;F) & \to \mathbb{R} \\ \qquad (P,L) & \to \widehat{W}(P,L). \end{cases} \tag{M.28}$$

At P, let $\widehat{W}_P(L) := \widehat{W}(P,L)$. The differential $d\widehat{W}_P(L) =^{\text{noted}} \partial_2\widehat{W}(P,L)$ in a direction $M \in \mathcal{L}(E;F)$ is

$$\partial_2\widehat{W}(P,L).M := d(\widehat{W}_P)(L).M = \lim_{h\to 0}\frac{\widehat{W}(P,L+hM) - \widehat{W}(P,L)}{h} \overset{\text{noted}}{=} \frac{\partial\widehat{W}}{\partial L}(P,L).M. \tag{M.29}$$

With a motion $\Phi := \Phi_t^{t_0} : \Omega_{t_0} \to \Omega_t$ define

$$f : \begin{cases} C^1(\overline{\Omega_{t_0}};\overline{\Omega_t}) & \to C^0(\overline{\Omega_{t_0}};\mathbb{R}) \\ \qquad \Phi & \to f(\Phi) := \widehat{W}(.,d\Phi(.)), \end{cases} \tag{M.30}$$

a function of Φ which only depends on its first (covariant) gradient; So, for all $P \in \Omega_{t_0}$,

$$f(\Phi)(P) = \widehat{W}(P, d\Phi(P)) \in \mathbb{R}. \tag{M.31}$$

(This kind of relation is generally deduced after application of the frame invariance principle, and the hypothesis of dependence on only the first order derivative $d\Phi = F$.)

Lemma M.12 (ω-lemma) *For all $\Phi,\Psi \in C^1(\overline{\Omega_{t_0}};\overline{\Omega_t})$,*

$$\boxed{df(\Phi)(\Psi) = \partial_2\widehat{W}(.,d\Phi)(d\Psi) \overset{noted}{=} \frac{\partial\widehat{W}}{\partial F}(.,d\Phi)(d\Psi)}, \tag{M.32}$$

i.e., for all $P \in \Omega_{t_0}$, $df(\Phi)(\Psi)(P) = \frac{\partial\widehat{W}}{\partial F}(P,d\Phi(P))(d\Psi(P))$.

With bases $(\vec{E}_i)$ and $(\vec{e}_i)$ in $\vec{\mathbb{R}}_{t_0}^n$ and $\vec{\mathbb{R}}_t^n$ and $d\Psi.\vec{E}_j = \sum_{i=1}^n \frac{\partial\Psi_i}{\partial X_j}\vec{e}_i$, we get

$$df(\Phi)(\Psi) = \sum_{i,j=1}^n \frac{\partial\widehat{W}}{\partial F_{ij}}(.,d\Phi)\frac{\partial\Psi_i}{\partial X_j}(.) \overset{noted}{=} [\frac{\partial\widehat{W}}{\partial F_{ij}}] : [\frac{\partial\Psi_i}{\partial X_j}]. \tag{M.33}$$

Marsden notations: $df(\Phi)(\Psi) = \sum_{i,J=1}^n \frac{\partial\widehat{W}}{\partial F_J^i}\frac{\partial\Psi^i}{\partial X^J} = [\frac{\partial\widehat{W}}{\partial F_J^i}] : [\frac{\partial\Psi^i}{\partial X^J}]$.

Proof. $C^1(\overline{\Omega_{t_0}};\overline{\Omega_t})$ is a vector space, so $df(\Phi)(\Psi) = \lim_{h\to 0}\frac{f(\Phi+h\Psi)-f(\Phi)}{h} \in C^0(\overline{\Omega_{t_0}};\overline{\Omega_t})$, i.e., for any $P \in \overline{\Omega_{t_0}}$ we have $df(\Phi)(\Psi)(P) = \lim_{h\to 0}\frac{f(\Phi+h\Psi)(P)-f(\Phi)(P)}{h} = \lim_{h\to 0}\frac{\widehat{W}(P,d\Phi(P)+h\,d\Psi(P))-\widehat{W}(P,d\Phi(P))}{h} = d\vec{W}_P(d\Psi(P)$, i.e. (M.32) ∴

M.3.4 Application to classical hyper-elasticity: $\mathbb{K} = \partial W/\partial F$

Let $(\cdot,\cdot)_g$ be a unique Euclidean dot product in $\vec{\mathbb{R}}^n_t$ at all times t, and let $(\vec{E}_i)$ and $(\vec{e}_i)$ be Euclidean bases at t_0 and at t. Let $\underline{\underline{\sigma}}_t(p) =$ the Cauchy stress tensor at t at $p = \Phi^{t_0}_t(P)$.

Let $\mathbb{K}(P) = J(P)\,\underline{\underline{\sigma}}_t(p).F^{-T}(P) =$ the first Piola–Kirchhoff (two point) tensor at P, cf. (M.16). Since $\mathbb{K}$ depends on Φ, the full notation is $\mathbb{K} = \mathbb{K}(\Phi)$ given by

$$\mathbb{K}(\Phi)(P) = J(P)\,\underline{\underline{\sigma}}_t(\Phi(P)).d\Phi(P)^{-T}. \tag{M.34}$$

Definition M.13 If there exists a function $\widehat{\mathbb{K}}$ such that $\mathbb{K}$ reads

$$\mathbb{K}(\Phi)(P) = \widehat{\mathbb{K}}(P, d\Phi(P)) \tag{M.35}$$

then $\widehat{\mathbb{K}}$ is called a constitutive function. (First order hypothesis: $\widehat{\mathbb{K}}$ only depends on $d\Phi = F$ the first order derivative of Φ.)

Definition M.14 The material is hyper-elastic iff there exists a function $\widehat{W} : \left\{\begin{array}{rl} \Omega_{t_0} \times \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t) & \to \mathbb{R} \\ (P,L) & \to \widehat{W}(P,L) \end{array}\right\}$ such that

$$(\mathbb{K}(\Phi) =) \qquad \widehat{\mathbb{K}}(.,d\Phi) = \frac{\partial \widehat{W}}{\partial F}(.,d\Phi), \quad \text{written} \quad \widehat{\mathbb{K}} = \frac{\partial \widehat{W}}{\partial F}, \tag{M.36}$$

that is, $\widehat{\mathbb{K}}(P,F(P)) = \frac{\partial \widehat{W}}{\partial F}(P,F(P))$ for all $P \in \Omega_{t_0}$, where $F = d\Phi$.

With bases $(\vec{E}_I)$ and $(\vec{e}_i)$ in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$, and (E^I) the dual basis of $(\vec{E}_I)$, and $\mathbb{K} = \sum_{i,J=1}^n \mathbb{K}^i{}_J \vec{e}_i \otimes E^J$,

$$[\mathbb{K}(\Phi)]_{|\vec{E},\vec{e}} = [\widehat{\mathbb{K}}(.,F)]_{|\vec{E},\vec{e}} = [\frac{\partial \widehat{W}}{\partial F}(.,F)]_{|\vec{E},\vec{e}}, \quad \text{i.e.} \quad [\mathbb{K}^i{}_J] = [\frac{\partial \widehat{W}}{\partial F^i_J}(.,F)]. \tag{M.37}$$

Thus, for any (virtual) motion $\Psi : \Omega_{t_0} \to \Omega_t$, with (M.32) and (M.27),

$$\widehat{\mathbb{K}}(d\Phi)(d\Psi) = \frac{\partial \widehat{W}}{\partial F}(d\Phi)(d\Psi) \qquad = \sum_{iJ} \frac{\partial \widehat{W}}{\partial F^i_J}(F)\frac{\partial \Psi^i}{\partial X^J} \overset{\text{noted}}{=} [\widehat{\mathbb{K}}] : [d\Psi], \tag{M.38}$$

that is, $\widehat{\mathbb{K}}(d\Phi)(d\Psi)(P) = \sum_{iJ} \frac{\partial \widehat{W}}{\partial F^i_J}(P, F^{t_0}_t(P))\frac{\partial \Psi^i}{\partial X^J}(P)$ for all $P \in \Omega_{t_0}$.

Exercice M.15 With a unique Euclidean dot product $(\cdot,\cdot)_g$ both in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$, with Euclidean bases $(\vec{E}_i) \in \vec{\mathbb{R}}^n_{t_0}$ and $(\vec{e}_i) \in \vec{\mathbb{R}}^n_t$, and with (E^i) the dual basis of $(\vec{E}_i)$, with $C = F^T.F$, prove (derivation in the direction $\vec{e}_i \otimes E^J$):

$$\frac{\partial C}{\partial F^i_J}(F) = \sum_K F^i_K \vec{E}_J \otimes E^K + \sum_K F^i_K \vec{E}_K \otimes E^J \quad (= dC(F).(\vec{e}_i \otimes E^J)). \tag{M.39}$$

$$\frac{\partial \sqrt{C}}{\partial F}(F) = \frac{1}{2}\big(\sqrt{C(F)}\big)^{-1}.\frac{\partial C}{\partial F}(F). \tag{M.40}$$

$$\frac{\partial \sqrt{C}}{\partial C} = \frac{1}{2}(\sqrt{C})^{-1}. \tag{M.41}$$

Answer. Let $F = \sum_{iJ} F^i_J \vec{e}_i \otimes E^J$, so $F^T = \sum_{Ij} (F^T)^I_j \vec{E}_I \otimes e^j = \sum_{Ij} F^j_I \vec{E}_I \otimes e^j$, and $C = \sum_{IJ} C^I_J \vec{E}_i \otimes E^J = F^T.F = \sum_{IJ}\sum_k (F^T)^I_k F^k_J \vec{E}_I \otimes E^J = \sum_{IJ}\sum_k F^k_I F^k_J \vec{E}_I \otimes E^J = C(F)$, so $C^I_J = \sum_k F^k_I F^k_J = C^I_J(F)$. And

$$\begin{aligned} C(F + h\vec{e}_i \otimes E^J) &= (F + h\vec{e}_i \otimes E^J)^T.(F + h\vec{e}_i \otimes E^J) = (F^T + h\vec{E}_J \otimes e^i).(F + h\vec{e}_i \otimes E^J) \\ &= C(F) + h\,(\vec{E}_J \otimes e^i).F + h\,F^T.(\vec{e}_i \otimes E^J) + h^2\,\vec{E}_J \otimes E^i \\ &= C(F) + h\,(\sum_K F^i_K \vec{E}_J \otimes E^K + \sum_K (F^T)^K_i \vec{E}_K \otimes E^J) + h^2\,\vec{E}_J \otimes E^J \end{aligned} \tag{M.42}$$

Thus (M.39). And $C(F + h\vec{e}_i \otimes E^J) - C(F) = (\sqrt{C}(F + h\vec{e}_i \otimes E^J) + \sqrt{C}(F)).(\sqrt{C}(F + h\vec{e}_i \otimes E^J) - \sqrt{C}(F))$ gives $dC(F)(\vec{e}_i \otimes E^J) = 2\sqrt{C}(F).d\sqrt{C}(F)(\vec{e}_i \otimes E^J)$. Thus $\frac{\partial \sqrt{C}}{\partial F^i_J}(F) = \frac{1}{2}(\sqrt{C(F)})^{-1}.\frac{\partial C}{\partial F^i_J}(F)$.

$(C + h\vec{e}_i \otimes e^j) - C = (\sqrt{C + h\vec{e}_i \otimes e^j} + \sqrt{C}).(\sqrt{C + h\vec{e}_i \otimes e^j} - \sqrt{C})$, divided by h, gives $\vec{e}_i \otimes e^j = 2\sqrt{C}.\lim_{h\to 0} \frac{\sqrt{C + h\vec{e}_i \otimes e^j} - \sqrt{C}}{h} = 2\sqrt{C}.d\sqrt{C}.(\vec{e}_i \otimes e^j)$, thus $L = 2\sqrt{C}.(d\sqrt{C}.L)$ for all L (linearity of $d\sqrt{C}$), thus $d\sqrt{C}.L = \frac{1}{2}(\sqrt{C})^{-1}.L$. ∴

M.3.5 Corollary (hyper-elasticity): $SK = \partial W/\partial C$

With the symmetry of the second Piola–Kirchhoff tensor $SK = F^{-1}.PK$, we deduce $SK_t^{t_0}(\Phi_t^{t_0})(P) = \widehat{SK_t^{t_0}}(P, F_t^{t_0}(P))$ (constitutive function). And we deduce the existence of a function $\widetilde{W}$: $\left\{\begin{array}{rl} \Omega_{t_0} \times \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0}) & \to \mathbb{R} \\ (P, L) & \to \widetilde{W}(P, L) \end{array}\right\}$ such that,

$$\widehat{SK_t^{t_0}}(.,C) = \frac{\partial \widetilde{W}}{\partial C}(.,C). \tag{M.43}$$

(See Marsden and Hughes [12] for details and the thermodynamical hypotheses required.)

N Conservation of mass

Let $\rho(t,p) = \rho_t(p)$ be the (Eulerian) mass density at t at $p \in \Omega_t$, supposed to be > 0; The mass $m(\omega_t)$ of a subset $\omega_t \subset \Omega_t$ is

$$m(\omega_t) = \int_{p \in \omega_t} \rho_t(p)\, d\omega_t. \tag{N.1}$$

Conservation of mass principle (no loss nor production of particles): For all $\omega_{t_0} \subset \Omega_{t_0}$ and all t,

$$m(\omega_t) = m(\omega_{t_0}), \quad \text{i.e.} \quad \int_{p \in \omega_t} \rho_t(p)\, d\omega_t = \int_{P \in \omega_{t_0}} \rho_{t_0}(P)\, d\omega_{t_0}. \tag{N.2}$$

Proposition N.1 *If (N.2) then, with $J_t^{t_0}(P) = \det(d\Phi_t^{t_0}(P))$ (positive Jacobian the motion being supposed regular) and $p = \Phi_t^{t_0}(P)$,*

$$\rho_t(p) = \frac{\rho_{t_0}(P)}{J_t^{t_0}(P)}. \tag{N.3}$$

Proof. The change of variable formula gives

$$\int_{p \in \omega_t} \rho_t(p)\, d\omega_t = \int_{P \in \omega_{t_0}} \rho_t(\Phi_t^{t_0}(P))\, J_t^{t_0}(P)\, d\omega_{t_0},$$

thus (N.2) gives $\rho_t(\Phi_t^{t_0}(P)) J_t^{t_0}(P) = \rho_{t_0}(P)$. ∴

Proposition N.2 *$\vec{v} = \vec{v}(t, p_t)$ being the Eulerian velocity at $(t, p_t) \in \mathbb{R} \times \Omega_t$, (N.2) gives*

$$\frac{D\rho}{Dt} + \rho\, \text{div}\vec{v} = 0, \quad \textit{i.e.} \quad \frac{\partial \rho}{\partial t} + \text{div}(\rho\, \vec{v}) = 0. \tag{N.4}$$

Thus, for all $\omega_t \subset \Omega_t$,

$$\int_{\omega_t} \frac{\partial \rho}{\partial t}\, d\omega_t = -\int_{\partial \omega_t} \rho \vec{v}.\vec{n}\, d\sigma_t. \tag{N.5}$$

Proof. (N.2) gives $\frac{d}{dt}(\int_{p(t) \in \omega_t} \rho(t, p(t))\, d\omega_t) = 0$, and Leibniz formula (K.41) applied for all ω_t gives (N.4). Then the Green formula $\int_{\Omega_t} \text{div}(\rho\, \vec{v})\, d\Omega_t = \int_{\partial\Omega_t} \rho \vec{v}.\vec{n}\, d\sigma_t$ gives (N.5). ∴

Exercice N.3 Use (N.3) to prove (N.4).

Answer. $J(t,P)\rho(t, \Phi(t,P)) = \rho_{t_0}(P)$ give, with $p_t = \Phi(t,P)$,

$$\frac{\partial J}{\partial t}(t,P)\, \rho(t, p_t) + J(t,P) \left(\frac{\partial \rho}{\partial t}(t, p_t) + d\rho(t, p_t).d\Phi(t,P) \right) = 0.$$

Thus $\frac{\partial J}{\partial t}(t,P) = J(t,P)\, \text{div}\vec{v}(t,p)$, cf. (K.40), gives (N.4). ∴

O Balance of momentum

O.1 Framework

$\widetilde{\Phi} : [t_0, T] \times Obj \to \mathbb{R}^n$ is a regular motion, $\Omega_t = \widetilde{\Phi}(t, Obj)$, $\Gamma_t = \partial\Omega_t$ (the boundary), $\vec{v}$ is the Eulerian velocity field, ω_t is a regular sub domain in Ω_t and $\partial\omega_t$ is its boundary.

An observer chooses a Euclidean basis $(\vec{e}_i)$ (e.g. made with the foot or the metre) and call $(\cdot,\cdot)_g$ the associated Euclidean dot product. And $\vec{n}(t,p) = \vec{n}_t(p)$ is the outer unit normal at t at $p \in \partial\omega_t$.

All the functions are assumed to be regular enough to validate the following calculations.

Let $\rho : \left\{ \begin{array}{ll} \bigcup\limits_{t\in[t_0,T]} (\{t\} \times \Omega_t) & \to \mathbb{R} \\ (t, p_t) & \to \rho(t, p_t) \end{array} \right\}$ (a mass density), let $\vec{f} : \left\{ \begin{array}{ll} \bigcup\limits_{t\in[t_0,T]} (\{t\} \times \Omega_t) & \to \vec{\mathbb{R}}^n \\ (t, p_t) & \to \vec{f}(t, p_t) \end{array} \right\}$ (a body force density), and let $\vec{T} : \left\{ \begin{array}{ll} \bigcup\limits_{t\in[t_0,T]} (\{t\} \times \partial\omega_t \times \vec{\mathbb{R}}^n_t) & \to \vec{\mathbb{R}}^n \\ (t, p_t, \vec{n}(p_t)) & \to \vec{T}(t, p_t, \vec{n}(p_t)) \end{array} \right\}$ (a surface force density) defined for any regular subset $\omega_t \subset \Omega_t$.

O.2 Master balance law

Definition O.1 The balance of momentum is satisfied by ρ, $\vec{f}$ and $\vec{T}$ iff, for all regular open subset ω_t in Ω_t,

$$\frac{d}{dt}\Big(\int_{\omega_t} \rho\vec{v}\, d\Omega_t\Big) = \int_{\omega_t} \vec{f}\, d\Omega_t + \int_{\partial\omega_t} \vec{T}_{\partial\omega_t}\, d\Gamma_t \quad \text{(master balance law).} \tag{O.1}$$

(It is in fact a linearity hypothesis, see theorem O.2.)

Thus, with (K.41),

$$\int_{\omega_t} \frac{D(\rho\vec{v})}{Dt} + \rho\vec{v}\,\mathrm{div}\vec{v}\, d\Omega_t = \int_{\omega_t} \vec{f}\, d\Omega_t + \int_{\partial\omega_t} \vec{T}_{\partial\omega_t}\, d\Gamma_t. \tag{O.2}$$

And with the conservation of mass hypothesis, cf. (N.4), we get

$$\int_{\omega_t} \rho\frac{D\vec{v}}{Dt}\, d\Omega_t = \int_{\omega_t} \vec{f}\, d\Omega_t + \int_{\partial\omega_t} \vec{T}_{\partial\omega_t}\, d\Gamma_t, \tag{O.3}$$

with $\frac{D\vec{v}}{Dt} = \vec{\gamma} =$ the Eulerian acceleration.

O.3 Cauchy theorem $\vec{T} = \underline{\underline{\sigma}}.\vec{n}$ (stress tensor $\underline{\underline{\sigma}}$)

Theorem O.2 (Cauchy first law: Cauchy stress tensor) *If the master balance law (O.1) is satisfied, then $\vec{T}$ is linear in $\vec{n}$, that is, there exists a Eulerian endomorphism $\underline{\underline{\sigma}}$, identified to a Eulerian tensor $\underline{\underline{\sigma}} \in T^1_1(\Omega_t)$, called the Cauchy stress tensor, s.t. on all $\partial\omega_t$, in short*

$$\vec{T} = \underline{\underline{\sigma}}.\vec{n}, \tag{O.4}$$

where $\vec{n}$ is the unit outward normal to $\partial\omega_t$ (i.e., $\vec{T}(t,p_t) = \underline{\underline{\sigma}}(t,p_t).\vec{n}(t,p_t)$ for all t and $p_t \in \partial\omega_t$).

The proof is based on:

Lemma O.3 *Let* $\varphi : \left\{ \begin{array}{ll} \overline{\Omega} & \to \mathbb{R} \\ p & \to \varphi(p) \end{array} \right\} \in C^1(\overline{\Omega}; \mathbb{R})$ *and* $\psi : \left\{ \begin{array}{ll} \overline{\Omega} \times \vec{\mathbb{R}^3} & \to \mathbb{R} \\ (p, \vec{w}) & \to \psi(p, \vec{w}) \end{array} \right\} \in C^1(\overline{\Omega}, \vec{\mathbb{R}^3}; \mathbb{R})$. *If*

$$\forall \omega \text{ open in } \Omega,\ \int_{p\in\omega} \varphi(p)\, d\Omega = \int_{p\in\partial\omega} \psi(p, \vec{n}(p))\, d\Gamma \tag{O.5}$$

(no dependence on the curvature or on higher derivatives since at any $p \in \partial\omega$, ψ only depends on $\vec{n}(p)$), then

$$\exists \vec{k} \in C^1(\overline{\Omega}; \vec{\mathbb{R}^3}) \text{ s.t. } \psi = (\vec{k}, \vec{n})_g, \text{ and } \varphi = \mathrm{div}\vec{k}, \tag{O.6}$$

i.e. ψ depends linearly on $\vec{n}$, and φ is a divergence.

Proof. (Lemma O.3.) (This proof is standard: We recall it.) Let $p \in \Omega \subset \mathbb{R}^3$. Consider the tetrahedral defined by its vertices p, $p + (h_1, 0, 0)$, $p + (0, h_2, 0)$ and $p + (0, 0, h_3)$, with $h_i > 0$ for all i. (On each face of a tetrahedron, the unit normal vector is uniform.) Let Σ_1 the side which outer unit normal is $-\vec{E}_1$: It area is $\sigma_1 = \frac{1}{2}h_2h_3$ (square triangle). Idem for Σ_2 and Σ_3. Let Σ be the fourth side: its area is $\sigma = \frac{1}{2}\sqrt{h_2^2h_3^2 + h_3^2h_1^2 + h_1^2h_2^2}$ and its outer unit normal is $\vec{n} = \frac{1}{2\sigma}(h_2h_3, h_3h_1, h_1h_2)$ (see exercise O.5), that is $\vec{n} = (n_1, n_2, n_3)$ with $n_i = \frac{\sigma_i}{\sigma}$ pour $i = 1, 2, 3$. The volume of the tetrahedral is $\frac{1}{6}h_1h_2h_3 =^{\text{noted}} \ell^3$. Let $M := \sup_{p\in\overline{\Omega}} |\varphi(p)|$; We have $M < \infty$, since φ is continuous in $\overline{\Omega}$. Then (O.6) give

$$M\ell^3 \geq |\int_{\partial\omega_t} \psi(p, \vec{n}(p))\, d\Gamma|, \quad \text{so} \quad \int_{\partial\omega_t} \psi(p, \vec{n}(p))\, d\Gamma = O(\ell^3). \tag{O.7}$$

And ψ being continuous, the mean value theorem applied on Σ_i gives: There exists $p_i \in \Sigma_i$ s.t.

$$\int_{\Sigma_i} \psi(p, \vec{n}(p))\, d\Gamma = \sigma_i \psi(p_i, \vec{n}_i).$$

Thus

$$\int_{\partial\omega_t} \psi(p, \vec{n}(p))\, d\Gamma = \Big(\sigma_1\psi(p_1, -\vec{E}_1) + \sigma_2\psi(t, p_2, -\vec{E}_2) + \sigma_3\psi(p_3, -\vec{E}_3) + \sigma\psi(p_4, \vec{n})\Big).$$

Then, Ψ being continous, (O.7) gives

$$\sigma_1\psi(p_1, -\vec{E}_1) + \sigma_2\psi(p_2, -\vec{E}_2) + \sigma_3\psi(p_3, -\vec{E}_3) + \sigma\psi(p_4, \vec{n}) = O(\ell^3). \tag{O.8}$$

We flatten the tetrahedron on the yz face by taking $h_2 = h_3 =^{\text{noted}} h$ and $h_1 = h^2$; Thus $\sigma_1 = \frac{1}{2}h^2$, $\sigma_2 = o(h^2)$, $\sigma_3 = o(h^2)$, $\sigma \sim \sigma_1$, $\ell^3 = \frac{1}{6}h^4$, with $\vec{n} \sim -\vec{n}_1 = \vec{E}_1$ and $p_i \sim p$; Then

$$\psi(p, -\vec{E}_1) + \psi(p, +\vec{E}_1) = 0. \tag{O.9}$$

Idem with xz and xy. And for a fixed tetrahedron with h_1, h_2, h_3 given, consider the smaller tetrahedron with $\varepsilon h_1, \varepsilon h_2, \varepsilon h_3$. Then as $\varepsilon \to 0$ (O.8) with (O.9) give

$$\psi(p, \vec{n}) = -\frac{\sigma_i}{\sigma}\psi(p, -\vec{E}_1) - \frac{\sigma_2}{\sigma}\psi(p, -\vec{E}_2) - \frac{\sigma_3}{\sigma}\psi(p, -\vec{E}_3) = \sum_{i=1}^{3} n_i\psi(p, \vec{E}_i),$$

since $n_i = \frac{\sigma_i}{\sigma}$ pour $i = 1, 2, 3$. The same steps can be done for any (inclined) tetrahedron (or apply a change of variable to get back to the above tetrahedron). Thus ψ_p is a linear map in $\vec{n}_p$, that is, there exists a linear form α_p s.t. $\psi_p(\vec{n}_p) = \alpha_p.\vec{n}_p$ for any $p \in \partial\omega$. And the Riesz representation theorem gives: $\exists \vec{k}_p$ s.t. $\alpha_p.\vec{n}_p = (\vec{k}_p, \vec{n}_p)_g =^{\text{noted}} \vec{k}_p \bullet \vec{n}_p$. ∴

Proof. (Theorem.) Apply Lemma O.3 component by component with $\vec{\varphi} = \rho\frac{D\vec{v}}{Dt} - \vec{f} = \sum_{i=1}^n \varphi^i\vec{e}_i$, cf. (O.3). ∴

Corollary O.4 *With* $\text{div}\underline{\underline{\sigma}} := \sum_{i=1}^n(\sum_{j=1}^n \frac{\partial\sigma_{ij}}{\partial x_j})\vec{e}_i$ *(definition of "the matrix divergence" see (S.65)),*

$$\begin{cases} \vec{f} + \text{div}\underline{\underline{\sigma}} = \rho\dfrac{D\vec{v}}{Dt} & \text{in } \Omega_t, \\ \underline{\underline{\sigma}}.\vec{n} = \vec{T} & \text{on } \Gamma_t \end{cases} \tag{O.10}$$

(matrix meaning). (With duality notations, $\text{div}\underline{\underline{\sigma}} := \sum_{i=1}^n(\sum_{j=1}^n \frac{\partial\sigma^i_j}{\partial x^j})\vec{e}_i$*.)*

Proof. Apply the divergence Formula to (O.3). ∴

Exercice O.5 Consider a triangle T in $\mathbb{R}^3$ which vertices are $A = (h_1, 0, 0)$, $B = (0, h_2, 0)$, $C = (0, 0, h_3)$. Prove that $\vec{n} = (h_2h_3, h_3h_1, h_1h_2)$ is orthogonal to T and that $\sigma = \frac{1}{2}\sqrt{h_2^2h_3^2 + h_3^2h_1^2 + h_1^2h_2^2}$ is its area.

Answer. Consider the parametric surface $\vec{r}(t, u) = A + t\vec{AB} + u\vec{AC}$ for $t, u \in [0, 1]$ describing the triangle. Thus $\vec{n} = \frac{\partial\vec{r}}{\partial t} \wedge \frac{\partial\vec{r}}{\partial u} = \vec{AB} \wedge \vec{AC} = \begin{pmatrix} -h_1 \\ h_2 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} -h_1 \\ 0 \\ h_3 \end{pmatrix} = \begin{pmatrix} h_2h_3 \\ h_3h_1 \\ h_1h_2 \end{pmatrix}$ is orthonormal. And $d\sigma = ||\frac{\partial\vec{r}}{\partial t} \wedge \frac{\partial\vec{r}}{\partial u}||dudt = \sqrt{h_2^2h_3^2 + h_3^2h_1^2 + h_1^2h_2^2}dudt$. Thus $\sigma = \int_{t=0}^1\int_{u=0}^1 d\sigma = \sqrt{h_2^2h_3^2 + h_3^2h_1^2 + h_1^2h_2^2}$ is twice the aera of the triangle. ∴

P Balance of moment of momentum

Definition P.1 The balance of moment of momentum is satisfied by ρ, $\vec{f}$ and $\vec{T}$ iff for all regular sub-open set $\omega_t \subset \Omega_t$

$$\frac{d}{dt}\int_{\omega_t} \rho\,\overrightarrow{OM} \wedge \vec{v}\,d\Omega_t = \int_{\omega_t} \rho\,\overrightarrow{OM} \wedge \vec{f}\,d\Omega_t + \int_{\partial\omega_t} \overrightarrow{OM} \wedge \vec{T}\,d\Gamma_t, \tag{P.1}$$

equality called the master balance of moment of momentum law. (This excludes e.g. Cosserat continua materials.)

Theorem P.2 *(Cauchy second law.) If the master balance law (so $\vec{T} = \underline{\underline{\sigma}}.\vec{n}$) and the master balance of moment of momentum law are satisfied then $\underline{\underline{\sigma}}$ is symmetric.*

Proof. (Standard proof.) Let $\vec{x} = \overrightarrow{OM} = \sum_i x_i\vec{E}_i$, and $\vec{T} = \sum_i T_i\vec{E}_i = \underline{\underline{\sigma}}.\vec{n} = \sum_{ij}\sigma_{ij}n_j\vec{E}_i$. Then (first component) $(\vec{x}\wedge\vec{T})_1 = x_2T_3 - x_3T_2 = x_2(\sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3) - x_3(\sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3) = (x_2\sigma_{31} - x_3\sigma_{21})n_1 + (x_2\sigma_{32} - x_3\sigma_{22})n_2 + (x_2\sigma_{33} - x_3\sigma_{23})n_3$. Thus $\int_{\partial\omega_t}(\vec{x}\wedge\vec{T})_1\,d\Gamma_t = \int_{\omega_t}\frac{\partial(x_2\sigma_{31}-x_3\sigma_{21})}{\partial x_1} + \frac{\partial(x_2\sigma_{32}-x_3\sigma_{22})}{\partial x_2} + \frac{\partial(x_2\sigma_{33}-x_3\sigma_{23})}{\partial x_3}\,d\Omega_t = \int_{\omega_t} x_2(\mathrm{div}\underline{\underline{\sigma}})_3 + x_3(\mathrm{div}\underline{\underline{\sigma}})_2 + \sigma_{32} - \sigma_{23}\,d\omega_t$.

(O.10) gives $\rho\frac{D\vec{v}}{Dt} - \vec{f} = \mathrm{div}\underline{\underline{\sigma}}$, thus $\vec{x}\wedge(\rho\vec{\gamma} - \vec{f}) = \vec{x}\wedge\mathrm{div}\underline{\underline{\sigma}}$, so the first component of $\vec{x}\wedge(\rho\vec{\gamma} - \vec{f})$ is $x_2(\mathrm{div}\underline{\underline{\sigma}})_3 - x_3(\mathrm{div}\underline{\underline{\sigma}})_2$, cf. (O.10). Thus (P.1) gives $\int_{\omega_t}\sigma_{32} - \sigma_{23}\,d\omega_t = 0$. True for all ω_t, thus $\sigma_{32} - \sigma_{23} = 0$. Idem for the other components: $\underline{\underline{\sigma}}$ is symmetric. ∎

Q Uniform tensors in $\mathcal{L}^r_s(E)$

Uniform tensors enable to define without ambiguity the "objective contraction rules". Uniform tensors are scalar valued multilinear functions acting on both vectors and linear forms.

NB: In classical mechanics courses, what is called a "tensor" generally not a tensor but a matrix. E.g. you may encounter the expression "Euclidean tensor" which means: The matrix representation of "something" with respect to a Euclidean basis (based on the foot, metre,...) chosen by some observer. (An "Euclidean tensor" is a non-sense, e.g. can you define a "Euclidean vector"?)

Q.1 Tensorial product and multilinear forms

Let $A_1, ..., A_n$ be n finite dimension vector spaces. And $A_i^* = \mathcal{L}(A_i;\mathbb{R})$ the set of linear forms.

Q.1.1 Tensorial product of functions

Let $f_1 : A_1 \to \mathbb{R}$, ..., $f_n : A_n \to \mathbb{R}$ be n functions. Their tensorial product is the function $f_1 \otimes ... \otimes f_n : A_1 \times ... \times A_n \to \mathbb{R}$ defined by (separate variable function)

$$(f_1 \otimes ... \otimes f_n)(\vec{x}_1, ..., \vec{x}_n) = f_1(\vec{x}_1)...f_n(\vec{x}_n). \tag{Q.1}$$

(E.g., $n = 2$ and $A_1 = A_2 = \mathbb{R}$ and $(\cos\otimes\sin)(x, y) = \cos(x)\sin(y)$.)

Q.1.2 Tensorial product of linear forms: multilinear forms

Let $\mathcal{L}(A_1, ..., A_n;\mathbb{R})$ be the set of $\mathbb{R}$-multilinear forms on the Cartesian product $A_1 \times ... \times A_n$, that is, the set of the functions $M : A_1 \times ... \times A_n \to \mathbb{R}$ s.t., for all $i = 1, ..., n$, all $\vec{x}_i, \vec{y}_i \in A_i$ and all $\lambda \in \mathbb{R}$,

$$M(..., \vec{x}_i + \lambda\vec{y}_i, ...) = M(..., \vec{x}_i, ...) + \lambda\,M(..., \vec{y}_i, ...), \tag{Q.2}$$

the other variables being unchanged.

Definition: An elementary tensor is multilinear form $M = \ell_1 \otimes .... \otimes \ell_n$, with $\ell_i \in A_i^*$ for all i; So

$$\forall(\vec{x}_i)_{i\in\mathbb{N}^*} \in \prod_{i=1}^n A_i, \quad (\ell_1 \otimes ... \otimes \ell_n)(\vec{x}_1, ..., \vec{x}_n) = (\ell_1.\vec{x}_1)...(\ell_n.\vec{x}_n) \ \in \mathbb{R}. \tag{Q.3}$$

(The dot in $\ell_i.\vec{x}_i$ is **not** an inner dot product: It is the duality "outer product" $\ell_i.\vec{x}_i := \ell_i(\vec{x}_i)$, cf. (A.45).)

Q.2 Uniform tensors in $\mathcal{L}^0_s(E)$

Let E be a real vector space, with $\dim(E) = n \in \mathbb{N}^*$. In this section we consider the first overlay on E made of multilinear forms M on E, called the uniform tensors of type 0 s or of type $\binom{0}{s}$.

E.g., $M \in \mathcal{L}^0_1(E)$ a linear form, $M \in \mathcal{L}^0_2(E)$ an inner dot product, $M \in \mathcal{L}^0_n(E)$ a determinant...

Notations for quantification purposes: $(\vec{e}_i)$ is a basis in E, (π_{ei}) is its (covariant) dual basis (basis in $E^* = \mathcal{L}(E;\mathbb{R})$), (∂_i) is its bidual basis (basis in $E^{**} = \mathcal{L}(E^*;\mathbb{R})$).

Q.2.1 Definition of type $\binom{0}{s}$ uniform tensors

$\mathcal{L}^0_0(E) := \mathbb{R}$, and if $s \in \mathbb{N}^*$ then

$$\mathcal{L}^0_s(E) := \mathcal{L}(\underbrace{E \times ... \times E}_{s \text{ times}};\mathbb{R}) \tag{Q.4}$$

is called the set of uniform tensors of type $\binom{0}{s}$ on E.

Q.2.2 Example: Type $\binom{0}{1}$ uniform tensor = linear forms

A type $\binom{0}{1}$ uniform tensor is an element of $\mathcal{L}^0_1(E) = \mathcal{L}(E;\mathbb{R}) = E^*$: It is a linear form $\ell \in \mathcal{L}^0_1(E) = E^*$.

Quantification: With $\ell_i := \ell(\vec{e}_i)$ we have, cf. (A.10),

$$\ell = \sum_{i=1}^n \ell_i \pi_{ei}, \quad \text{and} \quad [\ell]_{|\pi_e} = (\,\ell_1 \quad ... \quad \ell_n\,) \overset{\text{noted}}{=} [\ell]_{|\vec{e}} \tag{Q.5}$$

(row matrix for a linear form). Duality notations: (e^i) is the covariant dual basis and $\ell = \sum_{i=1}^n \ell_i e^i$.

Thus, if $\vec{v} \in E$, $\vec{v} = \sum_{i=1}^n v_i \vec{e}_i$, then $\vec{v}$ is represented by $[\vec{v}]_{|\vec{e}} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ (column matrix for a vector), and the matrix calculation rules give

$$\ell(\vec{v}) = [\ell]_{|\vec{e}}.[\vec{v}]_{|\vec{e}} = (\,\ell_1 \quad ... \quad \ell_n\,).\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i=1}^n \ell_i v_i \overset{\text{noted}}{=} \ell.\vec{v}. \tag{Q.6}$$

Duality notations: $\vec{v} = \sum_{i=1}^n v^i \vec{e}_i$ and $\ell(\vec{v}) = \sum_{i=1}^n \ell_i v^i$, and Einstein's convention is satisfied.

Q.2.3 Example: Type $\binom{0}{2}$ uniform tensor

A type $\binom{0}{2}$ uniform tensor is an element of $\mathcal{L}^0_2(E) = \mathcal{L}(E,E;\mathbb{R})$: It is a bilinear form $T \in \mathcal{L}(E,E;\mathbb{R})$.

Quantification: Let $T_{ij} := T(\vec{e}_i, \vec{e}_j)$. Then, with $\vec{v} = \sum_{i=1}^n v_i \vec{e}_i$ and $\vec{w} = \sum_{i=1}^n w_i \vec{e}_i$,

$$T(\vec{v},\vec{w}) = \sum_{i,j=1}^n T_{ij} v_i w_j = [\vec{v}]^T_{|\vec{e}}.[T]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}, \quad \text{i.e.} \quad T = \sum_{i,j=1}^n T_{ij} \pi_{ei} \otimes \pi_{ej}. \tag{Q.7}$$

Duality notations: $T(\vec{v},\vec{w}) = \sum_{i,j=1}^n T_{ij} v^i w^j$, and Einstein's convention is satisfied.

An elementary uniform tensor in $\mathcal{L}^0_2(E)$ is a tensor $T = \ell \otimes m$, where $\ell, m \in E^*$. And so, for all $\vec{v}, \vec{w} \in E$,

$$(\ell \otimes m)(\vec{v},\vec{w}) = (\ell.\vec{v})(m.\vec{w}). \tag{Q.8}$$

Q.2.4 Example: Determinant

The determinant is a alternating $\binom{0}{n}$ uniform tensor, cf. (K.2).

Q.3 Uniform tensors in $\mathcal{L}^r_s(E)$

In this section we consider an over-overlay on E: The multilinear forms acting on both vectors ($\in E$) and functions $\in E^*$ (linear forms).

Q.3.1 Definition of type $\binom{r}{s}$ uniform tensors

Let $r,s\in\mathbb{N}$ s.t. $r+s\geq 1$. The set of multilinear forms

$$\mathcal{L}^r_s(E):=\mathcal{L}(\underbrace{E^*\times\ldots\times E^*}_{r\text{ times}},\underbrace{E\times\ldots\times E}_{s\text{ times}};\mathbb{R}) \tag{Q.9}$$

is called the set of uniform tensors of type $\binom{r}{s}$ on E.

The case $r=0$ has been considered at § Q.2.

When $r\geq 1$, a tensor $T\in\mathcal{L}^r_s(E)$ is a functional: Its domain of definition contains a set of functions (the set $E^*=\mathcal{L}(E;\mathbb{R})$).

Q.3.2 Example: Type $\binom{1}{0}$ uniform tensor: Identified with a vector

A uniform $\binom{1}{0}$ tensor is a element $T\in\mathcal{L}^1_0(E)=\mathcal{L}(E^*;\mathbb{R})=\mathcal{L}(\mathcal{L}(E;\mathbb{R});\mathbb{R})=E^{**}$. With the natural canonical isomorphism

$$\mathcal{J}:\begin{cases}E & \rightarrow E^{**}=\mathcal{L}^1_0(E)\\ \vec{w} & \rightarrow \mathcal{J}(\vec{w})=w,\quad \text{defined by}\quad w(\ell):=\ell(\vec{w}),\quad \forall\ell\in E^*,\end{cases} \tag{Q.10}$$

cf. (T.9) and prop. T.5,

$$w\overset{\text{noted}}{=}\vec{w},\quad\text{so}\quad w.\ell\overset{\text{noted}}{=}\vec{w}.\ell\quad(=\ell.\vec{w}). \tag{Q.11}$$

So a $\binom{1}{0}$ type uniform tensor w is identified (natural canonical) to the vector $\vec{w}=\mathcal{J}^{-1}(w)$.

Interpretation: E^{**} is the set of directional derivatives. Indeed, if $\mathcal{E}$ is an affine space, if E is the associated vector space, if $p\in\mathcal{E}$, and if f is a differentiable function at p, then $w.df(p)=^{(Q.10)}df(p).\vec{w}$ is the directional derivative along $\vec{w}$.

Remark: In differential geometry, $w.df$ is written $\vec{w}(f)$, so $\vec{w}(f)(p):=df(p).\vec{w}$, the definition of a vector being a directional derivative.

Quantification: For all i,j,

$$\partial_i.\pi_{ej}=\delta_{ij}=\pi_{ej}.\vec{e}_i,\quad\text{thus}\quad\partial_i=\mathcal{J}(\vec{e}_i)\overset{\text{noted}}{=}\vec{e}_i. \tag{Q.12}$$

Duality notations: $\partial_i.e^j=\delta^j_i=e^j.\vec{e}_i$. E.g., if f is a C^1 function then $df(p)=\sum_{i=1}^n f_{|i}(p)\,\pi_{ei}$ $(=\sum_{i=1}^n f_{|i}(p)\,e^i)$ and

$$\partial_i(df(p))=df(p).\vec{e}_i=f_{|i}(p)\overset{\text{noted}}{=}\partial_i(f)(p)\overset{\text{noted}}{=}\vec{e}_i(f)(p). \tag{Q.13}$$

Q.3.3 Example: Type $\binom{1}{1}$ uniform tensor

An elementary uniform tensor in $\mathcal{L}^1_1(E)$ is a tensor $T=u\otimes\beta$, where $u\in E^{**}$ and $\beta\in E^*$. And, with $\vec{u}=J^{-1}(u)\in E$, cf. (Q.10), we also write $T=\vec{u}\otimes\beta$. Thus, for all $\ell\in E^*$ and $\vec{w}\in E$

$$(u\otimes\beta)(\ell,\vec{w})=u(\ell)\beta(\vec{w})=\ell(\vec{u})\beta(\vec{w})\overset{\text{noted}}{=}\vec{u}(\ell)\beta(\vec{w})\overset{\text{noted}}{=}(\vec{u}\otimes\beta)(\ell,\vec{w}). \tag{Q.14}$$

Quantification: Let $T(\pi_{ei},\vec{e}_j)$. So

$$T=\sum_{i,j=1}^n T_{ij}\;\vec{e}_i\otimes\pi_{ej},\quad\text{and}\quad[T]_{|\vec{e}}=[T_{ij}], \tag{Q.15}$$

$[T]_{|\vec{e}}=[T_{ij}]$ being the matrix of T relative to the basis $(\vec{e}_i)$. Duality notations: $T(e^i,\vec{e}_j)=T^i{}_j$, $[T]_{|\vec{e}}=[T^i{}_j]$, $T=\sum_{i,j=1}^n T^i{}_j\vec{e}_i\otimes e^j$, and Einstein's convention is satisfied.

Thus with $\ell\in E^*$, $\ell=\sum_{i=1}^n\ell_ie^i\in E^*$, and $\vec{w}\in E$, $\vec{w}=\sum_{i=1}^n w^i\vec{e}_i\in E$, (Q.15) gives

$$T(\ell,\vec{w})=\sum_{i,j=1}^n T_{ij}\vec{e}_i(\ell)\pi_{ej}(\vec{w})=\sum_{i,j=1}^n T_{ij}\ell_iw_j=[\ell]_{|\vec{e}}.[T]_{|\vec{e}}.[\vec{w}]_{|\vec{e}} \tag{Q.16}$$

($[\ell]_{|\vec{e}}$ is a row matrix). Duality notations: $T(\ell,\vec{w})=\sum_{i,j=1}^n T^i{}_j\ell_iw^j$ and Einstein convention is satisfied.

Q.3.4 Example: Type $\binom{1}{2}$ uniform tensor

The same steps are applied to any tensor. E.g., if $T \in \mathcal{L}^1_2(E)$, then with duality notations, ${T^i}_{jk} = T(e^i, \vec{e}_j, \vec{e}_k)$ and

$$T = \sum_{i,j,k=1}^n {T^i}_{jk} \vec{e}_i \otimes e^j \otimes e^k, \quad \text{and} \quad T(\ell, \vec{u}, \vec{w}) = \sum_{i,j,k=1}^n {T^i}_{jk} \ell_i u^j w^k. \tag{Q.17}$$

Q.4 Exterior tensorial products

Let $T_1 \in \mathcal{L}^{r_1}_{s_1}(E)$ and $T_2 \in \mathcal{L}^{r_2}_{s_2}(E)$. Their tensorial product is the tensor $T_1 \otimes T_2 \in \mathcal{L}^{r_1+r_2}_{s_1+s_2}(E)$ defined by

$$(T_1 \otimes T_2)(\ell_{1,1}, ..., \ell_{2,1}, ..., \vec{u}_{1,1}, ..., \vec{u}_{2,1}, ...) := T_1(\ell_{1,1}, ..., \vec{u}_{1,1}, ...) T_2(\ell_{2,1}, ..., \vec{u}_{2,1}, ...). \tag{Q.18}$$

Particular case: with $\lambda \in \mathcal{L}^0_0(E) = \mathbb{R}$ and $T \in \mathcal{L}^r_s(E)$,

$$\lambda \otimes T = T \otimes \lambda := \lambda T \in \mathcal{L}^r_s(E). \tag{Q.19}$$

Example Q.1 let $T_1, T_2 \in \mathcal{L}^1_1(E)$. Quantification: Let $T_1 = \sum_{i,j=1}^n (T_1)^i_j \vec{e}_i \otimes e^j$ and let $T_2 = \sum_{k,m=1}^n (T_2)^k_m \vec{e}_k \otimes e^m$; Then $T_1 \otimes T_2 = \sum_{i,j,k,m=1}^n (T_1)^i_k (T_2)^j_m \vec{e}_i \otimes \vec{e}_j \otimes e^k \otimes e^m \in \mathcal{L}^2_2(E)$. ⁂

Remark Q.2 Alternative definition: $T_1 \widetilde{\otimes} T_2 := \sum_{i,j,k,m=1}^n (T_1)^i_j (T_2)^k_m \vec{e}_i \otimes e^j \otimes \vec{e}_k \otimes e^m \in \mathcal{L}(E^*, E, E^*, E; \mathbb{R})$. And we get back to the previous definition thanks to the natural canonical isomorphism $\widetilde{J} : \mathcal{L}(E^*, E, E^*, E; \mathbb{R}) \to \mathcal{L}(E^*, E^*, E, E; \mathbb{R}) = \mathcal{L}^2_2(E)$ defined by $\widetilde{J}(\widetilde{T}) = T$ where $T(\ell, m, \vec{v}, \vec{w}) = \widetilde{T}(\ell, \vec{v}, m, \vec{w})$. ⁂

Q.5 Contractions

Q.5.1 Contraction of a linear form with a vector

Let $\ell \in \mathcal{L}^0_1(E) = E^*$ and $\vec{w} \in E$. Their contraction is the value

$$\ell(\vec{w}) \overset{\text{linearity}}{=} \ell.\vec{w} \overset{\text{noted}}{=} \vec{w}.\ell. \tag{Q.20}$$

And with a basis $(\vec{e}_i)$ and its dual basis (π_{ei}), $\ell = \sum_{i=1}^n \ell_i \pi_{ei}$ and $\vec{w} = \sum_{i=1}^n w_i \vec{e}_i$ give

$$\ell.\vec{w} = \sum_{i=1}^n \ell_i w_i = [\ell]_{|\vec{e}}.[\vec{w}]_{|\vec{e}} = \sum_{i=1}^n w_i \ell_i = \vec{w}.\ell = \mathrm{Tr}(\vec{w} \otimes \ell), \tag{Q.21}$$

where Tr is the objective trace operator $\mathrm{Tr} : \mathcal{L}(E;E) \simeq \mathcal{L}^1_1(E) \to \mathbb{R}$ (defined by $\mathrm{Tr}(\vec{e}_i \otimes \pi_{ej}) = \delta^i_j$). Duality notations: $\ell.\vec{w} = \sum_{i=1}^n \ell_i w^i$, and Einstein convention is satisfied.

Exercice Q.3 Use the change of coordinate formulas to prove that the computation $\ell.\vec{w}$ in (Q.21) gives a result independent of the basis.

Answer. Let P be the change of basis matrix. So $[\vec{w}]_{new} = P^{-1}.[\vec{w}]_{old}$ and $[\ell]_{new} = [\ell]_{old}.P$, cf. (A.28), thus $[\ell]_{new}.[\vec{w}]_{new} = ([\ell]_{old}.P).(P^{-1}.[\vec{w}]_{old}) = [\ell]_{old}.(P.P^{-1}).[\vec{w}]_{old} = [\ell]_{old}[\vec{w}]_{old}$ $(= \ell.\vec{w})$. ⁂

Q.5.2 Contraction of a $\binom{1}{1}$ tensor and a vector

Let $\ell \in E^*$ and $\vec{u} \in E$. The contraction of the elementary tensor $\vec{w} \otimes \ell \in \mathcal{L}^1_1(E)$ with $\vec{u}$ is defined by:

$$(\vec{w} \otimes \underbrace{\ell).\vec{u}}_{\text{contraction}} = (\ell.\vec{u})\vec{w}. \tag{Q.22}$$

Thus, if $(\vec{e}_i)$ is a basis in E and (π_{ei}) is the dual basis, and $T = \sum_{i,j=1}^n T_{ij} \vec{e}_i \otimes \pi_{ej} \in \mathcal{L}^1_1(E)$ and $\vec{u} = \sum_{j=1}^n u_j \vec{e}_j \in E$, then

$$T = \sum_{i,j=1}^n T_{ij} \vec{e}_i \otimes e^j \quad \Longrightarrow \quad T.\vec{u} = \sum_{i,j=1}^n T_{ij} u^j_j \vec{e}_i \tag{Q.23}$$

because $\pi_{ej}(\vec{u}) = u_j$. Duality notations: $T.\vec{u} = \sum_{i,j=1}^n T^i_j u^j \vec{e}_i$.

Then, with the natural canonical isomorphism ($\mathcal{L}^1_1(E)$ =) $\mathcal{L}(E,E^*;\mathbb{R}) \simeq \mathcal{L}(E;E)$, see (T.7), any endomorphism $L \in \mathcal{L}(E;E)$ defined by $L.\vec{e}_j = \sum_{i=1}^n L_{ij}\vec{e}_i$ can be written, for calculation purpose,

$$\widetilde{L} = \sum_{i,j=1}^n L_{ij}\vec{e}_i \otimes \pi_{ej} \overset{\text{noted}}{=} L, \quad \text{which means} \quad L.\vec{u} \overset{(Q.22)}{=} \sum_{i=1}^n L_{ij}u_j\vec{e}_i \qquad (Q.24)$$

when $\vec{u} = \sum_i u_j\vec{e}_j$, since $\pi_{ej}(\vec{u}) = u_j$. Duality notations: $L = \sum_{i,j=1}^n L^i{}_j\vec{e}_i \otimes e^j$.

Q.5.3 Contractions of uniform tensors

More generally, the contraction of two tensors, if meaningful, is defined thanks to (Q.20): Let $T_1 \in \mathcal{L}^{r_1}_{s_1}(E)$, $T_2 \in \mathcal{L}^{r_2}_{s_2}(E)$, $\ell \in E^*$ and $\vec{u} \in E$.

Definition Q.4 The objective contraction of $T_1 \otimes \ell \in \mathcal{L}^{r_2}_{s_2+1}(E)$ and $\vec{u} \otimes T_2 \in \mathcal{L}^{r_2+1}_{s_2}(E)$ is the tensor $(T_1 \otimes \ell).(\vec{u} \otimes T_2) \in \mathcal{L}^{r_1+r_2}_{s_1+s_2}$ given by

$$(T_1 \otimes \underbrace{\ell).(\vec{u}}_{\text{contraction}} \otimes T_2) := (\ell.\vec{u})\, T_1 \otimes T_2. \qquad (Q.25)$$

In particular $(T_1 \otimes \ell).\vec{u} = (\ell.\vec{u})\, T_1$ (as in (Q.22)), and $\ell.(\vec{u} \otimes T_2) = (\ell.\vec{u})\, T_2$.

And the objective contraction of $T_1 \otimes \vec{u} \in \mathcal{L}^{r_2+1}_{s_2}(E)$ and $\ell \otimes T_2 \in \mathcal{L}^{r_2}_{s_2+1}(E)$ is the tensor $(T_1 \otimes \vec{u}).(\ell \otimes T_2) \in \mathcal{L}^{r_1+r_2}_{s_1+s_2}$ given by

$$(T_1 \otimes \vec{u}).(\ell \otimes T_2) = (\vec{u}.\ell)\, T_1 \otimes T_2 \quad (= (\ell.\vec{u})\, T_1 \otimes T_2). \qquad (Q.26)$$

Quantification with a basis $(\vec{e}_i)$, examples to avoid cumbersome notations:

Example Q.5 Let $T \in \mathcal{L}^1_1(E) = \mathcal{L}^1_{0+1}(E)$, $T = \sum_{i,j=1}^n T^i_j\vec{e}_i \otimes e^j$. With $\vec{w} \in E \sim E^{**} = \mathcal{L}^1_0(E)$, $\vec{w} = \sum_{j=1}^n w^j\vec{e}_j$, (Q.25) gives $T.\vec{w} \in \mathcal{L}^1_0(E) \sim E$ and

$$T.\vec{w} = \sum_{i,j=1}^n T^i_j w^j\vec{e}_i, \quad \text{i.e.} \quad [T.\vec{w}]_{|\vec{e}} = [T]_{|\vec{e}}.[\vec{w}]_{|\vec{e}} \quad \text{(column matrix)}. \qquad (Q.27)$$

(Einstein's convention is satisfied.) Indeed, $T.\vec{w} = \sum_{i,j,k=1}^n T^i_j w^k(\vec{e}_i \otimes e^j).\vec{e}_k = \sum_{i,j,k=1}^n T^i_j w^k\vec{e}_i(e^j.\vec{e}_k) = \sum_{i,j,k=1}^n T^i_j w^k\vec{e}_i(\delta^j_k) = \sum_{i,j=1}^n T^i_j w^j\vec{e}_i$. With $\ell \in E^* = \mathcal{L}^0_1(E)$, $\ell = \sum_{i=1}^n \ell_i e^i$, (Q.25) gives $\ell.T \in \mathcal{L}^0_1(E) = E^*$ and

$$\ell.T = \sum_{i,j=1}^n \ell_i T^i_j e^j, \quad \text{i.e.} \quad [\ell.T]_{|\vec{e}} = [\ell]_{|\vec{e}}.[T]_{|\vec{e}} \quad \text{(row matrix)}. \qquad (Q.28)$$

(Einstein's convention is satisfied.) Indeed $\ell.T = (\sum_{i=1}^n \ell_i e^i).(\sum_{j,k=1}^n T^k_j\vec{e}_k \otimes e^j) = \sum_{i,j,k=1}^n \ell_i T^k_j(e^i.\vec{e}_k)e^j = \sum_{i,j=1}^n \ell_i T^i_j e^j$. ∴

Example Q.6 Let $S,T \in \mathcal{L}^1_1(E)$, $S = \sum_{i,k=1}^n S^i_k\vec{e}_i \otimes e^k$ and $T = \sum_{j,k=1}^n T^k_j\vec{e}_k \otimes e^j$. Then

$$S.T = \sum_{i,j,k=1}^n S^i_k T^k_j\vec{e}_i \otimes e^j, \quad \text{i.e.} \quad [S.T]_{|\vec{e}} = [S]_{|\vec{e}}.[T]_{|\vec{e}} \qquad (Q.29)$$

(Einstein's convention is satisfied.) Indeed $S.T = (\sum_{i,k=1}^n S^i_k\vec{e}_i \otimes e^k).(\sum_{j,m=1}^n T^m_j\vec{e}_m \otimes e^j) = \sum_{i,j,k,m=1}^n S^i_k T^m_j\vec{e}_i(e^k.\vec{e}_m) \otimes e^j = \sum_{i,j,k=1}^n S^i_k T^k_j\vec{e}_i \otimes e^j$. ∴

Example Q.7 Let $T \in \mathcal{L}^1_2(E)$, $T = \sum_{i,j,k=1}^n T^i_{jk}\vec{e}_i \otimes e^j \otimes e^k$, and $\vec{u},\vec{w} \in E \sim \mathcal{L}^1_0(E)$, $\vec{w} = \sum_{i=1}^n w^i\vec{e}_i$ and $\vec{u} = \sum_{i=1}^n u^i\vec{e}_i$. Then

$$T.\vec{w} = \sum_{i,j,k=1}^n T^i_{jk}w^k\vec{e}_i \otimes e^j \in \mathcal{L}^1_1(E), \quad \text{and} \quad (T.\vec{w}).\vec{u} = \sum_{i,j,k=1}^n T^i_{jk}w^k u^j\vec{e}_i \overset{\text{noted}}{=} T(\vec{u},\vec{w}). \qquad (Q.30)$$

(Einstein's convention is satisfied.) So $[T.\vec{w}]_{|\vec{e}} = [\sum_{k=1}^n T^i_{jk}w^k]_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$. And with $\ell \in E^*$, $\ell = \sum_{i=1}^n \ell_i e^i$,

$$((T.\vec{w}).\vec{u}).\ell = \sum_{i,j,k=1}^n T^i_{jk}w^k u^j\ell_i = T(\ell,\vec{u},\vec{w}) = \ell.T(\vec{u},\vec{w}) = \ell.(T.\vec{w}).\vec{u}. \qquad (Q.31)$$

∴

Q.5.4 Objective double contractions of uniform tensors

Definition Q.8 Let $S, T \in \mathcal{L}^1_1(E)$. And let $(\vec{e}_i)$ be a basis in E, (e^i) its dual basis, $S = \sum_{i,j=1}^n S^i_j \vec{e}_i \otimes e^j$ and $T = \sum_{i,j=1}^n T^i_j \vec{e}_i \otimes e^j$. The double objective contraction $S \mathbin{(\!:\!)} T$ of S and T is defined by

$$S \mathbin{(\!:\!)} T = \sum_{i,j=1}^n S^i_j T^j_i. \tag{Q.32}$$

(Einstein convention is satisfied.)

Proposition Q.9 *$S \mathbin{(\!:\!)} T$ defined in (Q.32) is an invariant: It is the trace $\mathrm{Tr}(L_S \circ L_T)$ of the endomorphisms $L_S, L_T \in \mathcal{L}(E;E)$ naturally canonically associated to S and T (given by $\ell.L_S.\vec{u} := S(\ell, \vec{u})$ and $\ell.L_T.\vec{u} := T(\ell, \vec{u})$ for all $(\vec{u}, \ell) \in E \times E^*$). So the real value $\sum_{i,j=1}^n S^i_j T^j_i$ has the same real value regardless of the chosen basis $(\vec{e}_i)$. (Which is not the case of the term to term matrix multiplication $S : T = \sum_{i,j=1}^n S^i_j T^i_j$, see next § Q.5.5 and example Q.13.)*

Proof. Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be two bases and $P = [P^i_j]$ be the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$, i.e., $\vec{b}_j = \sum_{i=1}^n P^i_j \vec{a}_i$ for all j. Let $Q = [Q^i_j] := P^{-1}$. Then $b^i = \sum_{i=1}^n Q^i_j a^i$. Let $S = \sum_{ij} (S_a)^i_j \vec{a}_i \otimes a^j = \sum_{ij} (S_b)^i_j \vec{b}_i \otimes b^j$. So $[(S_b)^i_j] = P^{-1}.[(S_a)^i_j].P$ (change of basis formula for $\binom{1}{1}$ tensors identified with endomorphisms), i.e. $(S_b)^i_j = \sum_{km} Q^i_k (S_a)^k_m P^m_j$ for all i, j. Idem with T. Thus $\sum_{i,j} (S_b)^i_j (T_b)^j_i = \sum_{i,j,k,m,\alpha,\beta} Q^i_k (S_a)^k_m P^m_j Q^j_\alpha (T_a)^\alpha_\beta P^\beta_i = \sum_{i,j,k,m,\alpha,\beta} (S_a)^k_m (T_a)^\alpha_\beta P^\beta_i Q^i_k P^m_j Q^j_\alpha = \sum_{k,m,\alpha,\beta} (S_a)^k_m (T_a)^\alpha_\beta \delta^\beta_k \delta^m_\alpha = \sum_{k,m} (S_a)^k_m (T_a)^m_k$. ∎

Definition Q.10 More generally, the objective double contractions $S \mathbin{(\!:\!)} T$ of uniform tensors, is obtained by applying the objective simple contraction twice consecutively, when applicable.

E.g., $T_1 \otimes \ell_{1,1} \otimes \ell_{1,2}$ and $\vec{u}_{2,1} \otimes \vec{u}_{2,2} \otimes T_2$ give

$$\begin{aligned}(T_1 \otimes \ell_{1,1} \otimes \underbrace{\ell_{1,2}).(\vec{u}_{2,1}}_{\text{first}} \otimes \vec{u}_{2,2} \otimes T_2) &= (\ell_{1,2}.\vec{u}_{2,1})(T_1 \otimes \underbrace{\ell_{1,1}) \otimes (\vec{u}_{2,2}}_{\text{second}} \otimes T_2) \\ &= (\ell_{1,2}.\vec{u}_{2,1})(\ell_{1,1}.\vec{u}_{2,2})\, T_1 \otimes T_2.\end{aligned} \tag{Q.33}$$

Example Q.11 Let $S \in \mathcal{L}^1_2(E)$, $T \in \mathcal{L}^2_1(E)$, $S = \sum_{i,j,k=1}^n S^i_{jk} \vec{e}_i \otimes e^j \otimes e^j$, $T = \sum_{\alpha,\beta,\gamma=1}^n T^{\alpha\beta}_\gamma \vec{e}_\alpha \otimes \vec{e}_\beta \otimes e^\gamma$. Then

$$S.T = \sum_{i,j,k,\beta,\gamma=1}^n S^i_{jk} T^{k\beta}_\gamma \vec{e}_i \otimes e^j \otimes \vec{e}_\beta \otimes e^\gamma, \quad \text{and} \quad S \mathbin{(\!:\!)} T = \sum_{i,j,k,\gamma=1}^n S^i_{jk} T^{kj}_\gamma \vec{e}_i \otimes e^\gamma. \tag{Q.34}$$

(Einstein's convention is satisfied.) ∎

Exercice Q.12 If $S \in \mathcal{L}(E, F; \mathbb{R})$, $T \in \mathcal{L}(F, G; \mathbb{R})$ and $U \in \mathcal{L}(G, E; \mathbb{R})$ then prove

$$S \mathbin{(\!:\!)} (T.U) = (S.T) \mathbin{(\!:\!)} U = (U.S) \mathbin{(\!:\!)} T \quad \text{(circular permutation).} \tag{Q.35}$$

Answer. If $S = \sum S^i_j \vec{a}_i \otimes b^j$, $T = \sum T^i_j \vec{b}_i \otimes c^j$ and $U = \sum U^i_j \vec{c}_i \otimes a^j$, then $T.U = \sum T^i_k U^k_j \vec{b}_i \otimes a^j$, thus $S \mathbin{(\!:\!)} (T.U) = \sum S^i_m T^m_k U^k_i$, and $S.T = \sum S^i_k T^k_j \vec{a}_i \otimes c^j$, so $(S.T) \mathbin{(\!:\!)} U = \sum S^i_k T^k_m U^m_i$. And the second equality thanks to the symmetry of $\mathbin{(\!:\!)}$, i.e. $(S.T) \mathbin{(\!:\!)} U = U \mathbin{(\!:\!)} (S.T) = (U.S) \mathbin{(\!:\!)} T$ with the previous calculation. ∎

We define in the same way the triple objective contraction (apply the simple contraction three times consecutively). E.g., with (Q.34) we get

$$S \mathbin{(\!\vdots\!)} T = \sum_{i,j,k=1}^n {S^i}_{jk} {T^{kj}}_i. \tag{Q.36}$$

(Einstein's convention is satisfied.)

Q.5.5 Non objective double contraction: Double matrix contraction

The engineers often use the double matrix contraction of second order tensors defined by (term to term multiplication): If $S = [S_{ij}] = [S^i_j]$ and $T = [T_{ij}] = [T^i_j]$ then

$$S : T := \sum_{i,j=1}^{n} S_{ij} T_{ij} = \sum_{i,j=1}^{n} S^i_j T^i_j \overset{\text{noted}}{=} \mathrm{Tr}(S.T^T). \tag{Q.37}$$

Einstein's convention is ***not*** satisfied, and the result is observer dependent for associated endomorphism:

Example Q.13 Let $(\vec{e}_i)$ be a basis, let $S \in \mathcal{L}(E;E)$ given by $[S]_{\vec{e}} = \begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}$ (so $S.\vec{e}_1 = 2\vec{e}_2$ and $S.\vec{e}_2 = 4\vec{e}_1$). Then the double matrix contraction (Q.37) gives

$$S : S = [S]_{\vec{e}} : [S]_{\vec{e}} = 4 * 4 + 2 * 2 = 20. \tag{Q.38}$$

Change of basis: let $\vec{b}_1 = \vec{e}_1$ and $\vec{b}_2 = 2\vec{e}_2$. The transition matrix from $(\vec{e}_i)$ to $(\vec{b}_i)$ is $P = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Thus $[S]_{\vec{b}} = P^{-1}.[S]_{\vec{e}}.P = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} . \begin{pmatrix} 0 & 8 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 8 \\ 1 & 0 \end{pmatrix}$. Thus

$$S : S = [S]_{\vec{b}} : [S]_{\vec{b}} = 8 * 8 + 1 * 1 = 65 \neq 20. \tag{Q.39}$$

To be compared with the double objective contraction: $[S]_{\vec{e}} \mathbin{0\mkern-10mu:} [S]_{\vec{e}} = 4{*}2{+}2{*}4 = 16 = [S]_{\vec{b}} \mathbin{0\mkern-10mu:} [S]_{\vec{b}} = S \mathbin{0\mkern-10mu:} S$ (observer independent result = objective result).

So it is absurd to use $S : S$ (double matrix contraction) if you need objectivity: Recall that the foot is the international vertical unit in aviation, and thus the use of the double **objective** contraction is vital, while the use of the double matrix contraction can be fatal (really). Also see the Mars climate orbiter probe crash. ⁂

Exercice Q.14 Let $S \in \mathcal{L}^0_2(E)$ (e.g. a metric), let $(\vec{a}_i)$ be a Euclidean basis in foot, and let $(\vec{b}_i) = (\lambda \vec{a}_i)$ be the related euclidean basis in metre (change of unit). Give $[S]_{|\vec{a}} : [S]_{|\vec{a}}$ and $[S]_{|\vec{b}} : [S]_{|\vec{b}}$ and compare. (The simple and double objective contractions are impossible here since S and T are not compatible.)

Answer. Let $S = \sum_{i,j=1}^n S_{a,ij} a^i \otimes a^j = \sum_{i,j=1}^n S_{b,ij} b^i \otimes b^j$. Since $(\vec{b}_i) = (\lambda \vec{a}_i)$ we have $b^i = \frac{1}{\lambda} a^i$. Thus $\sum_{i,j=1}^n S_{a,ij} a^i \otimes a^j = \sum_{i,j=1}^n S_{a,ij} \lambda^2 b^i \otimes b^j$, thus $\lambda^2 S_{a,ij} = S_{b,ij}$. Thus

$$[S]_{|\vec{b}} : [S]_{|\vec{b}} = \sum_{i,j=1}^{n} (S_{b,ij})^2 = \lambda^4 \sum_{i,j=1}^{n} (S_{a,ij})^2 = \lambda^4 [S]_{|\vec{a}} : [S]_{|\vec{a}}, \tag{Q.40}$$

with $\lambda^4 \geq 100$: Quite a difference isn't it? ⁂

Q.6 Kronecker (contraction) tensor, trace

Definition Q.15 The Kronecker tensor is the $\binom{1}{1}$ uniform tensor $\underline{\underline{\delta}} \in \mathcal{L}^1_1(E)$ defined by

$$\forall (\ell, \vec{u}) \in E^* \times E, \quad \underline{\underline{\delta}}(\ell, \vec{u}) := \ell.\vec{u}. \tag{Q.41}$$

And the Kronecker symbols relative to a basis $(\vec{e}_i)$ are the reals defined by, calling (π_{ei}) the dual basis,

$$\delta_{ij} := \delta(\pi_{ei}, \vec{e}_j) = \left\{ \begin{matrix} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j, \end{matrix} \right\} \quad \text{i.e.} \quad \underline{\underline{\delta}} := \sum_{i=1}^{n} \pi_{ei} \otimes e^i, \quad [\underline{\underline{\delta}}] = [\delta_j] = [I] \tag{Q.42}$$

(identity matrix whatever the basis). Duality notations: $\delta^i_j := \delta(e^i, \vec{e}_j)$, $\underline{\underline{\delta}} := \sum_{i=1}^n \vec{e}_i \otimes e^i$ and $[\underline{\underline{\delta}}] = [\delta^i_j]$.

Definition Q.16 The trace of a $\binom{1}{1}$ uniform tensor $T \in \mathcal{L}^1_1(E)$ is

$$\widetilde{\mathrm{Tr}}(T) = \underline{\underline{\delta}} \mathbin{0\mkern-10mu:} T \quad (= \mathrm{Tr}(L_T)) \tag{Q.43}$$

(with the natural canonical isomorphism $T \in \mathcal{L}^1_1(E) \simeq L_T \in \mathcal{L}(E;E)$ given by $T(\ell, \vec{v}) := \ell.L_T.\vec{v}$).

Thus $\widetilde{\mathrm{Tr}}(T) = \sum_{i=1}^n T^i{}_i$.

In particular $\widetilde{\mathrm{Tr}}(\underline{\underline{\delta}}) = n$, and $\widetilde{\mathrm{Tr}}(\vec{v} \otimes \ell) = \sum_i v^i \ell_i = \ell.\vec{v}$ when $\vec{v} = \sum_i v^i \vec{e}_i$ and $\ell = \sum_j \ell_j e^j$.

R Tensors in $T^r_s(U)$

R.1 Introduction, module, derivation

Let A and B be any sets, and let $\mathcal{F}(A;B)$ be the set of functions $A \to B$. The "plus" inner operation and the "dot" outer operation are defined by, for all $f,g \in \mathcal{F}(A;B)$, all $\lambda \in \mathbb{R}$ and all $p \in A$,

$$\begin{cases} (f+g)(p) := f(p) + g(p), \quad \text{and} \\ (\lambda.f)(p) := \lambda\, f(p), \quad \lambda.f \overset{\text{noted}}{=} \lambda f. \end{cases} \tag{R.1}$$

$(\mathcal{F}(A;B),+,.,\mathbb{R})$ is thus a vector space on the field $\mathbb{R}$ (see any elementary course) called $\mathcal{F}(A;B)$.

But the field $\mathbb{R}$ is "too small" to define a tensor which can be seen as "a linear tool that satisfies the change of coordinate system rules":

Example R.1 Fundamental counter-example: Derivation. Let U be an open set in $\mathbb{R}^n$. The derivation $d : \vec{w} \in C^1(U;\vec{\mathbb{R}}^n) \to d\vec{w} \in C^0(U;\mathcal{L}(\vec{\mathbb{R}}^n;\vec{\mathbb{R}}^n))$ is $\mathbb{R}$-linear: In particular $d(\lambda\vec{w}) = \lambda(d\vec{w})$ for all $\lambda \in \mathbb{R}$...

...but d doesn't satisfy the change of coordinate system rules, see (S.35).

So a derivation it ***not*** a tensor (it is a "spray", see Abraham–Marsden [1]).

In fact, one requirement for T to be a tensor is, e.g. with $T = \vec{w}$ a vector field: For all $\varphi \in C^\infty(U;\mathbb{R})$, and all $\vec{w} \in \Gamma(U)$ (C^∞-vector field),

$$T(\varphi\vec{w}) = \varphi\, T(\vec{w}). \tag{R.2}$$

While

$$d(\varphi\vec{w}) \neq \varphi\, d(\vec{w}), \quad \text{because} \quad d(\varphi\vec{w}) = \varphi\, d\vec{w} + d\varphi.\vec{w}. \tag{R.3}$$

Thus the elementary $\mathbb{R}$-linearity requirement "$T.(\lambda\vec{w}) = \lambda(T.\vec{w})$ for all $\lambda \in \mathbb{R}$ is not sufficient to characterize a tensor: The $\mathbb{R}$-linearity has to be replaced by the $C^\infty(U;\mathbb{R})$-linearity, cf. (R.2).

Thus we will have to replace a real vector space $(V,+,.,\mathbb{R})$ over the field $\mathbb{R}$ with the "module" $(V,+,.,C^\infty(U;\mathbb{R}))$ over the ring $C^\infty(U;\mathbb{R})$, which mainly amounts to consider (R.1) for all $\lambda = \varphi \in C^\infty(U;\mathbb{R})$. Remark: The use of a module is very similar to the use of a vector space, but for the use of the inverse: all real $\lambda \neq 0$ has a multiplicative inverse in $\mathbb{R}$ (namely $\frac{1}{\lambda}$), but a function $f \in C^\infty(U;\mathbb{R})$ s.t. "$f \neq 0$ and f vanishes at one point" doesn't have a multiplicative inverse in $C^\infty(U;\mathbb{R})$. ⁂

R.2 Field of functions and vector fields

Framework of classical mechanics: U is an open set in an affine space $\mathcal{E}$ which associated vector is E. And the definition of tensors is done at a fixed time t (concerns the space variables). As before, the approach is first qualitative, then quantitative with a basis $(\vec{e}_i(p))$ and its dual basis $(\pi_{ei}(p)) = (e^i(p))$, at any $p \in \mathcal{E}$.

R.2.1 Field of functions

Let $f \in C^\infty(U;\mathbb{R})$ be a function. The associated function field is

$$\widetilde{f} : \begin{cases} U & \to U \times \mathbb{R} \\ p & \to \widetilde{f}(p) := (p; f(p)), \end{cases} \tag{R.4}$$

and p is called the base point. So $\mathrm{Im}\widetilde{f} = \{(p; f(p)) : p \in U\}$ is the graph of f. Definition:

$$T^0_0(U) := \{\widetilde{f} : f \in C^\infty(U;\mathbb{R})\} = \{\text{field of functions}\} = \text{the set of } \tbinom{0}{0} \text{ type tensor on } U, \tag{R.5}$$

or the set of tensors of order 0 on U. Abusive short notations (to lighten the writings):

$$\widetilde{f}(p) \overset{\text{noted}}{=} f(p), \quad \text{and} \quad T^0_0(U) \overset{\text{noted}}{=} C^\infty(U;\mathbb{R}), \tag{R.6}$$

but keep the base point in mind (no ubiquity gift).

In $T^0_0(U)$, the internal sum is defined by, for all $\widetilde{f},\widetilde{g} \in T^0_0(U)$ with $\widetilde{f}(p) = (p; f(p))$ and $\widetilde{g}(p) = (p; g(p))$,

$$(\widetilde{f} + \widetilde{g})(p) := (p; (f+g)(p)) \quad (= (p; f(p) + g(p))), \tag{R.7}$$

and the external multiplication on the ring $C^\infty(U;\mathbb{R})$ is defined by, for all $\varphi \in C^\infty(U;\mathbb{R})$,

$$(\varphi\widetilde{f})(p) := (p; (\varphi f)(p)) \quad (= (p; \varphi(p) f(p))) \tag{R.8}$$

(the base point p remains unchanged). Thus $(T^0_0(U),+,.)$ is a module over the ring $C^\infty(U;\mathbb{R})$.

R.2.2 Vector fields

Let $\vec{w} \in C^\infty(U,E)$ be a vector valued function (at least Lipschitzian, to get integral curves, cf. Cauchy–Lipschitz theorem). The associated vector field is

$$\widetilde{\vec{w}} : \begin{cases} U & \to U \times E \\ p & \to \widetilde{\vec{w}}(p) = (p; \vec{w}(p)). \end{cases} \tag{R.9}$$

So $\mathrm{Im}\widetilde{\vec{w}} = \{(p;\vec{w}(p)) : p \in U\}$ is the graph of $\vec{w}$, and the definition of $\widetilde{\vec{w}}$ tells that the vector $\vec{w}(p)$ has to be drawn at p (the base point). Abusive short notation:

$$\widetilde{\vec{w}}(p) \overset{\text{noted}}{=} \vec{w}(p) \quad \text{instead of } \widetilde{\vec{w}}(p) = (p; \vec{w}(p)). \tag{R.10}$$

It lightens the notations, but keep the base point in mind. Let

$$\Gamma(U) := \text{ the set of vector fields on } U. \tag{R.11}$$

More precisely, we will use the following full definition of vector fields (see e.g. Abraham–Marsden [1]): A vector field is built from tangent vectors to curves. It makes sense on non planar surfaces, and more generally on differential manifolds.

R.3 Differential forms

The basic concept is that of vector fields. A first over-layer is made of differential forms (which "measure vector fields"):

Definition R.2 Let $\alpha \left\{ \begin{matrix} U & \to E^* \\ p & \to \alpha(p) \end{matrix} \right\}$ (so $\alpha(p)$ is a linear form at p). The associated differential form (also called a 1-form) is "the field of linear forms" defined by

$$\widetilde{\alpha} : \begin{cases} U & \to U \times E^* \\ p & \to \widetilde{\alpha}(p) = (p; \alpha(p)) \quad (= \text{"a pointed linear form at } p\text{"}). \end{cases} \tag{R.12}$$

And p is called the base point, and $\mathrm{Im}\widetilde{\alpha} = \{(p;\alpha(p)) : p \in U\}$ is the graph of α.

Thus, if $\widetilde{\alpha} \in \Omega^1(U)$ (differential form) and $\widetilde{\vec{w}} \in \Gamma(U)$ (vector field), then $\widetilde{\alpha}.\widetilde{\vec{w}} \in T_0^0(U)$ (field of scalar valued functions) satisfies

$$\widetilde{\alpha}.\widetilde{\vec{w}} : \begin{cases} U & \to U \times \mathbb{R} \\ p & \to (\widetilde{\alpha}.\widetilde{\vec{w}})(p) = (p; (\alpha.\vec{w})(p)) = (p; \alpha(p).\vec{w}(p)) \ \in U \times \mathbb{R}. \end{cases} \tag{R.13}$$

Short notation:

$$\widetilde{\alpha}(p) \overset{\text{noted}}{=} \alpha(p), \quad \text{instead of } \widetilde{\alpha}(p) = (p; \alpha(p)), \tag{R.14}$$

but keep the base point in mind. And

$$\Omega^1(U) := \text{ the set of differential forms } U. \tag{R.15}$$

R.4 Tensors

A second over-layer is introduced with the tensors with are "functions defined on vector fields and on differential forms" (which "measure vector fields and differential forms").

Let $r,s \in \mathbb{N}$, $r{+}s \geq 1$, and let $T : \left\{ \begin{matrix} U & \to \mathcal{L}^r_s(E) \\ p & \to T(p) \end{matrix} \right\}$ (so $T(p)$ is a uniform $\binom{r}{s}$ tensor for each p, cf. (Q.3.1)). And consider the associated function

$$\widetilde{T} : \begin{cases} U & \to U \times \mathcal{L}^r_s(E) \\ p & \to \widetilde{T}(p) = (p; T(p)) \end{cases} \tag{R.16}$$

Abusive short notation:

$$\widetilde{T}(p) \overset{\text{noted}}{=} T(p) \quad \text{instead of } \widetilde{T}(p) = (p; T(p)), \tag{R.17}$$

but keep the base point in mind.

Definition R.3 (Abraham–Marsden [1].) $\widetilde{T}$ is a tensor of type $\binom{r}{s}$ iff T is $C^\infty(U;\mathbb{R})$-multilinear (not only $\mathbb{R}$-multilinear), i.e., for all $f \in C^\infty(U;\mathbb{R})$, all z_1, z_2 vector field or differentiable form where applicable, and all $p \in U$,

$$\begin{cases} T(p)(...,z_1(p)+z_2(p),...) = T(p)(...,z_1(p),...)+T(p)(...,z_2(p),...), \quad \text{and} \\ T(p)(...,f(p)z_1(p),...) = f(p)\,T(p)(...,z_1(p),...), \end{cases} \tag{R.18}$$

written in short

$$\begin{cases} T(...,z_1+z_2,...) = T(...,z_1,...)+T(...,z_2,...), \quad \text{and} \\ T(...,fz_1,...) = f\,T(...,z_1,...). \end{cases} \tag{R.19}$$

And

$$T_s^r(U) := \text{ the set of } \tbinom{r}{s} \text{ type tensors on } U. \tag{R.20}$$

(Recall: $T_0^0(U) := C^\infty(U;\mathbb{R})$ the set of function fields, cf. (R.4).)

Remark R.4 Definition in differential geometry lessons: A tensor is a section of a certain bundle over a manifold. For classical mechanics, definition R.3 gives an equivalent definition. ⁂

R.5 First Examples

R.5.1 Type $\binom{0}{1}$ tensor = differential forms

If $T \in T_1^0(U)$ then $T(p) \in E^*$, so $T = \alpha \in \Omega^1(U)$ is a differential form: $T_1^0(U) \subset \Omega^1(U)$.

Converse: Does a differential form $\alpha \in \Omega^1(U)$ defines a $\binom{0}{1}$ type tensor on U? Yes: We have to check (R.18), which is trivial. So $\alpha \in T_1^0(U)$, so $\Omega^1(U) \subset T_1^0(U)$.

Thus

$$T_1^0(U) = \Omega^1(U). \tag{R.21}$$

R.5.2 Type $\binom{1}{0}$ tensor (identified to a vector field)

Let $T \in T_1^0(U)$, so $T(p) \in \mathcal{L}_0^1(E) = \mathcal{L}(E^*;\mathbb{R}) = E^{**}$ for all $p \in U$. Thus, thanks to the natural canonical isomorphism $E^{**} \simeq E$, $T(p)$ can be identified to a vector, thus $T_1^0(U) \subset \Gamma(U)$.

Converse: Does a vector field $\vec{w} \in \Gamma(U)$ defines a $\binom{1}{0}$ type tensor on U? Yes: We have to check (R.18), which is trivial. So $\Gamma(U) \subset T_0^1(U)$.

Thus

$$T_0^1(U) \simeq \Gamma(U). \tag{R.22}$$

R.5.3 A metric is a $\binom{0}{2}$ tensor

Let $T \in T_2^0(U)$, so $T(p) \in \mathcal{L}_2^0(E)$ for all $p \in U$, and $T(\vec{u},\vec{w}) \in T_0^0(U)$ for all $\vec{u},\vec{w} \in \Gamma(U)$.

Definition R.5 A metric g on U is a $\binom{0}{2}$ type tensor on U such that, for all $p \in E$, $g(p) =^{\text{noted}} g_p$ is an inner dot product on E.

R.6 $\binom{1}{1}$ tensor, identification with fields of endomorphisms

Let $T \in T_1^1(U)$, so $T(p) \in \mathcal{L}_1^1(E)$ for all $p \in U$, and $T(\alpha,\vec{w}) \in T_0^0(U)$ for all $\alpha \in \Omega^1(U)$ and $\vec{w} \in \Gamma(U)$ (so $T(p)(\alpha(p),\vec{w}(p)) \in \mathbb{R}$ for all p).

The associated field of endomorphisms on U is $\widetilde{L}_T : \left\{ \begin{array}{ll} U & \to U \times \mathcal{L}(E;E) \\ p & \to \widetilde{L}_T(p) = (p, L_T(p)) \end{array} \right\}$ where $L_T(p)$ is identified with $T(p)$ thanks to the natural canonical isomorphism $\mathcal{L}(E;E) \simeq \mathcal{L}(E^*,E;\mathbb{R}) = \mathcal{L}_1^1(E)$ given by

$$\forall \ell \in E^*,\ \forall \vec{w} \in E, \quad \ell.(L_T(p).\vec{w}) = T(p)(\ell,\vec{w}). \tag{R.23}$$

R.7 Unstationary tensor

Let $t \in [t_1,t_2] \subset \mathbb{R}$. Let $(T_t)_{t\in[t_1,t_2]}$ be a family of $\binom{r}{s}$ tensors, cf. (R.16). Then $T : t \to T(t) := T_t$ is called an unstationary tensor. And the set of unstationary tensors is also noted $T_s^r(U)$. E.g., a Eulerian velocity field is a $\binom{1}{0}$ unstationary vector field.

S Differential, its eventual gradients, divergences

S.1 Differential

The definition of the differential of a function is observer independent: All observers have the same definition (qualitative: no man made tool required, like a basis or an inner dot product).

S.1.1 Framework

Classical Framework: $\mathcal{E}$ are $\mathcal{F}$ affine spaces associated with vector spaces E and F, and $||.||_E$ and $||.||_F$ are norms in E and F such that $(E,||.||_E)$ and $(F,||.||_F)$ are complete (we need "limit that stay in the space as $h\to 0$",). U is an open set in $\mathcal{E}$, and $\Phi : \left\{ \begin{aligned} U &\to \mathcal{F} \\ p &\to p_{\mathcal{F}} = \Phi(p) \end{aligned} \right\}$ is a function. If applicable, $\mathcal{E}$ and/or $\mathcal{F}$ can be replaced by E and/or F. (The definitions can be generalized to manifolds.) Reminder:

Definition S.1 Let $p\in U$. The function Φ is said to be continuous at p iff $\Phi(q) \underset{q\to p}{\longrightarrow} \Phi(p)$ relative to the considered norms, i.e., $||\Phi(q)-\Phi(p)||_F \longrightarrow_{||q-p||_E\to 0} 0$, also written (Landau notation): Near p,

$$\Phi(q) = \Phi(p) + o(1), \tag{S.1}$$

called "the zero-th order Taylor expansion of Φ near p". In other words:

$\forall\varepsilon>0$, $\exists\eta>0$ s.t. $\forall q\in\mathcal{E}$ s.t. $||q-p||_E<\eta$ we have $||\Phi(q)-\Phi(p)||_F<\varepsilon$.
And $C^0(U;\mathcal{F})$ is the set of functions that are continuous at all $p\in U$.

S.1.2 Directional derivative and differential (observer independent)

Let $p\in U$, $\vec{u}\in E$, and let $f:\mathbb{R}\to\mathcal{F}$ defined by

$$f(h) := \Phi(p+h\vec{u}) \tag{S.2}$$

Definition S.2 The function Φ is differentiable at p in the direction $\vec{u}$ iff f is derivable at 0, i.e. iff the limit $f'(0)=\lim_{h\to 0}\frac{\Phi(p+h\vec{u})-\Phi(p)}{h} =^{\text{noted}} d\Phi(p)(\vec{u})$ exists in F, i.e. iff, near p,

$$\Phi(p+h\vec{u}) = \Phi(p) + h\,d\Phi(p)(\vec{u}) + o(h), \tag{S.3}$$

equation called the first order Taylor expansion of Φ at p in the direction $\vec{u}$ (it is the first order Taylor expansion of f near p).

Then $d\Phi(p)(\vec{u})$ is called the directional derivative of Φ at p in the direction $\vec{u}$.

And if, for all $\vec{u}\in E$, $d\Phi(p)(\vec{u})$ exists (in F) then Φ is called Gâteaux differentiable at p.

Exercice S.3 Prove: If Φ is Gâteaux differentiable at p then $d\Phi(p)$ is homogeneous, i.e., $d\Phi(p)(\lambda\vec{u}) = \lambda\,d\Phi(p)(\vec{u})$ for all $\vec{u}\in E$ and all $\lambda\in\mathbb{R}$.

Answer. $\lim_{h\to 0}\frac{\Phi(p+h(\lambda\vec{u}))-\Phi(p)}{h} = \lambda\lim_{h\to 0}\frac{\Phi(p+\lambda h\vec{u})-\Phi(p)}{\lambda h} = \lambda\lim_{k\to 0}\frac{\Phi(p+k\vec{u})-\Phi(p)}{k}$. ■

Definition S.4 If Φ is Gateaux differentiable and if moreover $d\Phi(p)$ is linear and continuous at p, then Φ is said to be differentiable at p (or Fréchet differentiable at p). So

$$\Phi(q) = \Phi(p) + h\,d\Phi(p).\vec{pq} + o(||\vec{pq}||_E), \tag{S.4}$$

since then $d\Phi(p)(\vec{u}) =^{\text{noted}} d\Phi(p).\vec{u}$ for all $\vec{u}\in E$ (linearity).

And the affine function $\mathit{aff}_p : q \to \mathit{aff}_p(q) := \Phi(p) + d\Phi(p).\vec{pq}$ is the affine approximation of Φ at p. (So, the graph of aff_p is the tangent plane of Φ at p.)

Definition S.5 $\Phi: U\to\mathcal{F}$ is said to be differentiable in U iff Φ is differentiable at all $p\in U$. Then its differential is the map

$$d\Phi : \left\{ \begin{aligned} U &\to \mathcal{L}(E;F) \\ p &\to d\Phi(p). \end{aligned} \right. \tag{S.5}$$

And $C^1(U;\mathcal{F})$ is the set of differentiable functions ψ such that $d\Phi\in C^0(U;\mathcal{L}(E;F))$.

And $C^2(U;\mathcal{F})$ is the set of differentiable functions ψ such that $d\Phi\in C^1(U;\mathcal{L}(E;F))$.

... And $C^k(U;\mathcal{F})$ is the set of differentiable functions ψ such that $d\Phi\in C^{k-1}(U;\mathcal{L}(E;F))$....

Proposition S.6 *The differentiation (or derivation) operator* $d : \begin{Bmatrix} C^1(U;\mathcal{F}) & \to C^0(U;\mathcal{L}(E;F)) \\ \Phi & \to d\Phi \end{Bmatrix}$ *is* $\mathbb{R}$*-linear ("a derivation is linear").*

Proof. $d(\Phi+\lambda\Psi)(p).\vec{u} = \lim_{h\to0}\frac{(\Phi+\lambda\Psi)(p+h\vec{u})-(\Phi+\lambda\Psi)(p)}{h} = \lim_{h\to0}\frac{\Phi(p+h\vec{u})-\Phi(p)+\lambda\Psi(p+h\vec{u})-\lambda\Psi(p)}{h} = \lim_{h\to0}\frac{\Phi(p+h\vec{u})-\Phi(p)}{h} + \lambda\lim_{h\to0}\frac{\Psi(p+h\vec{u})-\Psi(p)}{h} = d\Phi(p).\vec{u} + \lambda d\Psi(p).\vec{u} = (d\Phi(p)+\lambda d\Psi(p)).\vec{u}$ for all p and $\vec{u}$, thus $d(\Phi+\lambda\Psi) = d\Phi + \lambda d\Psi$ for all $\lambda\in\mathbb{R}$ and $\Phi,\Psi\in C^1(U;\mathcal{F})$. ⁂

Exercice S.7 Prove: if $f\in C^1(U;\mathbb{R})$ (scalar values) and $\Phi\in C^1(U;\mathcal{F})$ then, for all $\vec{u}\in E$,

$$d(f\Phi).\vec{u} = (df.\vec{u})\Phi + f(d\Phi.\vec{u}) \tag{S.6}$$

(and we also write $d(f\Phi) = \Phi\otimes df + f\,d\Phi$ for a use with contraction rules).

Answer.

$$\begin{aligned} d(f\Phi)(p).\vec{u} &= \lim_{h\to0}\frac{f(p+h\vec{u})\Phi(p+h\vec{u}) - f(p)\Phi(p)}{h} \\ &= \lim_{h\to0}\frac{f(p+h\vec{u})\Phi(p+h\vec{u}) - f(p)\Phi(p+h\vec{u})}{h} + \frac{f(p)\Phi(p+h\vec{u}) - f(p)\Phi(p)}{h} \\ &= \lim_{h\to0}\frac{f(p+h\vec{u}) - f(p)}{h}(\Phi(p)+o(1)) + \lim_{h\to0} f(p)\frac{\Phi(p+h\vec{u})-\Phi(p)}{h} \\ &= (df(p).\vec{u})\Phi(p) + f(p)(d\Phi(p).\vec{u}). \end{aligned} \tag{S.7}$$

Tensorial writing: $d(f\Phi).\vec{u} = (\Phi\otimes df).\vec{u} + (f\,d\Phi).\vec{u}$, thanks to the contraction rule which gives $(\Phi\otimes df).\vec{u} + (f\,d\Phi).\vec{u} = \Phi(df.\vec{u}) + f(d\Phi.\vec{u})$. ⁂

Remark S.8 In differential geometry, the definition of a tangent map is defined by, with definition S.4:

$$T\Phi : \begin{cases} U\times E & \to \mathcal{F}\times F \\ (p,\vec{u}) & \to T\Phi(p,\vec{u}) = (\Phi(p), d\Phi(p).\vec{u}). \end{cases} \tag{S.8}$$

The two points p (input) and $\Phi(p)$ (output) are the base points, and the two vectors $\vec{u}$ (input) and $d\Phi(p).\vec{u}$ (output) are the initial vector and its push-forward by Φ. ⁂

S.1.3 Notation for the second order Differential

Let $\Phi\in C^2(U;\mathcal{F})$; Thus $d\Phi\in C^1(U;\mathcal{L}(E;F))$, thus $d(d\Phi)\in C^0(U;\mathcal{L}(E;\mathcal{L}(E;F)))$; So, for $p\in U$ and $\vec{u}\in E$, we have $d(d\Phi)(p).\vec{u} = \lim_{h\to0}\frac{d\Phi(p+h\vec{u})-d\Phi(p)}{h} \in \mathcal{L}(E;F)$, and, with $\vec{v}\in E$ we have $(d(d\Phi)(p).\vec{u}).\vec{v}\in F$.

Definition S.9 The bilinear map $d^2\Phi(p)\in\mathcal{L}(E,E;F)$ is defined by

$$d^2\Phi(p)(\vec{u},\vec{v}) = (d(d\Phi)(p).\vec{u}).\vec{v}, \tag{S.9}$$

thanks to the natural canonical isomorphism $L\in\mathcal{L}(E;\mathcal{L}(E;F)) \leftrightarrow T_L\in\mathcal{L}(E,E;F)$ given by $T_L(\vec{u}_1,\vec{u}_2) := (L.\vec{u}_1).\vec{u}_2$ for all $\vec{u}_1,\vec{u}_2\in E$; Thus $L =^{\text{noted}} T_L$, thus $d(d\Phi) =^{\text{noted}} d^2\Phi(p)\in\mathcal{L}(E,E;F)$.

This gives the usual second order Taylor expansion of Φ (supposed C^2) near p in the direction $\vec{u}$:

$$\Phi(p+h\vec{u}) = \Phi(p) + h\,d\Phi(p).\vec{u} + \frac{h^2}{2}\,d^2\Phi(p)(\vec{u},\vec{u}) + o(h^2) \tag{S.10}$$

(=the second order Taylor expansion of $f : h\to f(h) = \Phi(p+h\vec{u})$ near $h=0$, cf. (S.2)).

And Schwarz's theorem tells that $d^2\Phi(p)$ is symmetric when Φ is C^2, i.e. $d^2\Phi(p)(\vec{u},\vec{v}) = d^2\Phi(p)(\vec{v},\vec{u})$.

S.2 A basis and the j-th partial derivative

Definition S.10 Let $\Phi\in C^1(U;\mathcal{F})$, $\vec{u}\in\Gamma(U)$ (a vector field), $p\in U$. The derivative of Φ at p along $\vec{u}$ is defined by

$$\partial_{\vec{u}}\Phi(p) := d\Phi(p).\vec{u}(p) \quad (= \lim_{h\to0}\frac{\Phi(p+h\vec{u}(p))-\Phi(p)}{h} \in F). \tag{S.11}$$

This defines the directional derivative operator along $\vec{u}$:

$$\partial_{\vec{u}} : \begin{cases} C^1(U;\mathcal{F}) & \to C^0(U;F) \\ \quad\Phi & \to \partial_{\vec{u}}(\Phi) := d\Phi.\vec{u}, \quad \text{i.e.}\quad \partial_{\vec{u}}(\Phi)(p) := d\Phi(p).\vec{u}(p). \end{cases} \tag{S.12}$$

(And $\partial_{\vec{u}}(\Phi)(p) =^{\text{noted}} \vec{u}(\Phi)(p)$ in differential geometry thanks to $E\simeq E^{**}$ which gives $\partial_{\vec{u}}\simeq\vec{u}$.)

In particular, if $(\vec{e}_i(p))$ is a basis at p, then the j-th partial derivative of Φ at p is $\partial_{\vec{e}_j}\Phi(p) =^{\text{noted}} \partial_j\Phi(p)$ (the derivative along $\vec{e}_j$), and the j-th directional derivative operator is

$$\partial_j : \left\{ \begin{array}{ll} C^1(U;\mathcal{F}) & \to C^0(U;F) \\ \Phi & \to \boxed{\partial_j\Phi := d\Phi.\vec{e}_j}, \quad \text{i.e.} \quad \partial_j(\Phi)(p) := d\Phi(p).\vec{e}_j(p). \end{array} \right. \tag{S.13}$$

(In differential geometry $\partial_j\Phi =^{\text{noted}} \vec{e}_j(\Phi)$, so $\vec{e}_j(\Phi)(p) := d\Phi(p).\vec{e}_j(p)$.)

S.3 Application 1: Scalar valued functions

S.3.1 Differential of a scalar valued function (objective)

Here $\Phi \overset{\text{noted}}{=} f : \left\{ \begin{array}{ll} U & \to \mathbb{R} \\ p & \to f(p) \end{array} \right\}$ is a C^1 scalar valued function, so $df \in \Omega^1(U) \cap C^0(U;E^*)$ (a C^0 differential form). So $df(p) \in E^*$ for all $p \in U$, and $df(p).\vec{u} = \lim_{h\to 0} \frac{f(p+h\vec{u})-f(p)}{h} \in \mathbb{R}$ for all $\vec{u} \in E$.

Exercice S.11 Prove: If $f, g \in C^1(U;\mathbb{R})$ then (derivative of a product)

$$d(fg) = (df)g + f(dg), \tag{S.14}$$

i.e., $d(fg).\vec{w} = (df.\vec{w})g + f(dg.\vec{w})$ for all $\vec{w} \in \Gamma(U)$.

Answer. $\lim_{h\to 0} \frac{f(p+h\vec{w})g(p+h\vec{w})-f(p)g(p)}{h} = \lim_{h\to 0} \frac{f(p+h\vec{w})g(p+h\vec{w})-f(p)g(p+h\vec{w})}{h} + \lim_{h\to 0} \frac{f(p)g(p+h\vec{w})-f(p)g(p)}{h} = \lim_{h\to 0} \frac{f(p+h\vec{w})-f(p)}{h}(g(p)+o(1)) + \lim_{h\to 0} f(p)\frac{g(p+h\vec{w})-g(p)}{h}$, calculation that only requires the first order (affine) approximation of f and g: We get the same result as with the affine functions $f(x) = a_0+a_1x$ and $g(x) = b_0+b_1x$, which give $(fg)(x) = a_0b_0 + (a_0b_1+a_1b_0)x + a_1b_1x^2$, and then $(fg)'(x) = a_0b_1+a_1b_0 + 2a_1b_1x$, which is indeed equal to $(f'g+fg')(x) = a_1(b_0+b_1x) + (a_0+a_1x)b_1$. ∴

S.3.2 Quantification

Let $(\vec{e}_i(p))$ be a basis at p. So $\partial_jf(p) =^{(S.13)} df(p).\vec{e}_j(p)$ $(= \lim_{h\to 0} \frac{f(p+h\vec{e}_j(p))-f(p)}{h})$, and we write

$$\partial_jf(p) \overset{\text{noted}}{=} f_{|j}(p). \tag{S.15}$$

So, with $(\pi_{ei}(p))$ the dual basis of the basis $(\vec{e}_i(p))$, and with $f_{|j}(p) := \pi_{ei}(p).df(p)$ (j-th component of $df(p)$ in the basis $(\pi_{ei}(p))$), we have

$$df = \sum_{j=1}^n f_{|j}\pi_{ej}, \quad \text{and} \quad [df(p)]_{|\vec{e}} = (\, f_{|1}(p) \quad \dots \quad f_{|n}(p)\,) \quad \text{(row matrix)}. \tag{S.16}$$

So $df.\vec{u} = \sum_{j=1}^n f_{|j}u_j = [df]_{|\vec{e}}.[\vec{u}]_{|\vec{e}}$ when $\vec{u}(p) = \sum_i u_i(p)\vec{e}_i(p)$. In particular with a Cartesian basis, $(\pi_{ei}(p)) =^{\text{noted}} (dx_j)$, and $df = \sum_{j=1}^n \frac{\partial f}{\partial x_j}dx_j$.

Duality notations: $\pi_{ei} = e^i$, $\vec{u} = \sum_{j=1}^n u^j\vec{e}_j$, $df = \sum_{j=1}^n f_{|j}\, e^j$, $df.\vec{u} = \sum_{j=1}^n f_{|j}u^j$, and with a Cartesian basis, $\pi_{ei} = dx^i$ and $df = \sum_{j=1}^n \frac{\partial f}{\partial x^j}dx^j$.

Exercice S.12 Prove: $(fg)_{|j} = f_{|j}\,g + f\,g_{|j}$ when $f, g : U \to \mathbb{R}$ are C^1 scalar valued functions.

Answer. Apply (S.7): here $d(fg) = g\,df + f\,dg$, i.e. $d(fg).\vec{e}_j = (df.\vec{e}_j)\,g + f\,(dg.\vec{e}_j)$ for all j. ∴

And $df(p) \in E^*$ satisfies the covariant change of basis formula for linear forms, i.e., if $(\vec{a}_i(p))$ and $(\vec{b}_i(p))$ are two bases at p and $P(p)$ is the transition matrix from $(\vec{a}_i(p))$ to $(\vec{b}_i(p))$, then $[df(p)]_{|\vec{b}} =^{(A.28)} [df(p)]_{|\vec{a}}.P(p)$, or in short:

$$[df]_{|\vec{b}} = [df]_{|\vec{a}}.P \quad \text{(covariance formula)}. \tag{S.17}$$

S.3.3 Gradients (subjective) associated with a differential through inner dot products

Let $f \in C^1(U;\mathbb{R})$ (a C^1 scalar valued function). Choose (subjective) an inner dot product $(\cdot,\cdot)_g$ in E.

Definition S.13 The conjugate gradient $\vec{\mathrm{grad}}_g f(p)$ of f at $p \in U$ relative to $(\cdot,\cdot)_g$, also called the $(\cdot,\cdot)_g$-conjugate gradient of f at p, is the $(\cdot,\cdot)_g$-Riesz representation vector of the linear form $df(p) \in E^*$:

$$\vec{\mathrm{grad}}_g f(p) := \vec{R}_g(df(p)). \tag{S.18}$$

I.e., the vector $\vec{\mathrm{grad}}_g f(p) \in E$ is characterized by, cf. (F.2),

$$\forall \vec{u} \in E, \quad \boxed{df(p).\vec{u} = (\vec{\mathrm{grad}}_g f(p), \vec{u})_g} = \vec{\mathrm{grad}}_g f(p) \underset{g}{\bullet} \vec{u}. \tag{S.19}$$

Fundamental: An English observer with his Euclidean dot product $(\cdot,\cdot)_a$ in foot and a French observer with his Euclidean dot product $(\cdot,\cdot)_b$ in metre have the same differential df (defined independently of any unit of measurement); But do ***not*** have the same gradient:

$$\vec{\mathrm{grad}}_b f \overset{(F.12)}{=} \lambda^2 \vec{\mathrm{grad}}_a f \quad \text{with} \quad \lambda^2 > 10. \tag{S.20}$$

Quite different vectors isn't it? The "gradient vector" strongly depends on the chosen inner dot product. And to forget this fact leads to accidents like the crash of the Mars Climate Orbiter probe, cf. remark A.14.

Subjective first order Taylor expansion: If an inner dot product $(\cdot,\cdot)_g$ exists and is used, then the first order Taylor expansion (S.3) gives

$$f(p+h\vec{u}) = f(p) + h\,(\vec{\mathrm{grad}}_g f(p), \vec{u})_g + o(h) \qquad (= f(p) + h\,\vec{\mathrm{grad}}_g f(p) \underset{g}{\bullet} \vec{u} + o(h)). \tag{S.21}$$

Fundamental once again (we insist):

- An inner dot product does not always exist (as a meaningful tool), see § B.3.2 (thermodynamics), thus, for a C^1 function, a gradient does not always exists (contrary to a differential).
- $df(p)$ is a linear form (covariant) while $\vec{\mathrm{grad}}_g f(p)$ is a vector (contravariant). In particular the change of basis formulas differ, cf. (A.28):

$$[df]_{|new} = [df]_{|old}.P, \quad \text{while} \quad [\vec{\mathrm{grad}}_g]_{|new} = P^{-1}.[\vec{\mathrm{grad}}_g]_{|old}. \tag{S.22}$$

- df cannot be identified $\vec{\mathrm{grad}} f$ (with one?) (Recall; there is no natural canonical isomorphims between E and E^*.) The differential df is also called the "covariant gradient", and any of its associated gradient vectors is also called the "contravariant gradient relative to an inner dot product".

Isometric Euclidean framework: If one Euclidean dot product can be imposed to all observers (foot? metre?) then $\vec{\mathrm{grad}}_g f =^{\text{noted}} \vec{\mathrm{grad}} f = \vec{\nabla} f$ and (S.19) is written $df.\vec{u} = \vec{\mathrm{grad}} f \bullet \vec{u} = \vec{\nabla} f \bullet \vec{u}$ (isometric framework).

Exercice S.14 Cartesian basis $(\vec{e}_i)$ and $(\cdot,\cdot)_g$ given by $[g]_{[\vec{e}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Give $[df]_{|\vec{e}}$ and $[\vec{\mathrm{grad}}_g f]_{|\vec{e}}$.

Answer. $[df]_{|\vec{e}} = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix}$ (row matrix) and (S.19) gives $[\vec{\mathrm{grad}}_g f]_{|\vec{e}} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{1}{2}\frac{\partial f}{\partial x_2} \end{pmatrix}$ (column matrix $\neq [df]^T$). ▲

S.4 Application 2: Coordinate system basis and Christoffel symbols

(Necessary when dealing with covariance.)

S.4.1 Coordinate system, and coordinate system basis

Consider a (open) set $U_{\text{par}} = \{\vec{q} \in]a_1,b_1[\times ... \times]a_n,b_n[\}$, called the set of parameters, in the Cartesian space $\mathbb{R}^n$, consider an open set $U \subset \mathbb{R}^n$, called the set of geometric positions, and consider a C^2-diffeomorphism $\Psi : \vec{q} \in U_{\text{par}} \to p \in U$, called a coordinate system.

Let $(\vec{a}_i)$ the canonical basis of the parameter space, let $\vec{q} = \sum_i q_i \vec{a}_i \in U_{\text{par}}$ (the q_i are called the parameters). E.g., see the polar coordinate system at § 6.6.2 where $\vec{q} = (q_1, q_2) = (r, \theta)$.

Ψ being a diffeomorphism, at any $p=\Psi(\vec{q})\in U$ the vectors

$$\vec{a}_{i*}(p):=d\Psi(\vec{q}).\vec{a}_i \tag{S.23}$$

make a basis in E at p, and $(\vec{a}_{i*}(p))$ is called the coordinate system basis at p. Its dual basis at p is made of the linear forms $dq_i(p)$, so where, for all i,j,

$$dq_i(p).\vec{a}_{j*}(p)=\delta_i^j. \tag{S.24}$$

Duality notations: $dq^i(p).\vec{a}_{j*}(p)=\delta_j^i$ for all i,j.

S.4.2 Parametric expression of the differential of a scalar valued function

With a coordinate system Ψ, a scalar valued function $f:\left\{\begin{matrix} U & \to \mathbb{R} \\ p & \to f(p)\end{matrix}\right\}$ defined in U can be described with the function $g=f\circ\Psi:\left\{\begin{matrix} U_{\text{par}} & \to \mathbb{R} \\ \vec{q} & \to g(\vec{q}):=f(p) \text{ when } p=\Psi(\vec{q})\end{matrix}\right\}$ defined in U_{par}, and g is called the parametric expression of f. Thus

$$dg(\vec{q})=df(p).d\Psi(\vec{q}) \quad \text{when} \quad p=\Psi(\vec{q}), \tag{S.25}$$

in particular,

$$\frac{\partial g}{\partial q_j}(\vec{q}):=dg(\vec{q}).\vec{a}_j=df(p).d\Psi(\vec{q}).\vec{a}_j=df(p).\vec{a}_{j*}(p)\overset{\text{noted}}{=}\frac{\partial f}{\partial q_j}(p). \tag{S.26}$$

Warning, pay attention: f is a function of p, not a function of $\vec{q}$, and the notations $\frac{\partial f}{\partial q_j}(p)$ means $:=\frac{\partial(f\circ\Psi)}{\partial q_j}(\vec{q})$ when $p=\Psi(\vec{q})$, and nothing else.

Thus with $(dq_j(p))$ the dual basis of the coordinate basis $(\vec{a}_{i*}(p))$ at p,

$$df(p)\overset{(S.26)}{=}\sum_{j=1}^n\frac{\partial f}{\partial q_j}(p)\,dq_j(p). \tag{S.27}$$

Duality notations: $df(p)=\sum_j\frac{\partial f}{\partial q^j}(p)\,dq^j(p)$.

Remark S.15 Pay attention to the notations that could contradict themselves:

1- In U_{par} the dual basis (π_{ai}) of the Cartesian basis $(\vec{a}_i)$ is a uniform basis (independent of $\vec{q}$)... and is (almost) never written (dq_i)...

2- Indeed, $(dq_i(p))$ is the name reserved for the dual basis of $(\vec{a}_{i*}(p))$ in the geometric space... Mind the notations! E.g. for polar coordinates $(dq_1(p),dq_2(p))=(dr(p),d\theta(p))$ is the dual basis of the polar coordinate system basis $(\vec{a}_{1*}(p),\vec{a}_{2*}(p))$ at p, cf. (6.6.2). ⁂

Exercice S.16 Bases $(\vec{a}_i)$ and $(\vec{b}_i)$ at p. A vector $\vec{x}$ is expressed as $\vec{x}=\sum_i x_{a,i}\vec{a}_i=\sum_i x_{b,i}\vec{b}_i$. Prove:

$$\vec{b}_i=\lambda\vec{a}_i,\ \forall i \implies \boxed{\frac{\partial f}{\partial x_{b,i}}=\lambda\frac{\partial f}{\partial x_{a,i}}} \quad \text{or} \quad \frac{\partial f}{\partial x_{a,i}}=\frac{\partial f}{\partial(\lambda x_{a,i})}. \tag{S.28}$$

(Change of unit formula.) Duality notations: $\frac{\partial f}{\partial x_b^j}=\lambda\frac{\partial f}{\partial x_a^j}$.

Answer. $df(p).\vec{b}_j(p)=\lambda df(p).\vec{a}_j(p)$ (linearity of $df(p)$) reads (S.28). (Or $[df]_{|\vec{b}}=[df]_{|\vec{a}}.P$ with $P=\lambda I$ here.) ⁂

Exercice S.17 $[df]_{|\vec{b}}=[df]_{|\vec{a}}.P$, cf. (S.17), i.e. $\frac{\partial f}{\partial x_b^j}=\sum_{i=1}^n\frac{\partial f}{\partial x_a^i}P_j^i$ is also noted

$$\frac{\partial f}{\partial x_b^j}=\sum_{i=1}^n\frac{\partial f}{\partial x_a^i}\frac{\partial x_a^i}{\partial x_b^j}. \tag{S.29}$$

Why?

Answer. **Quick answer.** $[\vec{x}]_{|\vec{b}} = P^{-1}.[\vec{x}]_{|\vec{a}}$, i.e. $[\vec{x}]_{|\vec{a}} = P.[\vec{x}]_{|\vec{b}}$, which means $[\vec{x}]_{|\vec{a}}([\vec{x}]_{|\vec{b}}) = P.[\vec{x}]_{|\vec{b}}$, i.e.

$$\begin{pmatrix} x_a^1(x_b^1,...,x_b^n) \\ \vdots \\ x_a^1(x_b^1,...,x_b^n) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n P_j^1 x_b^j \\ \vdots \\ \sum_{j=1}^n P_j^n x_b^j \end{pmatrix}, \quad \text{thus} \quad \frac{\partial x_a^i}{\partial x_b^j}(x_b^1,...,x_b^n) = P_j^i, \ \ \forall i,j. \tag{S.30}$$

Thus (S.29) means

$$\frac{\partial f}{\partial x_b^j}(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_a^i}(p) \frac{\partial x_a^i}{\partial x_b^j}(x_b^1,...,x_b^n) \quad \text{thus} \quad = \sum_{i=1}^n \frac{\partial f}{\partial x_a^i}(p)\, P_j^i, \tag{S.31}$$

as given in (S.17).
Detailed answer. Let O be a point (origin) in U. If $p \in U$, let $\vec{x} = \overrightarrow{Op} = \sum_{i=1}^n x_a^i \vec{a}_i = \sum_{i=1}^n x_b^i \vec{b}_i$.

This define the function $[\vec{x}]_{|\vec{a}} : [\vec{x}]_{|\vec{b}} \to [\vec{x}]_{|\vec{a}}([\vec{x}]_{|\vec{b}})$, and we have $[\vec{x}]_{|\vec{a}}([\vec{x}]_{|\vec{b}}) = P.[\vec{x}]_{|\vec{b}}$ (change of basis formula).

Then let $f_a, f_b : \mathbb{R}^n \to \mathbb{R}$ be defined by $f_a([\vec{x}]_{|\vec{a}}) := f(p)$ and $f_b([\vec{x}]_{|\vec{b}}) := f(p)$. (NB: f_a and f_b don't have the same definition domain: They are different).

Thus $f_a([\vec{x}]_{|\vec{a}}) = f_b([\vec{x}]_{|\vec{b}})$ $(= f(p))$ when $[\vec{x}([\vec{x}]_{|\vec{b}})]_{|\vec{a}} = P.[\vec{x}]_{|\vec{b}}$, so $(f_a \circ [\vec{x}]_{|\vec{a}})([\vec{x}]_{|\vec{b}}) = f_b([\vec{x}]_{|\vec{b}})$. Thus $\frac{\partial (f_a \circ [\vec{x}]_{|\vec{a}})}{\partial x_b^i}([\vec{x}]_{|\vec{b}}) = \frac{\partial f_b}{\partial x_b^i}([\vec{x}]_{|\vec{b}})$, thus the meaning of (S.29) is

$$\sum_{j=1}^n \frac{\partial f_a}{\partial x_a^j}([\vec{x}]_{|\vec{a}}) \frac{\partial x_a^j}{\partial x_b^i}([\vec{x}]_{|\vec{b}}) = \frac{\partial f_b}{\partial x_b^i}([\vec{x}]_{|\vec{b}}). \tag{S.32}$$

Question: Why did we introduce f_a and f_b (and not just keep f)?

Answer: Because a vector is not just a collection of components (is not just a matrix), and $\overrightarrow{Op}$ cannot be reduced to a matrix of components (which one: $[\vec{x}]_{|\vec{a}}$? $[\vec{x}]_{|\vec{b}}$?). Here f is a function acting on a point p (independent of a referential), while f_a and f_b are functions acting on matrices (dependent on the choice of a referential): The domain of definitions are different, so the functions f, f_a and f_b are different. ∎

S.4.3 Christoffel symbols

We use duality notations for readability and usage.

Definition S.18 In a coordinate system basis $(\vec{e}_i(p))$ in E (previously called $(\vec{a}_{i*}(p))$, the Christoffel symbol $\gamma_{jk}^i(p) \in \mathbb{R}$ are the components of the vector $d\vec{e}_k(p).\vec{e}_j(p)$, i.e. $d\vec{e}_k(p).\vec{e}_j(p) = \sum_{k=1}^n \gamma_{jk}^i(p)\vec{e}_i(p)$, so

$$\boxed{d\vec{e}_k.\vec{e}_j = \sum_{i=1}^n \gamma_{jk}^i \vec{e}_i}, \quad \text{or} \quad d\vec{e}_j.\vec{e}_i = \sum_{k=1}^n \gamma_{ij}^k \vec{e}_k. \tag{S.33}$$

(So, with $(e^i(p))$ the dual basis of $(\vec{e}_i(p))$, $\gamma_{jk}^i := e^i.d\vec{e}_k.\vec{e}_j$, and, for calculations with contractions, $d\vec{e}_k = \sum_{ij} \gamma_{jk}^i \vec{e}_i \otimes e^j$.)

(The Christoffel symbols vanish in a Cartesian framework.)

(Differential geometry in manifolds: $\nabla_{\vec{e}_j}\vec{e}_k = \sum_{i=1}^n \gamma_{jk}^i \vec{e}_i$, i.e. the $\gamma_{jk}^i = e^i.\nabla_{\vec{e}_j}\vec{e}_k$ are the component of the connection ∇, the usual connection in a surface in $\mathbb{R}^n$ being the Riemannian connection, in which case $\nabla_{\vec{e}_j}\vec{e}_k$ is the orthogonal projection of $d\vec{e}_k.\vec{e}_j$ on the surface relative to a Euclidean dot product.)

E.g. for the polar coordinate system, see remark 6.12, $d\vec{e}_2.\vec{e}_2 = -r\vec{e}_1$, thus $\gamma_{22}^1 = -r$ and $\gamma_{22}^2 = 0$.

Exercice S.19 Prove: If $(\vec{e}_i(p))$ is the coordinate system basis of a C^2 coordinate system, then:

$$\forall i,j,\ d\vec{e}_i.\vec{e}_j = d\vec{e}_j.\vec{e}_i \quad (= \frac{\partial^2 \Psi}{\partial q^i \partial q^j}), \quad \text{and} \quad \forall i,j,k,\ \gamma_{ji}^k = \gamma_{ij}^k \quad \text{(symmetry for lower indices).} \tag{S.34}$$

Answer. $\vec{e}_i(p) = (\vec{e}_i \circ \Psi)(\vec{q}) =^{(S.23)} d\Psi(\vec{q}).\vec{a}_i$ gives $d(\vec{e}_i \circ \Psi)(\vec{q}).\vec{a}_j = d(d\Psi(\vec{q}).\vec{a}_i).\vec{a}_j$, thus $d\vec{e}_i(\Psi(\vec{q})).d\Psi(\vec{q}).\vec{a}_j = \frac{\partial \frac{\partial \Psi}{\partial q^i}}{\partial q^j} = \frac{\partial \frac{\partial \Psi}{\partial q^j}}{\partial q^i}$ (Schwarz theorem since Ψ is C^2) $= d\vec{e}_j.\vec{e}_i =^{\text{noted}} \frac{\partial^2 \Psi}{\partial q^j \partial q^i}$, thus $\sum_{k=1}^n \gamma_{ij}^k \vec{e}_k = \sum_{k=1}^n \gamma_{ji}^k \vec{e}_k$. ∎

Exercice S.20 Consider two coordinate system bases $(\vec{a}_i(p))$ and $(\vec{b}_i(p))$ at p, $P(p) = [P_j^i(p)]$ the transition matrix from $(\vec{a}_i(p))$ to $(\vec{b}_i(p))$, and $Q = P^{-1}$. Using the generic notation $d\vec{e}_k.\vec{e}_j = \sum_{i=1}^n \gamma_{jk,e}^i \vec{e}_i$, prove the change of basis formula for the Christoffel symbols:

$$\gamma_{jk,b}^i = \sum_{\lambda,\mu,\nu=1}^n Q_\lambda^i P_j^\mu P_k^\nu \gamma_{\mu\nu,a}^\lambda + \sum_{\lambda,\mu=1}^n Q_\lambda^i P_j^\mu (dP_k^\lambda.\vec{a}_\mu) \quad (= \sum_{\lambda,\mu,\nu=1}^n Q_\lambda^i P_j^\mu P_k^\nu \gamma_{\mu\nu,a}^\lambda + \sum_{\lambda=1}^n Q_\lambda^i (dP_k^\lambda.\vec{b}_j)). \tag{S.35}$$

(Because of the term $\sum_{\mu\nu} Q_\lambda^i P_j^\mu (dP_k^\lambda.\vec{a}_\mu)$, a derivation is not a tensor.)

Answer. $\vec{b}_k(p) = \sum_\nu P_k^\nu(p)\vec{a}_\nu(p)$ gives $d\vec{b}_k.\vec{b}_j = \sum_\nu (dP_k^\nu.\vec{b}_j)\vec{a}_\nu + \sum_\nu P_k^\nu(d\vec{a}_\nu.\vec{b}_j) = \sum_{\mu\nu} P_j^\mu(dP_k^\nu.\vec{a}_\mu)\vec{a}_\nu + \sum_{\mu\nu} P_k^\nu P_j^\mu(d\vec{a}_\nu.\vec{a}_\mu)$; And $b^i = \sum_\lambda Q_\lambda^i a^\lambda$, thus

$$\gamma_{jk,b}^i = b^i.d\vec{b}_k.\vec{b}_j = \sum_{\lambda\mu\nu} Q_\lambda^i P_j^\mu(dP_k^\nu.\vec{a}_\mu)a^\lambda.\vec{a}_\nu + \sum_{\lambda\mu\nu} Q_\lambda^i P_j^\mu P_k^\nu a^\lambda.(d\vec{a}_\nu.\vec{a}_\mu) = \sum_{\lambda\mu} Q_\lambda^i P_j^\mu(dP_k^\lambda.\vec{a}_\mu) + \sum_{\lambda\mu\nu} Q_\lambda^i P_j^\mu P_k^\nu \gamma_{\mu\nu,a}^\lambda,$$

thus (S.35). ∎

S.5 Application 3: Differential of a vector field

Here $F = E = \vec{\mathbb{R}}^n$, $\Phi =^{\text{noted}} \vec{w} \in \Gamma(U)$ is a vector field. Thus $d\vec{w}(p) \in \mathcal{L}(E;E)$ and $d\vec{w}.\vec{u}$ is a vector field in E for all $\vec{u} \in \Gamma(U)$, given by $(d\vec{w}.\vec{u})(p) = d\vec{w}(p).\vec{u}(p) = \lim_{h\to 0} \frac{\vec{w}(p+h\vec{u}(p))-\vec{w}(p)}{h} \in E$.

Quantification: $(\vec{e}_i(p))$ is a basis at p in E. Call $w_i(p) \in \mathbb{R}$ the components of $\vec{w}(p)$, i.e. $\vec{w}(p) = \sum_{i=1}^n w_i(p)\vec{e}_i(p)$. And call $w_{i|j}(p)$ the components of $d\vec{w}(p)$ (endomorphism in E):

$$\vec{w} = \sum_{i=1}^n w_i\vec{e}_i, \quad d\vec{w}.\vec{e}_j = \sum_{i=1}^n w_{i|j}\vec{e}_i, \quad [d\vec{w}]_{|\vec{e}} = [w_{i|j}] \quad \text{(Jacobian matrix)}. \tag{S.36}$$

And tensorial notations for calculations with contractions: $(\pi_{ei}(p))$ being the dual basis,

$$d\vec{w} = \sum_{i,j=1}^n w_{i|j}\vec{e}_i \otimes \pi_{ej}. \tag{S.37}$$

Duality notations: $\vec{w} = \sum_{i=1}^n w^i\vec{e}_i$, $d\vec{w}.\vec{e}_j = \sum_{i,j=1}^n w_{|j}^i\vec{e}_i$, $[d\vec{w}]_{|\vec{e}} = [w_{|j}^i]$, and $d\vec{w} = \sum_{i,j=1}^n w_{|j}^i\vec{e}_i \otimes e^j$.

In a Cartesian basis: Here $(\vec{e}_i)$ is uniform, so $\vec{w}(p) = \sum_{i=1}^n w_i(p)\vec{e}_i$ gives $d\vec{w}(p).\vec{e}_j = \sum_{i=1}^n (dw_i(p).\vec{e}_j)\vec{e}_i$, thus (S.36) gives

$$w_{i|j} = \frac{\partial w_i}{\partial x_j}(p) \overset{\text{noted}}{=} w_{i,j}, \quad \text{so} \quad [d\vec{w}]_{|\vec{e}} = [\frac{\partial w_i}{\partial x_j}]. \tag{S.38}$$

Duality notations: $w_{|j}^i = \frac{\partial w^i}{\partial x^j}$ and $[d\vec{w}]_{|\vec{e}} = [\frac{\partial w^i}{\partial x^j}]$.

In a coordinate system basis: With the coordinate system described in § S.4 and the duality notations for readability (and usage). $\vec{w}(p) = \sum_{i=1}^n w^i(p)\vec{e}_i(p)$ gives, for all j,

$$d\vec{w}.\vec{e}_j = \sum_{i=1}^n (dw^i.\vec{e}_j)\vec{e}_i + \sum_{i=1}^n w^i(d\vec{e}_i.\vec{e}_j) \quad (= \sum_{i=1}^n w_{|j}^i\vec{e}_i). \tag{S.39}$$

(Tensorial notations to be used with contractions: $d\vec{w} = \sum_i \vec{e}_i \otimes dw^i + \sum_i w^i\, d\vec{e}_i = \sum_{ij} w_{|j}^i\vec{e}_i \otimes e^j$.)

And $\sum_i w^i(d\vec{e}_i.\vec{e}_j) =^{(S.33)} \sum_{ik} w^i\gamma_{ji}^k\vec{e}_k = \sum_{ik} w^k\gamma_{jk}^i\vec{e}_i$, thus, for all i,j,

$$\boxed{w_{|j}^i = \frac{\partial w^i}{\partial q^j} + \sum_{k=1}^n w^k\gamma_{jk}^i} \quad \text{where} \quad \frac{\partial w^i}{\partial q^j} := dw^i.\vec{e}_j. \tag{S.40}$$

($\frac{\partial w^i}{\partial q^j} := dw^i.\vec{e}_j$ is the derivation along the j-th coordinate line of the scalar valued function w^i).

(In particular, if $\vec{w} = \vec{e}_\ell = \sum_i \delta_\ell^i\vec{e}_i$, we recover $d\vec{e}_\ell.\vec{e}_j = \sum_i 0\vec{e}_i + \sum_{ik} \delta_\ell^k\gamma_{jk}^i\vec{e}_i = \sum_i \gamma_{j\ell}^i\vec{e}_i$, cf. (S.33).)

Exercice S.21 With exercise S.20, and $\vec{w} = \sum_{i=1}^n u^i\vec{a}_i = \sum_n v^i\vec{b}_i$, check with calculations ($d\vec{w}$ is an endomorphism defined independently of any basis):

$$[d\vec{w}]_{|\vec{b}} = P^{-1}.[d\vec{w}]_{|\vec{a}}.P, \quad \text{i.e.} \quad v_{|j}^i = \sum_{k,\ell=1}^n Q_k^i u_{|\ell}^k P_j^\ell. \tag{S.41}$$

Answer. $\vec{b}_j = \sum_\ell P_j^\ell\vec{a}_\ell$ for all i, $Q = P^{|1}$, and $[\vec{w}]_{|\vec{b}} = Q.[\vec{w}]_{|\vec{a}}$ reads $v^i = \sum_k Q_k^i u^k$ for all i.

Cartesian basis: $dv^i.\vec{b}_j = d(\sum_k Q_k^i u^k).(\sum_\ell P_j^\ell\vec{a}_\ell) = \sum_{k\ell} Q_k^i(du^k.\vec{a}_\ell)P_j^\ell$, qed (here the Q_k^i are uniform i.e. independent of p).

Coordinate system basis: $v^i=\sum_\lambda Q^i_\lambda u^\lambda$ gives $dv^i.\vec{b}_j=\sum_\lambda(dQ^i_\lambda.\vec{b}_j)u^\lambda+\sum_\lambda Q^i_\lambda(du^\lambda.\vec{b}_j)$; Thus

$$\begin{aligned} v^i_{|j} &\overset{(S.33)}{=} dv^i.\vec{b}_j+\sum_k v^k\gamma^i_{jk,b} \\ &= \sum_{\lambda\mu}u^\lambda P^\mu_j(dQ^i_\lambda.\vec{a}_\mu)+\sum_{\lambda\mu}Q^i_\lambda P^\mu_j(du^\lambda.\vec{a}_\mu)+\sum_{k\lambda\mu\nu}(Q^k_\lambda u^\lambda)Q^i_\nu P^\mu_j(dP^\nu_k.\vec{a}_\mu)+\sum_{k\omega\lambda\mu\nu}(Q^k_\omega u^\omega)Q^i_\lambda P^\mu_j P^\nu_k\gamma^\lambda_{\mu\nu,a} \end{aligned}$$

And $Q^k_\omega P^\lambda_k=\delta^\lambda_\omega$ gives $(dQ^k_\omega.\vec{a}_\mu)P^\lambda_k+Q^k_\omega(dP^\lambda_k.\vec{a}_\mu)=0$, thus the third term reads

$$\sum_{k\lambda\mu\nu}u^\lambda Q^i_\nu P^\mu_j Q^k_\lambda(dP^\nu_k.\vec{a}_\mu)=-\sum_{k\lambda\mu\nu}u^\lambda Q^i_\nu P^\mu_j P^\nu_k(dQ^k_\lambda.\vec{a}_\mu)=-\sum_{\lambda\mu}u^\lambda P^\mu_j(dQ^i_\lambda.\vec{a}_\mu),$$

which cancels the first term: Thus $v^i_{|j}=\sum_{\lambda\mu}Q^i_\lambda P^\mu_j(du^\lambda.\vec{a}_\mu)+\sum_{\lambda\mu\nu}u^\nu Q^i_\lambda P^\mu_j\gamma^\lambda_{\mu\nu}=\sum_{\lambda\mu}Q^i_\lambda u^i_{|j}P^\mu_j$, i.e. (S.41). ∎

S.6 Application 4: Differential of a differential form

Here $F=\mathbb{R}$, $\Phi =^{\text{noted}} \ell\in\Omega^1(U)$ (differential form) supposed C^1, $p\in U$, so $\ell(p)\in E^*$. Its differential at p in a direction $\vec{u}$ is $d\ell(p).\vec{u}=\lim_{h\to0}\frac{\ell(p+h\vec{u})-\ell(p)}{h}\in E^*$. And $(d\ell(p).\vec{u}).\vec{v}=\lim_{h\to0}\frac{\ell(p+h\vec{u}).\vec{v}-\ell(p).\vec{v}}{h}\in\mathbb{R}$ for all $\vec{u},\vec{v}\in E$.

Quantification: $(\pi_{ei}(p)$ its the dual basis.

Call $\ell_i(p)\in\mathbb{R}$ the components of $\ell(p)$, i.e. $\ell(p)=\sum_{i=1}^n\ell_i(p)\pi_{ei}(p)$. And call $\ell_{i|j}(p)$ the components of $d\ell(p)\in\mathcal{L}(E;E^*)$:

$$\ell=\sum_{i=1}^n\ell_i\pi_{ei},\quad d\ell.\vec{e}_j=\sum_{i=1}^n\ell_{i|j}\pi_{ei},\quad [d\ell]_{|\vec{e}}=[\ell_{i|j}]. \tag{S.42}$$

Tensorial notations, to be used with contractions: $d\ell=\sum_{i,j=1}^n\ell_{i|j}\pi_{ei}\otimes\pi_{ej}$.

Duality notations: $\ell=\sum_i\ell_ie^i$, $d\ell.\vec{e}_j=\sum_{i=1}^n\ell_{i|j}e^i$, $[d\ell]_{|\vec{e}}=[\ell_{i|j}]$, and $d\ell=\sum_{i,j=1}^n\ell_{i|j}e^i\otimes e^j$.

In a Cartesian basis: Here $(\vec{e}_i)$ is uniform, so

$$\ell_{i|j}=\frac{\partial\ell_i}{\partial x_j}(p)\overset{\text{noted}}{=}\ell_{i,j},\quad\text{so}\quad[d\ell]_{|\vec{e}}=[\frac{\partial\ell_i}{\partial x_j}]. \tag{S.43}$$

Duality notations: $\ell_{i|j}=d\ell_i.\vec{e}_j=\frac{\partial\ell_i}{\partial x^j}$ and $[d\ell]_{|\vec{e}}=[\frac{\partial\ell_i}{\partial x^j}]$.

In a coordinate system basis: With duality notations and Christoffel symbols:

$$\boxed{de^i.\vec{e}_j=-\sum_{k=1}^n\gamma^i_{jk}e^k}. \tag{S.44}$$

Indeed, $e^i.\vec{e}_k=\delta^i_k$ gives $(de^i.\vec{e}_j).\vec{e}_k+e^i.(d\vec{e}_k.\vec{e}_j)=0$, thus $(de^i.\vec{e}_j).\vec{e}_k=-e^i.\sum_\ell\gamma^\ell_{jk}\vec{e}_\ell=-\gamma^i_{jk}$. Thus

$$\boxed{\ell_{i|j}=\frac{\partial\ell_i}{\partial q^j}-\sum_{k=1}^n\ell_k\gamma^k_{ji}}\quad\text{where}\quad\frac{\partial\ell_i}{\partial q^j}(p):=d\ell_i(p).\vec{e}_i(p). \tag{S.45}$$

Indeed, $\ell=\sum_i\ell_ie^i$ gives $d\ell.\vec{e}_j=\sum_i(d\ell_i.\vec{e}_j)e^i+\sum_i\ell_i(de^i.\vec{e}_j)=\sum_i(d\ell_i.\vec{e}_j)e^i-\sum_{ik}\ell_i\gamma^i_{jk}e^k$.

S.7 Application 5: Differential of a 1 1 tensor

Consider a C^1 $\binom{1}{1}$ tensor $\underline{\underline{\tau}}:\left\{\begin{array}{ll}U&\to\mathcal{L}(E^*,E;\mathbb{R})\\p&\to\underline{\underline{\tau}}(p)\end{array}\right\}$. Its differential $d\underline{\underline{\tau}}:\left\{\begin{array}{ll}U&\to\mathcal{L}(E;\mathcal{L}(E^*,E;\mathbb{R}))\\p&\to d\underline{\underline{\tau}}(p)\end{array}\right\}$ is defined by $d\underline{\underline{\tau}}(p).\vec{u}=\lim_{h\to0}\frac{\underline{\underline{\tau}}(p+h\vec{u})-\underline{\underline{\tau}}(p)}{h}\in\mathcal{L}(E^*,E;\mathbb{R})$, so $(d\underline{\underline{\tau}}(p).\vec{u})(\ell,\vec{v})=\lim_{h\to0}\frac{\underline{\underline{\tau}}(p+h\vec{u})(\ell,\vec{v})-\underline{\underline{\tau}}(p)(\ell,\vec{v})}{h}$ $(\in\mathbb{R})$, for all $\vec{u},\vec{v}\in E$ and $\ell\in E^*$.

Quantification (duality notations): Basis $(\vec{e}_i(p))$ in E at p, dual basis $(e^i(p))$, call $\tau^i_j(p)$ the components of $\underline{\underline{\tau}}(p)$, call $\tau^i_{j|k}(p)$ the components of $d\underline{\underline{\tau}}(p)$:

$$\underline{\underline{\tau}}=\sum_{ij}\tau_{ij}\vec{e}_i\otimes e^j,\quad\boxed{d\underline{\underline{\tau}}.\vec{e}_k=\sum_{i,j=1}^n\tau^i_{j|k}\vec{e}_i\otimes e^j}. \tag{S.46}$$

Tensorial notations, to be used with contractions: $d\underline{\underline{\tau}}=\sum_{i,j,k=1}^n\tau^i_{j|k}\vec{e}_i\otimes e^j\otimes e^k$.

(Classical notations: $\underline{\underline{\tau}}=\sum_{ij}\tau_{ij}\vec{e}_i\otimes\pi_{ej}$, $d\underline{\underline{\tau}}.\vec{e}_k=\sum_{ij}\tau_{ij|k}\vec{e}_i\otimes\pi_{ej}$, and $d\underline{\underline{\tau}}=\sum_{ijk}\tau_{ij|k}\vec{e}_i\otimes\pi_{ej}\otimes\pi_{ek}$.)

Cartesian basis: $d\underline{\underline{\tau}}(p).\vec{e}_k = \sum_{ij}(d\tau^i_j(p).\vec{e}_k)\vec{e}_i \otimes e^j$, so

$$\tau^i_{j|k} = \frac{\partial \tau^i_j}{\partial x^k} \overset{\text{noted}}{=} \tau^i_{j,k} \quad (:= d\tau^i_j.\vec{e}_k). \tag{S.47}$$

Coordinate system basis: $\underline{\underline{\tau}}(p) = \sum_{i,j=1}^n \tau^i_j(p)\vec{e}_i(p) \otimes e^j(p)$ gives, for all k,

$$\begin{aligned} d\underline{\underline{\tau}}.\vec{e}_k &= \textstyle\sum_{ij}(d\tau^i_j.\vec{e}_k)\vec{e}_i \otimes e^j + \sum_{ij} \tau^i_j(d\vec{e}_i.\vec{e}_k) \otimes e^j + \sum_{ij} \tau^i_j\vec{e}_i \otimes (de^j.\vec{e}_k) \\ &= \textstyle\sum_{ij}(d\tau^i_j.\vec{e}_k)\vec{e}_i \otimes e^j + \sum_{ij\ell} \tau^i_j\gamma^\ell_{ki}\vec{e}_\ell \otimes e^j - \sum_{ij\ell} \tau^i_j\gamma^j_{k\ell}\vec{e}_i \otimes e^\ell \\ &= \textstyle\sum_{ij}(d\tau^i_j.\vec{e}_k)\vec{e}_i \otimes e^j + \sum_{ij\ell} \tau^\ell_j\gamma^i_{k\ell}\vec{e}_i \otimes e^j - \sum_{ij\ell} \tau^i_\ell\gamma^\ell_{kj}\vec{e}_i \otimes e^j \end{aligned} \tag{S.48}$$

thus

$$\boxed{\tau^i_{j|k} = \frac{\partial \tau^i_j}{\partial q^k} + \sum_{\ell=1}^n \tau^\ell_j\gamma^i_{k\ell} - \sum_{\ell=1}^n \tau^i_\ell\gamma^\ell_{kj}} \quad \text{where} \quad \frac{\partial \tau^i_j}{\partial q^k} := d\tau^i_j.\vec{e}_k. \tag{S.49}$$

(We have the + sign from vector fields, cf. (S.40), and the − sign from differential forms, cf. (S.45).)

Exercice S.22 If $\vec{u} \in E$, $\ell \in E^*$ then for the elementary $\binom{1}{1}$ tensor $\underline{\underline{\tau}} = \vec{u} \otimes \ell$ prove:

$$d(\vec{u} \otimes \ell).\vec{e}_k = (d\vec{u}.\vec{e}_k) \otimes \ell + \vec{u} \otimes (d\ell.\vec{e}_k), \quad \text{and} \quad (\vec{u} \otimes \ell)^i_{j|k} = u^i_{|k}\ell_j + u^i\ell_{j|k}, \tag{S.50}$$

when $\vec{u} = \sum_i u^i\vec{e}_i$, $\ell = \sum_j \ell_je^j$, $d\vec{u}.\vec{e}_k = \sum_i u^i_{|k}\vec{e}_i$, $d\ell.\vec{e}_k = \sum_j \ell_{j|k}e^j$.

Answer. $\underline{\underline{\tau}} = \vec{u} \otimes \ell = \sum_{ij} \tau^i_j\vec{e}_i \otimes e^j$. where $\tau^i_j = u^i\ell_j$, and $d\underline{\underline{\tau}}.\vec{e}_k = \sum_{i,j=1}^n \tau^i_{j|k}\vec{e}_i \otimes e^j$ where $\tau^i_{j|k} = (u^i\ell_j)_{|k} = u^i_{|k}\ell_j + u^i\ell_{j|k} = (\vec{u} \otimes \ell)^i_{j|k}$. Thus (similar to the derivation of a product):

$$\begin{aligned} d(\vec{u} \otimes \ell)(p).\vec{e}_k(p) &= \lim_{h\to0} \frac{(\vec{u} \otimes \ell)(p{+}h\vec{e}_k(p)) - (\vec{u} \otimes \ell)(p)}{h} = \lim_{h\to0} \frac{\vec{u}(p{+}h\vec{e}_k(p)) \otimes \ell(p{+}h\vec{e}_k(p)) - \vec{u}(p) \otimes \ell(p)}{h} \\ &= \lim_{h\to0} \frac{\vec{u}(p{+}h\vec{e}_k(p)) \otimes \ell(p{+}h\vec{e}_k(p)) - \vec{u}(p{+}h\vec{e}_k(p)) \otimes \ell(p)}{h} + \lim_{h\to0} \frac{\vec{u}(p{+}h\vec{e}_k(p)) \otimes \ell(p) - \vec{u}(p) \otimes \ell(p)}{h} \\ &= \lim_{h\to0}(\vec{u}(p{+}h\vec{e}_k(p)) \otimes (\frac{\ell(p{+}h\vec{e}_k(p)) - \ell(p)}{h}) + \lim_{h\to0}(\frac{\vec{u}(p{+}h\vec{e}_k(p)) - \vec{u}(p)}{h}) \otimes \ell(p) \\ &= \vec{u}(p) \otimes (d\ell(p).\vec{e}_k(p)) + (d\vec{u}(p).\vec{e}_k(p)) \otimes \ell(p), \end{aligned}$$

thus (S.50)$_1$. Which gives $d(\vec{u} \otimes \ell).\vec{e}_k = (\sum_i u^i\vec{e}_i) \otimes (\sum_j \ell_{j|k}e^j) + (\sum_i u^i_{|k}\vec{e}_i) \otimes (\sum_j \ell_je^j)$, thus (S.50)$_2$. ∎

S.8 Divergence of a vector field: Invariant

$\Gamma(U)$ is the set of C^1 vector fields in U, and $\text{Tr} : \mathcal{L}(E;E) \to \mathbb{R}$ is the trace operator.

Definition S.23 The divergence operator is

$$\text{div} := \text{Tr} \circ d : \begin{cases} \Gamma(U) & \to C^0(U;\mathbb{R}) \\ \quad \vec{w} & \to \text{div}\vec{w} := \text{Tr}(d\vec{w}), \quad \text{i.e.} \quad \text{div}\vec{w}(p) := \text{Tr}(d\vec{w}(p)). \end{cases} \tag{S.51}$$

(So $\text{div}\vec{w}(p)$ = trace of the endomorphism $d\vec{w}(p)$).

Tr and d are linear, hence div = Tr ∘ d is $\mathbb{R}$-linear (composed of two $\mathbb{R}$-linear maps).

Proposition S.24 *The divergence of a vector field is objective (is an invariant): Same value for all observers (objective quantity) intrinsic to $\vec{w}$.*

Proof. The differential and the trace are objective. (Or computation: $\vec{w} = \sum_i u^i\vec{a}_i = \sum_i v^i\vec{b}_i$ gives $v^i_{|j} = \sum_{k\ell} Q^i_ku^k_{|\ell}P^\ell_j$, see (S.41), thus $\sum_i v^i_{|i} = \sum_{ik\ell} P^\ell_iQ^i_ku^k_{|\ell} = \sum_{k\ell} \delta^\ell_ku^k_{|\ell} = \sum_k u^k_{|k}$.) ∎

Quantification: $\vec{w} \in \Gamma(U)$, $(\vec{e}_i)$ is a basis, $\vec{w} = \sum_{i=1}^n w_i\vec{e}_i$ with classical notations, and $w_{i|j}(p)$ are the components of the vector $d\vec{w}(p).\vec{e}_j(p)$ in the basis $(\vec{e}_i(p))$. Thus

$$\boxed{\text{div}\vec{w} = \sum_{i=1}^n w_{i|i}}. \tag{S.52}$$

Duality notations: $\vec{w} = \sum_{i=1}^n w^i\vec{e}_i$, $d\vec{w}.\vec{e}_j = \sum_{i=1}^n w^i_{|j}\vec{e}_i$, $[d\vec{w}]_{|\vec{e}} = [w^i_{|j}]$, $\text{div}\vec{w} = \sum_{i=1}^n w^i_{|i}$.

Cartesian basis $(\vec{e}_i)$ (classical notations): $dw_i.\vec{e}_j =^{\text{noted}} \frac{\partial w_i}{\partial x_j}$ and

$$w_{i|j} = \frac{\partial w_i}{\partial x_i}, \quad \text{thus} \quad \text{div}\vec{w} = \sum_{i=1}^{n} \frac{\partial w_i}{\partial x_i}. \tag{S.53}$$

Coordinate system basis $(\vec{e}_i)$ (duality notations): With the Christoffel symbols, cf. (S.33), (S.40) gives

$$w^i_{|i} = \frac{\partial w^i}{\partial q^i} + \sum_{i=1}^{n} w^k \gamma^i_{ik}, \quad \text{thus} \quad \text{div}\vec{w} = \sum_{i=1}^{n} \frac{\partial w^i}{\partial q^i} + \sum_{i,k=1}^{n} w^k \gamma^i_{ik}. \tag{S.54}$$

Exercice S.25 Prove:

$$\text{div}(f\vec{w}) = df.\vec{w} + f\,\text{div}\vec{w}. \tag{S.55}$$

Answer. $d(f\vec{w}) = \vec{w} \otimes df + f\,d\vec{w}$, thus $\text{Tr}(d(f\vec{w})) = \text{Tr}(\vec{w} \otimes df) + \text{Tr}(f\,d\vec{w}) = df.\vec{w} + f\,\text{Tr}(d\vec{w})$. Use a coordinate system if you prefer. ∴

Remark S.26 If α is a differential form, if $(\vec{e}_i)$ is a basis and (e^i) its dual basis, and if $\alpha = \sum_{i=1}^n \alpha_i e^i$, then $d\alpha = \sum_{i=1}^n \alpha_{i|j} e^i \otimes e^j$, with $\alpha_{i|j} := \vec{e}_i.d\alpha.\vec{e}_j$. Here it is impossible to define an objective trace of $d\alpha$ like $\sum_{i=1}^n \alpha_{i|i}$: The result depends on the choice of the basis (the Einstein convention is not satisfied, and e.g. with a Euclidean basis the result depends on the choice of unit of length: Foot? Meter?). Thus the objective (or intrinsic) divergence of a differential form is a nonsense. ∴

S.9 Objective divergence for 1 1 tensors

To create an objective divergence for a second order $\binom{1}{1}$ tensor $\underline{\underline{\tau}} \in T^1_1(U)$, in (S.46) we have to contract an admissible index with the "differential index k", So, no choice: Contract i and k to get $\widetilde{\text{div}}\underline{\underline{\tau}} := \sum_{i,j=1}^n \tau^i_{j|i} e^j$. Let us start with:

Definition S.27 Let $\vec{u} \in \Gamma(U)$ and $\ell \in \Omega^1(U)$ be C^1. The objective divergence of the elementary $\binom{1}{1}$ tensor $\vec{u} \otimes \ell \in T^1_1(U)$ is the differential form $\widetilde{\text{div}}(\vec{u} \otimes \ell) \in \Omega^1(U)$ defined by

$$\widetilde{\text{div}}(\vec{u} \otimes \ell) = (\text{div}\vec{u})\ell + d\ell.\vec{u}, \tag{S.56}$$

i.e. defined by $\widetilde{\text{div}}(\vec{u} \otimes \ell).\vec{w} = (\text{div}\vec{u})(\ell.\vec{w}) + (d\ell.\vec{w})\vec{u}$ for all $\vec{w} \in E$. And the objective divergence operator $\widetilde{\text{div}} : \left\{ \begin{matrix} T^1_1(U) & \to \Omega^1(U) \\ \underline{\underline{\tau}} & \to \widetilde{\text{div}}\underline{\underline{\tau}} \end{matrix} \right\}$ is the linear map defined on elementary tensors with (S.56).

Quantification: $(\vec{e}_i)$ is a basis, (e^i) its dual basis, $\vec{u} = \sum_i u^i \vec{e}_i$, $\ell = \sum_j \ell_j e^j$. Thus $\vec{u} \otimes \ell = \sum_{ij} u^i \ell_j \vec{e}_i \otimes e^j$, and (S.50)-(S.56) give

$$\widetilde{\text{div}}(\vec{u} \otimes \ell) = \sum_{i,j=1}^{n} (u^i_{|i} \ell_j + \ell_{j|i} u^i) e^j. \tag{S.57}$$

So for an elementary tensor $\underline{\underline{\tau}} = \vec{u} \otimes \ell$, $\underline{\underline{\tau}} = \sum_{ij} \tau^i_j \vec{e}_i \otimes e^j$, and $d\underline{\underline{\tau}}.\vec{e}_k = \sum_{ijk} \tau^i_{j|k} \vec{e}_i \otimes e^j$ and $\widetilde{\text{div}}(\underline{\underline{\tau}}) = \sum_{ij} \tau^i_{j|i} e^i$, with $\tau^i_{j|k} = u^i_{|k} \ell_j + u^i \ell_{j|k}$, here with $\tau^i_j = u^i \ell_j$, and $\tau^i_{j|k} = u^i_{|k} \ell_j + u^i \ell_{j|k}$, so $\tau^i_{j|i} = u^i_{|i} \ell_j + u^i \ell_{j|i}$.

Thus, by linearity of $\widetilde{\text{div}}$, for all tensors $\underline{\underline{\tau}} \in T^1_1(U)$, we have with (S.46):

$$\boxed{\widetilde{\text{div}}\underline{\underline{\tau}} = \sum_{i,j=1}^{n} \tau^i_{j|i} e^j}, \quad \text{i.e.} \quad [\widetilde{\text{div}}\underline{\underline{\tau}}]_{|\vec{e}} = \big(\textstyle\sum_i \tau^i_{1|i} \quad ... \textstyle\sum_i \tau^i_{n|i} \big) \tag{S.58}$$

(row matrix since $\widetilde{\text{div}}\underline{\underline{\tau}}$ is a differential form). I.e., we have contracted i and k in (S.46).

(Classical notations: $\widetilde{\text{div}}\underline{\underline{\tau}} := \sum_{i,j=1}^n \tau_{ij|i} \pi_{ej}$, i.e. $[\widetilde{\text{div}}\underline{\underline{\tau}}]_{|\vec{e}} = (\sum_i \tau_{i1|i} \quad ... \sum_i \tau_{in|i})$.)

So

$$\begin{aligned} &\text{Cartesian bases: } \widetilde{\text{div}}\underline{\underline{\tau}} = \sum_{i,j=1}^{n} \frac{\partial \tau^i_j}{\partial x^i} e^j, \\ &\text{Coord. sys. bases: } \widetilde{\text{div}}\underline{\underline{\tau}} = \sum_{i,j=1}^{n} \Big(\frac{\partial \tau^i_j}{\partial q^i} + \sum_{k=1}^{n} \tau^k_j \gamma^i_{ik} - \sum_{k=1}^{n} \tau^k_i \gamma^i_{kj} \Big) e^j. \end{aligned} \tag{S.59}$$

Indeed: With $\underline{\underline{\tau}} = \sum_j (\sum_i \tau^i_j \vec{e}_i) \otimes e^j = \sum_j \vec{w}_j \otimes e^j$ where $\vec{w}_j = \sum_i \tau^i_j \vec{e}_i$, the linearity of $\widetilde{\text{div}}$ gives $\widetilde{\text{div}}\underline{\underline{\tau}} = \sum_j \widetilde{\text{div}}(\vec{w}_j \otimes e^j)$; Thus, with (S.56): 1- Cartesian basis: $\text{div}\vec{w}_j = \sum_i \frac{\partial \tau^i_j}{\partial x^i}$ and $de^j = 0$ give $\widetilde{\text{div}}\underline{\underline{\tau}} =$

$\sum_j \sum_i \frac{\partial\tau^i_j}{\partial x^i} e^j = \sum_{ij} \tau^i_{j,i} e^j$, thus (S.59)$_1$; And 2- Coordinate system basis: $\mathrm{div}\vec{w}_j =^{(S.54)} \sum_i \frac{\partial\tau^i_j}{\partial q^i} + \sum_{ik} \tau^k_j \gamma^i_{ik}$ and $de^j.\vec{w}_j = \sum_k \tau^k_j de^j.\vec{e}_k = \sum_k \tau^k_j(-\sum_i \gamma^j_{ki} e^i)$, thus $\sum_j de^j.\vec{w}_j = -\sum_{ijk} \tau^k_j \gamma^j_{ki} e^i = -\sum_{ijk} \tau^k_i \gamma^i_{kj} e^j$, thus (S.59)$_2$.

Exercice S.28 Prove: If $f \in C^1(U;\mathbb{R})$ and $\underline{\underline{\tau}} = \sum_{i,j=1}^n \tau^i_j \vec{e}_i \otimes e^j \in T^1_1(U) \cap C^1$ then

$$\widetilde{\mathrm{div}}(f\underline{\underline{\tau}}) = df.\underline{\underline{\tau}} + f\,\widetilde{\mathrm{div}}\underline{\underline{\tau}}. \tag{S.60}$$

Answer. $f\underline{\underline{\tau}} = \sum_{ij} f\tau^i_j \vec{e}_i \otimes e^j$ gives $d(f\underline{\underline{\tau}}) = \sum_{ijk} (f\tau^i_j)_{|k} \vec{e}_i \otimes e^j \otimes e^k = \sum_{ijk} (f_{|k}\tau^i_j + f\tau^i_{j|k}) \vec{e}_i \otimes e^j \otimes e^k$, thus $\widetilde{\mathrm{div}}(f\underline{\underline{\tau}}) = \sum_{ij} (f_{|i}\tau^i_j + f\tau^i_{j|i}) e^j$; And $df.\underline{\underline{\tau}} + f\,\widetilde{\mathrm{div}}\underline{\underline{\tau}} = \sum_{ij} f_{|i}\tau^i_j e^j + f\,\sum_{ij} \tau^i_{j|i} e^j$. ∎

Exercice S.29 Prove: If $\underline{\underline{\tau}} \in T^1_1(U)$ and $\vec{w} \in \Gamma(U)$ then

$$\boxed{\mathrm{div}(\underline{\underline{\tau}}.\vec{w}) = \widetilde{\mathrm{div}}(\underline{\underline{\tau}}).\vec{w} + \underline{\underline{\tau}} \;(\!:\!)\; d\vec{w}}. \tag{S.61}$$

Answer. $\underline{\underline{\tau}} = \sum_{ij} \tau^i_j \vec{e}_i \otimes e^j$ and $\vec{w} = \sum_i w^i \vec{e}_i$ give $\underline{\underline{\tau}}.\vec{w} = \sum_{ij} \tau^i_j w^j \vec{e}_i$, thus $\mathrm{div}(\underline{\underline{\tau}}.\vec{w}) = \sum_{ij} \tau^i_{j|i} w^j + \tau^i_j w^j_{|i}$. ∎

Exercice S.30 If $\underline{\underline{\tau}} \in T^1_1(U)$ check with component calculations (since $\widetilde{\mathrm{div}}(\underline{\underline{\tau}})$ is objective):

$$[\widetilde{\mathrm{div}}(\underline{\underline{\tau}})]_{|b} = [\widetilde{\mathrm{div}}(\underline{\underline{\tau}})]_{|a}.P \quad \text{(covariance formula)}, \tag{S.62}$$

where P is the transition matrix from a basis $(\vec{a}_i)$ to a basis $(\vec{b}_i)$.

Answer. Let $\underline{\underline{\tau}} = \sum_{ij} \sigma^i_j \vec{a}_i \otimes a^j = \sum_{ij} \tau^i_j \vec{b}_i \otimes b^j$, so $\tau^i_j = \sum_{\lambda\mu} Q^i_\lambda \sigma^\lambda_\mu P^\mu_j$.

1- Cartesian bases: $\sum_i \tau^i_{j|i} = \sum_i d\tau^i_j.\vec{b}_i = \sum_i d(\sum_{\lambda\mu} Q^i_\lambda \sigma^\lambda_\mu P^\mu_j).(\sum_\nu P^\nu_i.\vec{a}_\nu) = \sum_{i\lambda\mu\nu} Q^i_\lambda P^\mu_j P^\nu_i (d\sigma^\lambda_\mu.\vec{a}_\nu) = \sum_{\lambda\mu\nu} \delta^\nu_\lambda P^\mu_j (d\sigma^\lambda_\mu.\vec{a}_\nu) = \sum_{\lambda\mu} P^\mu_j (d\sigma^\lambda_\mu.\vec{a}_\lambda) = \sum_\mu (\sum_\lambda \sigma^\lambda_{\mu|\lambda}) P^\mu_j$ as desired.

2- Coordinate system bases: $\sum_i \tau^i_{j|i} =^{(S.59)} \sum_i d\tau^i_j.\vec{e}_i + \sum_{i\ell} \tau^\ell_j \gamma^i_{i\ell,b} - \sum_{i\ell} \tau^i_\ell \gamma^\ell_{ij,b}$ (with j fixed); With

$$\begin{aligned}
\sum_i (d\tau^i_j.\vec{b}_i) &= \sum_{i\lambda\mu} Q^i_\lambda\,(d\sigma^\lambda_\mu.\vec{b}_i)\,P^\mu_j + \sum_{i\lambda\mu} (dQ^i_\lambda.\vec{b}_i)\,\sigma^\lambda_\mu\,P^\mu_j + \sum_{i\lambda\mu} Q^i_\lambda\,\sigma^\lambda_\mu\,(dP^\mu_j.\vec{b}_i) \\
&= \sum_{i\lambda\mu\nu} Q^i_\lambda P^\mu_j P^\nu_i (d\sigma^\lambda_\mu.\vec{a}_\nu) + \sum_{i\lambda\mu\nu} \sigma^\lambda_\mu\,P^\mu_j P^\nu_i (dQ^i_\lambda.\vec{a}_\nu) + \sum_{i\lambda\mu\nu} \sigma^\lambda_\mu\,Q^i_\lambda P^\nu_i (dP^\mu_j.\vec{a}_\nu) \\
&= \sum_{\lambda\mu} P^\mu_j (d\sigma^\lambda_\mu.\vec{a}_\lambda) - \sum_{i\lambda\mu\nu} \sigma^\lambda_\mu\,P^\mu_j Q^i_\lambda (dP^\nu_i.\vec{a}_\nu) + \sum_{\lambda\mu} \sigma^\lambda_\mu\,(dP^\mu_j.\vec{a}_\lambda)
\end{aligned}$$

since $P^\nu_i Q^i_\lambda = \delta^\nu_\lambda$ gives $P^\nu_i (dQ^i_\lambda.\vec{a}_\nu) - Q^i_\lambda (dP^\nu_i.\vec{a}_\nu)$. And, with (S.35),

$$\begin{aligned}
\sum_{i\ell} \tau^\ell_j \gamma^i_{i\ell,b} &= \sum_{i\ell} (\sum_{\lambda\mu} Q^\ell_\lambda \sigma^\lambda_\mu P^\mu_j)(\sum_{\alpha\beta\omega} Q^i_\alpha P^\beta_i P^\omega_\ell \gamma^\alpha_{\beta\omega,a} + \sum_{\alpha\beta} Q^i_\alpha P^\beta_i (dP^\alpha_\ell.\vec{a}_\beta)) \\
&= \sum_{\lambda\mu\alpha} \sigma^\lambda_\mu P^\mu_j \gamma^\alpha_{\alpha\lambda,a} + \sum_{\ell\lambda\mu\alpha} \sigma^\lambda_\mu Q^\ell_\lambda P^\mu_j (dP^\alpha_\ell.\vec{a}_\alpha),
\end{aligned} \tag{S.63}$$

and

$$\begin{aligned}
-\sum_{i\ell} \tau^i_\ell \gamma^\ell_{ij,b} &= -\sum_{i\ell} (\sum_{\lambda\mu} Q^i_\lambda \sigma^\lambda_\mu P^\mu_\ell)(\sum_{\alpha\beta\omega} P^\alpha_i P^\beta_j Q^\ell_\omega \gamma^\omega_{\alpha\beta,a} + \sum_{\alpha\omega} P^\alpha_i Q^\ell_\omega (dP^\omega_j.\vec{a}_\alpha)) \\
&= -\sum_{\lambda\mu\beta} \sigma^\lambda_\mu P^\beta_j \gamma^\mu_{\lambda\beta,a} - \sum_{\lambda\mu} \sigma^\lambda_\mu (dP^\mu_j.\vec{a}_\lambda).
\end{aligned} \tag{S.64}$$

Thus $\sum_i \tau^i_{j|i} = \sum_{\lambda\mu} P^\mu_j (d\sigma^\lambda_\mu.\vec{a}_\lambda) + \sum_{\lambda\mu\alpha} \sigma^\lambda_\mu P^\mu_j \gamma^\alpha_{\alpha\lambda,a} - \sum_{\lambda\mu\beta} \sigma^\lambda_\mu P^\beta_j \gamma^\mu_{\lambda\beta,a} = \sum_{\lambda\mu} P^\mu_j \sigma^\lambda_{\mu|\lambda}$ as desired. ∎

S.9.1 Divergence of a 2 0 tensor

Let $\underline{\underline{\tau}} \in T^2_0(U)$ and $\underline{\underline{\tau}} = \sum_{i,j=1}^n \tau^{ij} \vec{e}_i \otimes \vec{e}_j$, thus $d\underline{\underline{\tau}} = \sum_{i,j,k=1}^n \tau^{ij}{}_{|k} \vec{e}_i \otimes \vec{e}_j \otimes e^k$; Then two objective divergences may be defined: by contracting k with i, or k with j. (The Einstein convention is then satisfied.)

S.9.2 Divergence of a 0 2 tensor

Let $\underline{\underline{\tau}} = \sum_{i,j=1}^n \tau_{ij} e^i \otimes e^j \in T^0_2(U)$. Thus $d\underline{\underline{\tau}} = \sum_{i,j,k=1}^n \tau_{ij|k} e^i \otimes e^j \otimes e^k$, and there are no indices to contract to satisfy Einstein convention: There is no objective divergence of 0 2 tensors.

S.10 Euclidean framework and “classic divergence” of a tensor (subjective)

Let $\underline{\underline{\sigma}}$ be a C^1 tensor of order 2 of any kind. An observer chooses a (Cartesian) Euclidean basis $(\vec{e}_i)$ and call $(\cdot,\cdot)_g$ the associated Euclidean dot product. And he calls σ_{ij} the components of $\underline{\underline{\sigma}}$, e.g. writes $\underline{\underline{\sigma}}.\vec{e}_j = \sum_i \sigma_{ij}\vec{e}_i$.

Definition S.31 (Usual divergence in classical mechanics.) The divergence $\mathrm{div}\underline{\underline{\sigma}}$ of $\underline{\underline{\sigma}}$ relative to the basis $(\vec{e}_i)$, is the column matrix (it is not a vector)

$$\mathrm{div}_e\underline{\underline{\sigma}} = \begin{pmatrix} \sum_{j=1}^n \frac{\partial\sigma_{1j}}{\partial x^j} \\ \vdots \\ \sum_{j=1}^n \frac{\partial\sigma_{nj}}{\partial x^j} \end{pmatrix} \overset{\text{noted}}{=} \mathrm{div}\underline{\underline{\sigma}} \quad \text{(a matrix).} \tag{S.65}$$

(Take the divergences of the “row vectors” of $[\underline{\underline{\sigma}}]_{|e} = [\sigma_{ij}]$ to make the “column vector” $[\mathrm{div}_e\underline{\underline{\sigma}}]$.)

Proposition S.32 *The “so called vector”* $\mathrm{div}\underline{\underline{\sigma}}$*, in (S.67), is not a vector: It does not satisfy the change of basis formula: If* $(\vec{a}_i)$ *and* $(\vec{b}_i)$ *are bases, if* P *is the transition matrix from* $(\vec{a}_i)$ *to* $(\vec{b}_i)$*, if* $[\underline{\underline{\sigma}}]_{|\vec{a}} = [A_{ij}]$ *and* $[\underline{\underline{\sigma}}]_{|\vec{b}} = [B_{ij}]$*, with the divergence of* $\underline{\underline{\sigma}}$ *relative to* $(\vec{a}_i)$ *and* $(\vec{b}_i)$ *called* $\mathrm{div}_a\underline{\underline{\sigma}}$ *and* $\mathrm{div}_b\underline{\underline{\sigma}}$*, then neither a contravariant nor a covariant change of basis formula applies in general:*

$$\textit{neither}\quad [\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec{b}} \neq P^{-1}.[\mathrm{div}_a\underline{\underline{\sigma}}]_{|\vec{a}} \quad \textit{nor} \quad [\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec{b}}^T = [\mathrm{div}_a\underline{\underline{\sigma}}]_{|\vec{a}}^T.P \tag{S.66}$$

(compare with (S.62)). So $\mathrm{div}\underline{\underline{\sigma}}$ *as given in (S.67) is neither a contravariant vector nor a covariant vector (it is just a matrix which depends on an observer).*

Proof. Consider the simple case $\vec{b}_i = \lambda\vec{a}_i$, for all i, $\lambda > 1$: Transition matrix $P = \lambda I$, and $P^{-1} = \frac{1}{\lambda}I$.

For a $\binom{1}{1}$ tensor: $\underline{\underline{\sigma}} = \sum_{ij}(\sigma_b)^i_j\vec{b}_i \otimes b^j = \sum_{ij}(\sigma_a)^i_j\vec{a}_i \otimes a^j$, $[\underline{\underline{\sigma}}]_{|\vec{b}} = P^{-1}.[\underline{\underline{\sigma}}]_{|\vec{a}}.P = \frac{1}{\lambda}.[\underline{\underline{\sigma}}]_{|\vec{a}}.\lambda = [\underline{\underline{\sigma}}]_{|\vec{a}}$, i.e. $(\sigma_a)^i_j = (\sigma_b)^i_j$ for all i,j. Thus (S.67) gives $\mathrm{div}_b\underline{\underline{\sigma}} = \sum_{ij}(d(\sigma_b)^i_j.\vec{b}_j)\vec{b}_i = \sum_{ij}(d(\sigma_a)^i_j.(\lambda\vec{a}_j))(\lambda\vec{a}_i) = \lambda^2\mathrm{div}_a\underline{\underline{\sigma}}$. Thus $[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec{b}} \neq P^{-1}.[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec{a}}$ and $[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec{b}}^T \neq [\mathrm{div}_a\underline{\underline{\sigma}}]_{|\vec{a}}^T.P$.

For a $\binom{0}{2}$ tensor: $\underline{\underline{\sigma}} = \sum_{ij}\sigma_{b,ij}b^i \otimes b^j = \sum_{ij}\sigma_{a,ij}a^i \otimes a^j$, and $[\underline{\underline{\sigma}}]_{|\vec{b}} = P^T.[\underline{\underline{\sigma}}]_{|\vec{a}}.P = \lambda^2[\underline{\underline{\sigma}}]_{|\vec{a}}$, i.e. $\sigma_{b,ij} = \lambda^2\sigma_{a,ij}$ for all i,j. Thus (S.67) gives $\mathrm{div}_b\underline{\underline{\sigma}} = \sum_{ij}(d\sigma_{b,ij}.\vec{b}_j)\vec{b}_i = \lambda^2\sum_{ij}(d\sigma_{a,ij}.(\lambda\vec{a}_j))(\lambda\vec{a}_i) = \lambda^4\mathrm{div}_a\underline{\underline{\sigma}}$. Thus $[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec{b}} \neq P^{-1}.[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec{a}}$ and $[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec{b}}^T \neq [\mathrm{div}_a\underline{\underline{\sigma}}]_{|\vec{a}}^T.P$.

For a $\binom{2}{0}$ tensor: $\underline{\underline{\sigma}} = \sum_{ij}\sigma_b^{ij}\vec{b}_i \otimes \vec{b}_j = \sum_{ij}\sigma_a^{ij}\vec{a}_i \otimes \vec{a}_j$, and $[\underline{\underline{\sigma}}]_{|\vec{b}} = P^{-T}.[\underline{\underline{\sigma}}]_{|\vec{a}}.P^{-1} = \frac{1}{\lambda^2}[\underline{\underline{\sigma}}]_{|\vec{a}}$, i.e. $\sigma_b^{ij} = \frac{1}{\lambda^2}\sigma_a^{ij}$ for all i,j. Thus (S.67) gives $\mathrm{div}_b\underline{\underline{\sigma}} = \sum_{ij}(d\sigma_b^{ij}.\vec{b}_j)\vec{b}_i = \frac{1}{\lambda^2}\sum_{ij}(d\sigma_a^{ij}.(\lambda\vec{a}_j))(\lambda\vec{a}_i) = \mathrm{div}_a\underline{\underline{\sigma}}$. Thus $[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec{b}} \neq P^{-1}.[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec{a}}$ and $[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec{b}}^T \neq [\mathrm{div}_a\underline{\underline{\sigma}}]_{|\vec{a}}^T.P$. ∎

Remark: (S.65) can be written

$$\mathrm{div}\underline{\underline{\sigma}} = \sum_{ij}\frac{\partial\sigma_{ij}}{\partial x_j}\vec{E}_i \tag{S.67}$$

where $(\vec{E}_i)$ is the canonical basis in $\mathcal{M}_{n1}$ the space of $n*1$ column vectors.

T Natural canonical isomorphisms

T.1 The adjoint of a linear map

Setting of § A.12: E and F are vector spaces, $E^* = \mathcal{L}(E;\mathbb{R})$ and $F^* = \mathcal{L}(F;\mathbb{R})$ are their dual spaces, and the adjoint of a linear map $\mathcal{P} \in \mathcal{L}(E;F)$ is the linear map $\mathcal{P}^* \in \mathcal{L}(F^*;E^*)$ canonically defined by

$$\forall\ell \in F^*, \quad \mathcal{P}^*(\ell) := \ell \circ \mathcal{P}, \quad \text{written} \quad \mathcal{P}^*.\ell = \ell.\mathcal{P} \tag{T.1}$$

(dot notations $\mathcal{P}^*(\ell) =^{\text{noted}} \mathcal{P}^*.\ell$ and $\ell\circ\mathcal{P} =^{\text{noted}} \ell.\mathcal{P}$ since ℓ and $\mathcal{P}^*$ are linear), i.e., for all $(\ell,\vec{u}) \in F^* \times E$,

$$\mathcal{P}^*(\ell)(\vec{u}) = \ell(\mathcal{P}(\vec{u})), \quad \text{written} \quad (\mathcal{P}^*.\ell).\vec{u} = \ell.\mathcal{P}.\vec{u}. \tag{T.2}$$

Interpretation: If $\mathcal{P}$ is the push-forward of vector fields, then $\mathcal{P}^*$ is the pull-back of differential forms, see remark 7.5. In particular, it will be interpreted with $\mathcal{P} \in \mathcal{L}_i(E;F)$ (linear and invertible = a change of observer).

T.2 An isomorphism $E \simeq E^*$ is never natural (never objective)

Two observers A and B consider a linear map $L \in \mathcal{L}(E;E^*)$; Let $\mathcal{P} \in \mathcal{L}(E;E)$ be the change of observer endomorphism. Willing to work together, A and B ("naturally") consider the diagram

$$\begin{tikzcd} E \arrow[r, "L"] \arrow[d, "\mathcal{P}"'] & E^* & \leftarrow \text{considered by observer } A \\ E \arrow[r, "L"'] & E^* \arrow[u, "\mathcal{P}^*"'] & \leftarrow \text{considered by observer } B \end{tikzcd} \tag{T.3}$$

Definition T.1 (Spivak [17].) A linear map $L \in \mathcal{L}(E;E^*)$ is natural iff the diagram (T.3) commutes for all $\mathcal{P} \in \mathcal{L}(E;E)$:

$$L \in \mathcal{L}(E;E^*) \text{ is natural} \iff \forall \mathcal{P} \in \mathcal{L}(E;E),\quad \mathcal{P}^* \circ L \circ \mathcal{P} = L. \tag{T.4}$$

(In that case, if A computes $L.\vec{u}$ with the top line of the diagram, if B computes with the bottom line of the diagram, then they can easily check their results since here $L.\vec{u} = (\mathcal{P}^* \circ L \circ \mathcal{P}).\vec{u}$.)

Question: Does there exist an endomorphism L such that the diagram (T.3) commutes for all change of observers? That is, do we have

$$\exists ? L \in \mathcal{L}(E;E),\ \forall \mathcal{P} \in \mathcal{L}_i(E;E),\quad \mathcal{P}^* \circ L \circ \mathcal{P} = L\ ? \tag{T.5}$$

Answer: Always ***no*** (if $L \neq 0$):

Theorem T.2 *A (non-zero) linear map $L \in \mathcal{L}(E;E^*)$ is not natural: If $L \in \mathcal{L}(E;E^*) - \{0\}$, then*

$$\exists \mathcal{P} \in \mathcal{L}_i(E;E) \quad s.t. \quad L \neq \mathcal{P}^* \circ L \circ \mathcal{P}. \tag{T.6}$$

Proof. (Spivak [17].) It suffices to prove this proposition for $E = \vec{\mathbb{R}}$. Let $L \in \mathcal{L}(\vec{\mathbb{R}};(\vec{\mathbb{R}})^*)$, $L \neq 0$.

Let $(\vec{a}_1)$ be a basis in $\vec{\mathbb{R}}$ (chosen by A). Let $(\vec{b}_1)$ be a basis in $\vec{\mathbb{R}}$ (chosen by B).

Consider $\mathcal{P} \in \mathcal{L}_i(\vec{\mathbb{R}};\vec{\mathbb{R}})$ defined by $\mathcal{P}(\vec{a}_1) = \vec{b}_1$ (change of observer), and let $\lambda \in \mathbb{R}$ s.t. $\vec{b}_1 = \lambda \vec{a}_1$. Then (T.1) gives $\mathcal{P}^*(\ell)(\vec{a}_1) := \ell(\mathcal{P}(\vec{a}_1)) = \ell(\vec{b}_1) = \ell(\lambda\vec{a}_1) = \lambda\ell(\vec{a}_1)$, thus $\mathcal{P}^*(\ell) = \lambda\ell$ for all $\ell \in (\vec{\mathbb{R}})^*$.

Thus $\mathcal{P}^*(L(\mathcal{P}(\vec{a}_1))) = \mathcal{P}^*(L(\lambda\vec{a}_1)) = \lambda\mathcal{P}^*(L(\vec{a}_1)) = \lambda^2 L(\vec{a}_1) \neq L(\vec{a}_1)$ when $\lambda^2 \neq 1$. E.g., $\mathcal{P} = 2I$ gives $L \neq \mathcal{P}^* \circ L \circ \mathcal{P}$ $(= 4L)$, thus (T.6): A (non-zero) linear map $E \to E^*$ cannot be natural. ∴

Example T.3 Consider E s.t. $\dim E = 1$, and consider the linear map $L \in \mathcal{L}(E;E^*)$ which sends a basis $(\vec{a}_1)$ onto its dual basis (π_{a1}), so L is defined by $L.\vec{a}_1 := \pi_{a1}$.

Question: If $(\vec{b}_1)$ is another basis, $\lambda \neq \pm 1$ and $\vec{b}_1 = \lambda\vec{a}_1$ (change of unit of measurement), does $L.\vec{b}_1 = \pi_{b1}$, i.e. does L also sends $(\vec{b}_1)$ onto its dual basis?

Answer: No. Indeed, $\vec{b}_1 = \lambda\vec{a}_1$ gives $\pi_{b1} = \frac{1}{\lambda}\pi_{a1}$, thus $L.\vec{b}_1 = \lambda L.\vec{a}_1 = \lambda\pi_{a1} = \lambda^2\pi_{b1} \neq \pi_{b1}$ since $\lambda^2 \neq 1$. In words: L is not natural, cf. (T.6).

A different presentation: Let L_A and L_B be defined by $L_A.\vec{a}_j = \pi_{aj}$ and $L_B.\vec{b}_j = \pi_{bj}$ for all j. And suppose that $\vec{b}_j = \lambda\vec{a}_j$ for all j. Then, $L_A.\vec{b}_j = \lambda L_A.\vec{a}_j = \lambda\pi_{aj} = \lambda^2\pi_{bj} = \lambda^2 L_B.\vec{b}_j \neq L_B.\vec{b}_j$ when $\lambda^2 \neq 1$, that is, $L_A \neq L_B$ when $\lambda^2 \neq 1$: An operator that sends a basis onto its dual basis is not natural. ∴

Example T.4 Let $(\cdot,\cdot)_g$ be an inner dot product in $E = \vec{\mathbb{R}}^n$. Let $\vec{R}_g \in \mathcal{L}(E^*;E)$ be the Riesz representation map, that is, defined by $\vec{R}_g(\ell) = \vec{\ell}_g$ where $\vec{\ell}_g$ is defined by $(\vec{\ell}_g,\vec{v})_g = \ell.\vec{v}$ for all $\vec{v} \in \vec{\mathbb{R}}^n$, cf (F.3).

Question: Is $\vec{R}_g$ natural?

Answer: No: Consider the diagram

$$\left(\begin{tikzcd} E^* \arrow[r, "\vec{R}_g"] \arrow[d, "\mathcal{P}^*"'] & E \\ E^* \arrow[r, "\vec{R}_g"'] & E \arrow[u, "\mathcal{P}"'] \end{tikzcd}\right)$$

with $\mathcal{P} = \lambda I$, $\lambda \neq \pm 1$. Then $\mathcal{P}^* = \lambda I$, and $\mathcal{P}.\vec{R}_g.\mathcal{P}^*.\ell = \lambda^2\vec{R}_g.\ell \neq \vec{R}_g.\ell$ gives $\mathcal{P}.\vec{R}_g.\mathcal{P}^* \neq \vec{R}_g$: So $\vec{R}_g$ is not natural, cf. (T.6). (You may prefer to consider the diagram (T.3) with $L = \vec{R}_g^{-1}$.)

A different presentation: Consider two distinct Euclidean dot products $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ (e.g., built with a foot and built with a metre). So $(\cdot,\cdot)_h = \lambda^2(\cdot,\cdot)_g$ with $\lambda^2 \neq 1$. Let $\vec{R}_g, \vec{R}_h \in \mathcal{L}(\mathbb{R}^{n*};\mathbb{R}^n)$ be the Riesz operators relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$, that is $\vec{R}_g.\ell = \vec{\ell}_g$ and $\vec{R}_h.\ell = \vec{\ell}_h$ are given by $\ell.\vec{v} = (\vec{\ell}_g,\vec{v})_g = (\vec{\ell}_h,\vec{v})_h$ for all $\vec{v} \in \vec{\mathbb{R}}^n$. We have $\vec{\ell}_h = \lambda^2\vec{\ell}_g$, cf. (F.11), thus $\vec{R}_h = \lambda^2\vec{R}_g \neq \vec{R}_g$ since $\lambda^2 \neq 1$: A Riesz representation operator is not natural (it is observer dependent). ∴

T.3 Natural canonical isomorphism $E \simeq E^{**}$

Two observers A and B consider the same linear map $L \in \mathcal{L}(E;E^{**})$ (where $E^{**} = (E^*)^* = \mathcal{L}(E^*;\mathbb{R})$). Willing to work together, they ("naturally") consider the diagram

$$\begin{tikzcd}
E \arrow[r, "L"] \arrow[d, "\mathcal{P}"'] & E^{**} \arrow[d, "\mathcal{P}^{**}"] & \leftarrow \text{considered by observer } A \\
E \arrow[r, "L"'] & E^{**} & \leftarrow \text{considered by observer } B
\end{tikzcd} \tag{T.7}$$

where $\mathcal{P} \in \mathcal{L}(E;E)$ is a linear diffeomorphism, $\mathcal{P}^* \in \mathcal{L}(E^*;E^*)$ its adjoint, given by $\mathcal{P}^*(\ell) = \ell \circ \mathcal{P}$ cf. (T.1), and $\mathcal{P}^{**} \in \mathcal{L}_i(E^{**};E^{**})$ the adjoint of $\mathcal{P}^*$, thus given by $\mathcal{P}^{**}(u) = u \circ \mathcal{P}^*$ for all $u \in E^{**}$ cf. (T.1), i.e. $\mathcal{P}^{**}$ is given by, for all $(\ell,u) \in E^* \times E^{**}$,

$$(\mathcal{P}^{**}(u))(\ell) = u(\ell \circ \mathcal{P}), \quad \text{i.e.} \quad (\mathcal{P}^{**}.u).\ell = u.(\ell.\mathcal{P}). \tag{T.8}$$

Question: Does there exist a linear map $L \in \mathcal{L}(E;E^{**})$ that is natural?
Answer: Yes (particular case of the next proposition):

Proposition T.5 *The canonical isomorphism*

$$\mathcal{J}_E : \begin{cases} E & \to E^{**} \\ \vec{u} & \to u = \mathcal{J}_E(\vec{u}) \quad \textit{defined by} \quad \mathcal{J}_E(\vec{u})(\ell) := \ell.\vec{u}, \quad \forall \ell \in E^*, \end{cases} \tag{T.9}$$

is natural, that is, F being another finite dimensional vector space, the diagram

$$\begin{tikzcd}
E \arrow[r, "\mathcal{J}_E"] \arrow[d, "\mathcal{P}"'] & E^{**} \arrow[d, "\mathcal{P}^{**}"] & & E \arrow[r, "\mathcal{J}"] \arrow[d, "\mathcal{P}"'] & E^{**} \arrow[d, "\mathcal{P}^{**}"] \\
F \arrow[r, "\mathcal{J}_F"'] & F^{**} & \textit{written} & F \arrow[r, "\mathcal{J}"'] & F^{**}
\end{tikzcd} \tag{T.10}$$

commutes for all $\mathcal{P} \in \mathcal{L}(E;F)$, i.e.

$$\forall \mathcal{P} \in \mathcal{L}(E;F), \quad \mathcal{P}^{**} \circ \mathcal{J}_E = \mathcal{J}_F \circ \mathcal{P}, \quad \textit{and we write} \quad E \simeq E^{**}. \tag{T.11}$$

Thus we can use the unambiguous notation (observer independent)

$$\mathcal{J}(\vec{u}) \overset{noted}{=} \vec{u}, \quad \textit{and} \quad \mathcal{J}(\vec{u}).\ell \overset{noted}{=} \vec{u}.\ell \quad (= \ell.\vec{u}). \tag{T.12}$$

(And $u = \mathcal{J}(\vec{u})$ is the derivation operator in the direction $\vec{u}$.)

Proof. (Spivak [17].) It is trivial that $\mathcal{J}_E$ is linear and bijective (E is finite dimensional): It is an isomorphism. Then $(\mathcal{P}^{**} \circ \mathcal{J}_E(\vec{u}))(\ell) \overset{(T.8)}{=} \mathcal{J}_E(\vec{u})(\ell.\mathcal{P}) \overset{(T.9)}{=} (\ell \circ \mathcal{P})(\vec{u}) = \ell(\mathcal{P}(\vec{u})) \overset{(T.9)}{=} \mathcal{J}_F(\mathcal{P}(\vec{u}))(\ell)$, for all $\ell \in F^*$ and all $\vec{u} \in E$, thus $\mathcal{P}^{**} \circ \mathcal{J}_E(\vec{u}) = \mathcal{J}_F(\mathcal{P}(\vec{u}))$, for all $\vec{u} \in E$, thus $\mathcal{P}^{**} \circ \mathcal{J}_E = \mathcal{J}_F \circ \mathcal{P}$. ∴

Proposition T.6 *(Characterization of $\mathcal{J}_E$.) $\mathcal{J}_E$ sends any basis $(\vec{a}_i)$ onto its bidual basis. (Expected, since $\mathcal{J}_E(\vec{u})$ is the directional derivative in the direction $\vec{u}$, whatever $\vec{u}$.)*

Proof. Let $(\vec{a}_i)$ be a basis and (π_{ai}) be its dual basis (defined by $\pi_{ai}.\vec{a}_j = \delta_{ij}$ for all i,j). Then (T.9) gives $\mathcal{J}_E(\vec{a}_j).\pi_{ai} = \pi_{ai}.\vec{a}_j = \delta_{ij}$ for all i,j, thus $(\mathcal{J}_E(\vec{a}_j))$ is the dual basis of (π_{ai}), i.e., is the bidual basis of $(\vec{a}_i)$; True for all basis: $\mathcal{J}_E(\vec{b}_j).\pi_{bi} = \pi_{bi}.\vec{b}_j = \delta_{ij}$ for all i,j. ∴

T.4 Natural canonical isomorphisms $\mathcal{L}(E;F) \simeq \mathcal{L}(F^*,E;\mathbb{R}) \simeq \mathcal{L}(E^*;F^*)$

E,F,A,B are finite dimensional vector spaces. Consider the canonical isomorphism

$$\mathcal{J}_{EF} : \begin{cases} \mathcal{L}(E;F) & \to \mathcal{L}(F^*,E;\mathbb{R}) \\ L & \to \widetilde{L} = \mathcal{J}_{EF}(L) \quad \text{where} \quad \widetilde{L}(\ell,\vec{u}) := \ell.L.\vec{u}, \quad \forall (\ell,\vec{u}) \in F^* \times E. \end{cases} \tag{T.13}$$

Let $\mathcal{P}_1\in\mathcal{L}_i(E;A)$ and $\mathcal{P}_2\in\mathcal{L}(F;B)$, and consider the diagram

$$\begin{tikzcd}
\mathcal{L}(E;F) \arrow[r, "\mathcal{J}_{EF}"] \arrow[d, "\mathcal{I}_{\mathcal{P}}"'] & \mathcal{L}(F^*,E;\mathbb{R}) \arrow[d, "\widetilde{\mathcal{I}_{\mathcal{P}}}"] \\
\mathcal{L}(A;B) \arrow[r, "\mathcal{J}_{AB}"'] & \mathcal{L}(B^*,A;\mathbb{R})
\end{tikzcd} \tag{T.14}$$

where

$$\mathcal{I}_{\mathcal{P}}(L)=\mathcal{P}_2.L.\mathcal{P}_1^{-1}\quad\text{and}\quad\widetilde{\mathcal{I}_{\mathcal{P}}}(\widetilde{L})(b,\vec{a})=\widetilde{L}(b.\mathcal{P}_2,\mathcal{P}_1^{-1}.\vec{a})\quad\forall(b,\vec{a})\in B^*\times A. \tag{T.15}$$

($\mathcal{I}_{\mathcal{P}}$ and $\widetilde{\mathcal{I}_{\mathcal{P}}}$ are the push-forwards for linear maps $L\in\mathcal{L}(E;F)$ and for bilinear forms $\widetilde{L}\in\mathcal{L}(F^*,E;\mathbb{R})$.)

Proposition T.7 *The canonical isomorphism $\mathcal{J}_{EF}$ is natural, that is, the diagram (T.14) commutes for all $\mathcal{P}_1\in\mathcal{L}_i(E,A)$ and all $\mathcal{P}_2\in\mathcal{L}(F,B)$:*

$$\widetilde{\mathcal{I}_{\mathcal{P}}}\circ\mathcal{J}_{EF}=\mathcal{J}_{AB}\circ\mathcal{I}_{\mathcal{P}},\quad\text{and}\quad\mathcal{L}(E;F)\overset{natural}{\simeq}\mathcal{L}(F^*,E;\mathbb{R}). \tag{T.16}$$

Thus $\mathcal{L}(E^*;F^*)\overset{natural}{\simeq}\mathcal{L}(E;F)$.

Proof. $\mathcal{J}_{AB}(\mathcal{I}_{\mathcal{P}}(L))(b,\vec{a})\overset{(T.13)}{=}b.\mathcal{I}_{\mathcal{P}}(L).\vec{a}\overset{(T.15)}{=}b.(\mathcal{P}_2.L.\mathcal{P}_1^{-1}).\vec{a}=(b.\mathcal{P}_2).L.(\mathcal{P}_1^{-1}.\vec{a})\overset{(T.13)}{=}\mathcal{J}_{EF}(L)(b.\mathcal{P}_2,\mathcal{P}_1^{-1}.\vec{a})$ $\overset{(T.15)}{=}\widetilde{\mathcal{I}_{\mathcal{P}}}(\mathcal{J}_{EF}(L))(b,\vec{a})$, true for all $L\in\mathcal{L}(E;F)$, $b\in B^*$, $\vec{a}\in A$, thus (T.16).

Thus $\mathcal{L}(E^*;F^*)\overset{(T.16)}{\simeq}\mathcal{L}((F^*)^*,E^*;\mathbb{R})\overset{(T.11)}{\simeq}\mathcal{L}(F,E^*;\mathbb{R})\overset{(T.16)}{\simeq}\mathcal{L}(E^{**};F)\overset{(T.11)}{\simeq}\mathcal{L}(E;F)$. ∴

Consider the canonical isomorphism (defines the transposed of a bilinear map)

$$\mathcal{K}_{EF}:\left\{\begin{array}{rl}\mathcal{L}(E,F;\mathbb{R})&\to\mathcal{L}(F,E;\mathbb{R})\\ T&\to\mathcal{K}_{EF}(T)\end{array}\right\},\quad\mathcal{K}_{EF}(T)(\vec{u},\vec{v}):=T(\vec{v},\vec{u}),\quad\forall(\vec{u},\vec{v})\in E\times F, \tag{T.17}$$

and $Z_{AB}\in\mathcal{L}(E,F;\mathbb{R})\to\mathcal{L}(A,B;\mathbb{R})$ defined by $Z_{AB}(T)(\vec{a},\vec{b}):=T(\mathcal{P}_1^{-1}.\vec{a},\mathcal{P}_2^{-1}.\vec{b})$ for all $(\vec{a},\vec{b})\in A\times B$.

Proposition T.8 *The canonical isomorphism $\mathcal{K}_{EF}$ is natural: For all $(\mathcal{P}_1,\mathcal{P}_2)\in\mathcal{L}_i(E;A)\times\mathcal{L}(F;B)$, the diagram*

$$\begin{tikzcd}
\mathcal{L}(E,F;\mathbb{R}) \arrow[r, "\mathcal{K}_{EF}"] \arrow[d, "Z_{AB}"'] & \mathcal{L}(F,E;\mathbb{R}) \arrow[d, "Z_{BA}"] \\
\mathcal{L}(A,B;\mathbb{R}) \arrow[r, "\mathcal{K}_{AB}"'] & \mathcal{L}(B,A;\mathbb{R})
\end{tikzcd}$$

commutes: $\mathcal{L}(E,F;\mathbb{R})\overset{natural}{\simeq}\mathcal{L}(F,E;\mathbb{R})$.

Proof. $\mathcal{K}_{EF}(Z_{AB}(T))(\vec{b},\vec{a})=Z_{AB}(T)(\vec{a},\vec{b})=T(\mathcal{P}_2^{-1}.\vec{b},\mathcal{P}_1^{-1}.\vec{a})$ and $Z_{BA}(\mathcal{K}_{EF}(T))(\vec{a},\vec{b})=\mathcal{K}_{EF}(T)(\mathcal{P}_1^{-1}.\vec{a},\mathcal{P}_2^{-1}.\vec{b})=$ $T(\mathcal{P}_2^{-1}.\vec{b},\mathcal{P}_1^{-1}.\vec{a})$, thus $\mathcal{K}_{AB}\circ Z_{AB}=Z_{BA}\circ\mathcal{K}_{EF}$. ∴

T.5 Natural canonical isomorphisms $\mathcal{L}(E;\mathcal{L}(E;F))\simeq\mathcal{L}(E,E;F)\simeq\mathcal{L}(F^*,E,E;\mathbb{R})$

For application to the second order derivative $d(d\vec{u})\simeq d^2\vec{u}$ and, with $\vec{u}\in T_0^1(U)$, the notation $d\vec{u}\in T_1^1(U)$, then $d^2\vec{u}\in T_2^1(U)$, ..., $d^k\vec{u}\in T_k^1(U)$, ...

Consider the canonical isomorphism

$$\mathcal{J}_{12E}:\left\{\begin{array}{rl}\mathcal{L}(E;\mathcal{L}(E;F))&\to\mathcal{L}(E,E;F)\\ T_1&\to T_2=\mathcal{J}_{12E}(T_1)\end{array}\right\},\quad\mathcal{J}_{12E}(T_1)(\vec{u}_1,\vec{u}_2):=T_1(\vec{u}_1).\vec{u}_2\in F,\quad\forall\vec{u}_1,\vec{u}_2\in E, \tag{T.18}$$

and the canonical isomorphism

$$\mathcal{J}_{23E}:\left\{\begin{array}{rl}\mathcal{L}(E,E;F)&\to\mathcal{L}(F^*,E,E;\mathbb{R})\\ T_2&\to\mathcal{J}_{23E}(T_2)=T_3\end{array}\right\},\quad T_3(\ell,\vec{u},\vec{v}):=\ell.T_2(\vec{u}_1,\vec{u}_2),\quad\forall\vec{u}_1,\vec{u}_2\in E,\ \forall\ell\in F^*. \tag{T.19}$$

Proposition T.9 *$\mathcal{J}_{12}$ and $\mathcal{J}_{23}$ are natural. Thus $\mathcal{J}_{23}\circ\mathcal{J}_{12}$ is natural.*

Proof. 1- We have to prove that the following diagram commutes:

$$\begin{tikzcd} \mathcal{L}(E;\mathcal{L}(E;F)) \arrow[r, "\mathcal{J}_{12E}"] \arrow[d, "Z_{AB}"'] & \mathcal{L}(E,E;F) \arrow[d, "Y_{AB}"] \\ \mathcal{L}(A;\mathcal{L}(A;B)) \arrow[r, "\mathcal{J}_{12A}"] & \mathcal{L}(A,A;B) \end{tikzcd} \quad \text{where} \quad \begin{aligned} &Z_{AB}(T_1)(\vec{a}_1).\vec{a}_2 := T_1(\mathcal{P}_1^{-1}.\vec{a}_1).(\mathcal{P}_1^{-1}.\vec{a}_2),\\ &Y_{AB}(T_2)(\vec{a}_1,\vec{a}_2) = T_2(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2), \end{aligned} \tag{T.20}$$

(the “push-forwards) for all $\vec{a}_1,\vec{a}_2 \in A$ and $L_{AB} \in \mathcal{L}(A;B)$.

Let $T_1 \in \mathcal{L}(E;\mathcal{L}(E;F))$. We have
$\mathcal{J}_{12A}(Z_{AB}(T_1))(\vec{a}_1).\vec{a}_2 = Z_{AB}(T_1)(\vec{a}_1).\vec{a}_2 = T_1(\mathcal{P}_1^{-1}.\vec{a}_1).(\mathcal{P}_1^{-1}.\vec{a}_2)$, and
$Y_{AB}(\mathcal{J}_{12E}(T_1))(\vec{a}_1,\vec{a}_2) = \mathcal{J}_{12E}(T_1)(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2) = T_1(\mathcal{P}_1^{-1}.\vec{a}_1).(\mathcal{P}_1^{-1}.\vec{a}_2)$,
thus $\mathcal{J}_{12A} \circ Z_{AB} = Y_{AB} \circ \mathcal{J}_{12E}$, thus $\mathcal{J}_{12}$ is natural.

2- We have to prove that the following diagram commutes:

$$\begin{tikzcd} \mathcal{L}(E,E;F) \arrow[r, "\mathcal{J}_{23E}"] \arrow[d, "Z_{AB}"'] & \mathcal{L}(F^*,E,E;\mathbb{R}) \arrow[d, "Y_{AB}"] \\ \mathcal{L}(A,A;B) \arrow[r, "\mathcal{J}_{23A}"] & \mathcal{L}(B^*,A,A;\mathbb{R}) \end{tikzcd} \quad \text{where} \quad \begin{aligned} &\ell_B.Z_{AB}(T_2)(\vec{a}_1,\vec{a}_2) := (\ell_B.\mathcal{P}_2).T_2(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2),\\ &Y_{AB}(T_3)(\ell_B,\vec{a}_1,\vec{a}_2) = T_3(\ell_B.\mathcal{P}_2,\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2), \end{aligned} \tag{T.21}$$

(the “push-forwards) for all $\vec{a}_1,\vec{a}_2 \in A$ and $\ell_B \in B^*$.

Let $T_2 \in \mathcal{L}(E,E;F)$. We have
$\mathcal{J}_{23A}(\ell_B, Z_{AB}(T_2)(\vec{a}_1,\vec{a}_2)) = \ell_B.Z_{AB}(T_2)(\vec{a}_1,\vec{a}_2) = (\ell_B.\mathcal{P}_2).T_2(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2)$, and
$Y_{AB}(\mathcal{J}_{23A}(T_2))(\ell_B,\vec{a}_1,\vec{a}_2) = \mathcal{J}_{23A}(T_2)(\ell_B.\mathcal{P}_2,\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2) = \ell_B.\mathcal{P}_2.T_2(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2)$
thus $\mathcal{J}_{23A} \circ Z_{AB} = Y_{AB} \circ \mathcal{J}_{23E}$, thus $\mathcal{J}_{23}$ is natural. ∴

U Distribution in brief: A covariant concept

We refer to the books of Laurent Schwartz for a full description. In continuum mechanics, with Ω an open set in $\mathbb{R}^n$ and for the space of the finite energy functions $L^2(\Omega)$ and its sub-spaces, a distribution gives a covariant formulation for the virtual power, as used by Germain.

U.1 Definitions

Usual notations: Let $p \in [1,\infty[$ (e.g. $p = 2$ for finite energy functions), and let

$$L^p(\Omega) := \{f : \Omega \to \mathbb{R} : \int_\Omega |f(x)|^p \, d\Omega < \infty\} \quad \text{and} \quad ||f||_p = (\int_\Omega |f(x)|^p \, d\Omega)^{\frac{1}{p}}, \tag{U.1}$$

the space of functions such that $|f|^p$ is Lebesgue integrable with $||.||_p$ its usual norm. Then $(L^p(\Omega), ||.||_{L^p})$ is a Banach space (a complete normed space). And let

$$L^\infty(\Omega) := \{f : \Omega \to \mathbb{R} : \sup_{x\in\Omega}(|f(x)|) < \infty\}, \quad \text{and} \quad ||f||_\infty = \sup_{x\in\Omega}(|f(x)|), \tag{U.2}$$

the space of Lebesgue measurable bounded functions with $||.||_\infty$ its usual norm. Then $(L^\infty(\Omega), ||.||_{L^\infty})$ is a Banach space (a complete normed space).

Definition U.1 If $f \in \mathcal{F}(\Omega;\mathbb{R})$, then its support is the set

$$\text{supp}(f) := \overline{\{x \in \Omega : f(x) \neq 0\}} = \text{the closure of } \{x \in \Omega : f(x) \neq 0\} \tag{U.3}$$

$=$ the set where it is interesting to study f.

The closure is required: E.g., if $\Omega =]0,2\pi[$ and $f(x) = \sin x$ for all $x \in \omega$, then $\{x \in \Omega : f(x) \neq 0\} =]0,\pi[\cup]\pi,2\pi[$; And the point π is a point of interest since sin varies in its vicinity: $f'(\pi) = \cos(\pi) = -1 \neq 0$; So it is the closure $\text{supp}(f) := \overline{]0,\pi[\cup]\pi,2\pi[} = [0,2\pi]$ that is considered.

Definition U.2 (Schwartz notation, D being the letter after C:) Let

$$\mathcal{D}(\Omega) := C_c^\infty(\Omega;\mathbb{R}) = \{\varphi \in C^\infty(\Omega;\mathbb{R}) \text{ s.t. supp}(\varphi) \text{ is compact in } \Omega\}. \tag{U.4}$$

E.g., $\Omega = \mathbb{R}$, $\varphi(x) := e^{-\frac{1}{1-x^2}}$ if $x \in]-1,1[$ and $\varphi(x) := 0$ elsewhere: $\varphi \in \mathcal{D}(\mathbb{R})$ with $\mathrm{supp}(\varphi) = [-1,1]$.

And $\mathcal{D}(\Omega)$ is a vector space which is dense in $(L^p(\Omega), ||.||_{L^p})$ for any $p \in [1,\infty[$.

Definition U.3 A distribution in Ω is a linear $\mathcal{D}(\Omega)$-continuous[3] function

$$T : \begin{cases} \mathcal{D}(\Omega) & \to \mathbb{R} \\ \quad \varphi & \to T(\varphi) \overset{\text{noted}}{=} \langle T, \varphi \rangle \end{cases} \tag{U.5}$$

The space of distribution in Ω is named $\mathcal{D}'(\Omega)$ (the dual of $\mathcal{D}(\Omega)$).

The notation $\langle T, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle T, \varphi \rangle$ is the "duality bracket" = the "covariance–contravariance bracket" between a linear function $T \in \mathcal{D}'(\Omega)$ and a vector $\varphi \in \mathcal{D}(\Omega)$.

Definition U.4 Let $f \in L^p(\Omega)$. The regular distribution $T_f \in \mathcal{D}'(\Omega)$ associated to f is defined by

$$T_f(\varphi) := \int_\Omega f(x)\varphi(x)\, d\Omega, \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{U.6}$$

So T_f is a measuring instrument with density $dm_f(x) = f(x)\, d\Omega$, i.e. $T_f(\varphi) := \int_\Omega \varphi(x)\, dm_f(x)$.

Definition U.5 Let $x_0 \in \mathbb{R}^n$. The Dirac measure at x_0 is the distribution $T \overset{\text{noted}}{=} \delta_{x_0} \in \mathcal{D}'(\mathbb{R})$ defined by, for all $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\delta_{x_0}(\varphi) = \varphi(x_0), \quad \text{i.e.} \quad \langle \delta_{x_0}, \varphi \rangle = \varphi(x_0). \tag{U.7}$$

And δ_{x_0} is not a regular distribution (δ_{x_0} is not a density measure): There is no integrable function f such that $T_f = \delta_{x_0}$. Interpretation: δ_{x_0} corresponds to an ideal measuring device: The precision is perfect at x_0 (gives the exact value $\varphi(x_0)$ at x_0). In real life δ_{x_0} is the ideal approximation of T_{f_n} where f_n is e.g. given by $f_n(x) = n1_{[x_0, x_0+\frac{1}{n}]}$ (drawing): For all $\varphi \in \mathcal{D}(\Omega)$, $T_{f_n}(\varphi) \longrightarrow_{n\to\infty} \delta_{x_0}(\varphi) = \varphi(x_0)$.

Generalization of the definition: In (U.5) $\mathcal{D}(\Omega) = C_c^\infty(\Omega; \mathbb{R})$ is replaced by $C_c^\infty(\Omega; \vec{\mathbb{R}}^n)$. So if you consider a basis $(\vec{e}_i)$ then $\vec{\varphi} \in C_c^\infty(\Omega; \vec{\mathbb{R}}^n)$ reads $\vec{\varphi} = \sum_{i=1}^n \varphi^i \vec{e}_i$ with $\varphi^i \in \mathcal{D}(\Omega)$ for all i.

Example U.6 Power: Let $\alpha : \Omega \to T_1^0(\Omega)$ be a differential form. Then $P = T_\alpha$ defined by $P(\vec{v}) = \int_\Omega \alpha.\vec{v}\, d\Omega$ gives the virtual power associated to α relative to the vector field $\vec{v}$ (mechanics and thermodynamics). ⁂

U.2 Derivation of a distribution

Let O be a point in $\mathbb{R}^n$ (an origin). If $p \in \mathbb{R}^n$ and if $(\vec{e}_i)$ is a basis in $\vec{\mathbb{R}}^n$, let $\vec{x} = \overrightarrow{Op} = \sum_{i=1}^n x_i \vec{e}_i$.

Definition U.7 The derivative $\frac{\partial T}{\partial x_i}$ of a distribution $T \in \mathcal{D}'(\Omega)$ is the distribution $\in \mathcal{D}'(\Omega)$ defined by, for all $\varphi \in \mathcal{D}(\Omega)$,

$$\frac{\partial T}{\partial x_i}(\varphi) := -T(\frac{\partial \varphi}{\partial x_i}), \quad \text{i.e.} \quad \langle \frac{\partial T}{\partial x_i}, \varphi \rangle := -\langle T, \frac{\partial \varphi}{\partial x_i} \rangle. \tag{U.8}$$

($\frac{\partial T}{\partial x_i}$ is indeed a distribution: Easy check.)

Example U.8 If $T = T_f$ is a regular distribution with $f \in C^1(\Omega)$, then $\frac{\partial (T_f)}{\partial x_i} = T_{(\frac{\partial f}{\partial x_i})}$. Indeed, for all $\varphi \in \mathcal{D}(\Omega)$, $\frac{\partial (T_f)}{\partial x_i}(\varphi) = -T_f(\frac{\partial \varphi}{\partial x_i}) = -\int_\Omega f(x) \frac{\partial \varphi}{\partial x_i}\, d\Omega = +\int_\Omega \frac{\partial f}{\partial x_i} \varphi(x)\, d\Omega + \int_\Gamma 0\, d\Gamma$, since φ vanishes on $\Gamma = \partial\Omega$ (the support of φ is compact in Ω), thus $\frac{\partial (T_f)}{\partial x_i}(\varphi) = T_{(\frac{\partial f}{\partial x_i})}(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$. ⁂

Example U.9 Consider the Heaviside function (the unit step function) $H_0 := 1_{\mathbb{R}_+}$ and the associated distribution $T = T_{H_0}$. Then $\langle (T_{H_0})', \varphi \rangle := -\langle T_{H_0}, \varphi' \rangle = -\int_\Omega H_0(x)\varphi'(x)\, dx = -\int_0^\infty \varphi'(x)\, dx = \varphi(0) = \langle \delta_0, \varphi \rangle$ for any $\varphi \in \mathcal{D}(\mathbb{R})$, thus $(T_{H_0})' = \delta_0$. Written ${H_0}' = \delta_0$ in $\mathcal{D}'(\Omega)$, which is not in a equality between functions, because H_0 is not derivable at 0 as a function, and δ_0 is not a function; It is equality between distributions: The notation ${H_0}'$ can only be used to compute ${H_0}'(\varphi)$ $(= \langle {H_0}', \varphi \rangle := -\langle H_0, \varphi' \rangle)$. ⁂

---

[3] The $\mathcal{D}(\Omega)$-continuity of T is defined by: 1- A sequence $(\varphi_n)_{\mathbb{N}^*}$ in $\mathcal{D}(\Omega)$ converges in $\mathcal{D}(\Omega)$ towards a function $\varphi \in \mathcal{D}(\Omega)$ iff there exists a compact $K \subset \Omega$ s.t. $\mathrm{supp}(\varphi_n) \subset K$ for all n, and $||\frac{\partial^k \varphi}{\partial x_{i_1} \ldots \partial x_{i_k}} - \frac{\partial^k \varphi_n}{\partial x_{i_1} \ldots \partial x_{i_k}}||_\infty \longrightarrow_{n\to\infty} 0$ for all $k \in \mathbb{N}$ and all i_j; 2- T is continuous at $\varphi \in \mathcal{D}(\Omega)$ iff $T(\varphi_n) \underset{n\to\infty}{\longrightarrow} T(\varphi)$ for any sequence $(\varphi_n)_\mathbb{N} \in \mathcal{D}(\Omega)^\mathbb{N} \underset{n\to\infty}{\longrightarrow} \varphi$ in $\mathcal{D}(\Omega)$.

U.3 Hilbert space $H^1(\Omega)$

U.3.1 Motivation

Consider the hat function $\Lambda(x)\begin{cases} = x+1 \text{ if } x\in[-1,0], \\ = 1-x \text{ if } x\in[0,1], \\ = 0 \text{ otherwise}\end{cases}$ (drawing). When applying the finite element method, it is well-known that, if you use integrals (if you use the virtual power principle which makes you compute average values), then you can consider the derivative of the hat function Λ as if it was the usual derivative, i.e. at the points where the usual computation of Λ' is meaningful, that is,

$$\Lambda'(x)\begin{cases} = 1 \text{ if } x\in]-1,0[, \\ = -1 \text{ if } x\in]0,1[, \\ = 0 \text{ if } x\in\mathbb{R}-\{-1,0,1\}\end{cases} \tag{U.9}$$

(drawing).

Problem: Λ' is not defined at $-1,0,1$ (the function Λ is not derivable at $-1,0,1$);
Question: So does the "usual" computation $I=\int_{\mathbb{R}}\Lambda'(x)\varphi(x)\,dx$ with (U.9) gives the good result? (This is not a trivial question: E.g., with $H_0=1_{\mathbb{R}_+}$ instead of Λ, we would get the absurd result $H_0'=0$, absurd since $H_0'=\delta_0$.)

Answer: Yes:

1- Consider T_Λ the regular distribution associated to Λ, cf. (U.6);

2- Then consider $(T_\Lambda)'$, cf. (U.8): We get $\langle (T_\Lambda)',\varphi\rangle \overset{(U.8)}{=} -\langle T_\Lambda,\varphi'\rangle = -\int_{\mathbb{R}}\Lambda(x)\varphi'(x)\,dx = -\int_{-1}^0\Lambda(x)\varphi'(x)\,dx-\int_0^1\Lambda(x)\varphi'(x)\,dx = +\int_{-1}^0 1_{]-1,0[}(x)\varphi(x)\,dx+\int_0^1 1_{]0,1[}\varphi(x)\,dx$, for any $\varphi\in\mathcal{D}(\mathbb{R})$;

3- Thus $(T_\Lambda)'=T_f$ where $f=1_{]-1,0[}+1_{]0,1[}$, that is $(T_\Lambda)'$ is a regular distribution, And its is named $f=\Lambda'$ within the distribution framework, i.e., for computations $\langle\Lambda',\varphi\rangle := \langle (T_\Lambda)',\varphi\rangle$ with $\varphi\in\mathcal{D}(\mathbb{R})$ (value $=\int_{\mathbb{R}}f(x)\varphi(x)\,dx$).

U.3.2 Definition of $H^1(\Omega)$

The space $C^1(\Omega;\mathbb{R})$ is too small in many applications (e.g., for the Λ function above); We need a larger space where the functions are "derivable is a weaker sense" which is the distribution sense. Consider a basis in $\mathbb{R}^n$:

Definition U.10 The Sobolev space $H^1(\Omega)$ is the subspace of $L^2(\Omega)$ restricted to functions whose generalized derivatives are in $L^2(\Omega)$:

$$H^1(\Omega)=\{v\in L^2(\Omega) : \frac{\partial v}{\partial x_i}\in L^2(\Omega),\ \forall i=1,...,n\}. \tag{U.10}$$

Usual shortened notation: $H^1(\Omega)=\{v\in L^2(\Omega) : \vec{\text{grad}} v\in L^2(\Omega)^n\}$.

So to check that $v\in H^1(\Omega)$, even if $\frac{\partial v}{\partial x_i}$ does not exists in the classic way (see the above hat function Λ), you have to:

1- Consider its associated regular distribution T_v,

2- Compute $\frac{\partial T_v}{\partial x_i}$ in $\mathcal{D}'(\Omega)$,

3- And if, for all i, there exists $f_i\in L^2(\Omega)$ s.t. $\frac{\partial T_v}{\partial x_i}=T_{f_i}$, then $v\in H^1(\Omega)$.

4- And then f_i is noted $\frac{\partial v}{\partial x_i}$ when used within the Lebesgue integrals $\int_\Omega\frac{\partial v}{\partial x_i}(x)\varphi(x)\,dx$ with $\varphi\in\mathcal{D}(\Omega)$.

E.g., $\Lambda\in H^1(\mathbb{R})$ since $(T_\Lambda)'=T_f$ with $f=1_{]-1,0[}+1_{]0,1[}\in L^2(\mathbb{R})$; And $(T_\Lambda)' =^{\text{noted}} \Lambda'$ $(=f)$ in the distribution context (integral computations).

Let $(\cdot,\cdot)_{L^2}$ and $||.||_{L^2}$ be the usual inner dot product and norm in $L^2(\Omega)$, i.e.

$$(u,v)_{L^2}=\int_\Omega u(x)v(x)\,d\Omega, \quad\text{and}\quad ||v||_{L^2}=\sqrt{(v,v)_{L^2}}=(\int_\Omega v(x)^2\,d\Omega)^{\frac12}. \tag{U.11}$$

$(L^2(\Omega),(\cdot,\cdot)_{L^2})$ is a Hilbert space. Then define, for all $u,v\in H^1(\Omega)$,

$$(u,v)_{H^1}=(u,v)_{L^2}+\sum_{i=1}^n(\frac{\partial u}{\partial x_i},\frac{\partial v}{\partial x_i})_{L^2}, \quad\text{and}\quad ||v||_{H^1}=(v,v)_{H^1}^{\frac12}. \tag{U.12}$$

Then $(H^1(\Omega),(\cdot,\cdot)_{H^1})$ is a Hilbert space (Riesz–Fisher theorem).

With a Euclidean dot product $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n$ and a $(\cdot,\cdot)_g$-orthonormal basis,

$$(u,v)_{H^1} = (u,v)_{L^2} + (\overrightarrow{\mathrm{grad}}u, \overrightarrow{\mathrm{grad}}v)_{L^2}. \tag{U.13}$$

U.3.3 Subspace $H_0^1(\Omega)$ and its dual space $H^{-1}(\Omega)$

The boundary $\Gamma = \partial\Omega$ of Ω is supposed to be regular. Let

$$H_0^1(\Omega) := \{v \in H^1(\Omega) : v_{|\Gamma} = 0\}. \tag{U.14}$$

Then $(H_0^1(\Omega), (\cdot,\cdot)_{H^1})$ is a Hilbert space.

More generally (without any regularity assumption on Γ), $H_0^1(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^1}$ = the closure of $\mathcal{D}(\Omega)$ in $(H^1(\Omega), ||.||_{H^1})$: This closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ enables the use of the distribution framework.

Notation : The dual space of $H_0^1(\Omega)$ is the space

$$H^{-1}(\Omega) = (H_0^1(\Omega))' = \mathcal{L}(H_0^1(\Omega);\mathbb{R}) \tag{U.15}$$

equipped with the (usual) norm $||T||_{H^{-1}} := \sup_{||v||_{H_0^1}=1} |T(v)|$. And (duality bracket), if $v \in H_0^1(\Omega)$ and $T \in H^{-1}(\Omega)$ then

$$T(v) \overset{\text{noted}}{=} \langle T, v\rangle_{H^{-1},H_0^1} \overset{\text{noted}}{=} \langle T, v\rangle. \tag{U.16}$$

Theorem U.11 **(Characterization of $H^{-1}(\Omega) = (H_0^1(\Omega))'$.)** *A distribution T is in $H^{-1}(\Omega)$ iff*

$$\exists (f,\vec{g}) \in L^2(\Omega) \times L^2(\Omega)^n \quad s.t. \quad T = f - \mathrm{div}\vec{g} \ \ (\in \mathcal{D}'(\Omega)), \tag{U.17}$$

that is, for all $v \in H_0^1(\Omega)$,

$$\langle T, v\rangle_{H^{-1},H_0^1} = \int_\Omega fv\,d\Omega + \int_\Omega dv.\vec{g}\,d\Omega. \tag{U.18}$$

And if Ω is bounded then we can choose $f = 0$. If moreover $\vec{g} \in H^1(\Omega)^n$ then

$$\langle T, v\rangle_{H^{-1},H_0^1} = \int_\Omega f(x)v(x)\,dx - \int_\Omega \mathrm{div}\vec{g}(x)v(x)\,dx. \tag{U.19}$$

(In fact we only need $\vec{g} \in H_{\mathrm{div}}(\Omega) = \{\vec{g} \in L^2(\Omega)^n : \mathrm{div}\vec{g} \in L^2(\Omega)\}$.)

Proof. E.g., see Brezis [4]. ∴

For boundary value problems with Neumann boundary conditions, we then need $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$. Characterization of $(H^1(\Omega))'$: We still have (U.18), but we have to replace (U.17) or (U.19) by, with a Euclidean dot product in $\vec{\mathbb{R}}^n$, see Brezis [4],

$$\langle T, v\rangle_{(H^1)',H^1} = \int_\Omega f(x)v(x)\,dx - \int_\Omega \mathrm{div}\vec{g}(x)v(x)\,dx + \int_\Gamma \vec{g}(x) \bullet \vec{n}(x)\,v(x)\,dx. \tag{U.20}$$

References

[1] Abraham R., Marsden J.E.: *Foundation of mechanics, 2nd edition.* Addison-Wesley, 1985.

[2] Arnold V.I.: *Equations différentielles ordinaires.* Editions Mir, 1974.

[3] Arnold V.I.: *Mathematical Methods of Classical Mechanics.* Second Edition, Springer 1989.

[4] Brézis H.: *Functional Analysis, Sobolev spaces and Partial differential equations.* Springer, 2011.

[5] Cartan H.: *Formes différentielles.* Hermann 1967.

[6] Cartan H.: *Cours de calcul différentiel.* Hermann 1990.

[7] Forest et al: *Mécanique de milieux continus.* Ecole des Mines de Paris. 2020. `http://mms2.ensmp.fr/mmc_paris/poly/MMC2020web.pdf`

[8] Germain P.: *Cours de Mécanique des Milieux Continus. Tome 1: théorie générale.* Masson. Paris. 1973.

[9] Germain P.: The Method of Virtual Power in Continuum Mechanics. Part 2: Microstructure. SIAM Journal on Applied Mathematics, Vol. 25, No. 3 (Nov., 1973), pp. 556-575.

[10] Hughes T.J.R., Marsden J.E.: *A short Course in Fluid Mechanics.* Publish or Perish, 1976.

[11] Le Dret Hervé: *Méthodes mathématiques en élasticité, notes de cours de DEA 2003–2004.* Université Pierre et Marie Curie.

[12] Marsden J.E., Hughes T.J.R.: *Mathematical Foundations of Elasticity.* Dover publications, 1993.

[13] Maxwell: A treatise on electricity and magnetism, Clarendon press series, 1873.

[14] Misner C.W., Thorne K.S., Wheeler J.A.: *Gravitation.* W.H. Freeman and Cie, 1973

[15] Petropoulos M.: Relativité Générale. La gravitation en une leçon et demie. `https://gargantua.polytechnique.fr/siatel-web/linkto/mICYYYTEsNY`

[16] Sadd Martin H.: *Elasticity. Theory, Applications, and Numerics.* Elsevier 2005.

[17] Spivak M.: *A comprehensive introduction to differential geometry, Volume 1.* Publish or Perish, Inc. 1979.

[18] Strang G.: *Introduction to linear algebra.* 5th edition. Wellesley - Cambridge Press 2016.

[19] Truesdell C., Noll W.: *The Non-Linear Field Theories of Mechanics.* Springer, 2004.

[20] http://planet-terre.ens-lyon.fr/article/force-de-coriolis.xml, 2018.